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A method for constructing exact solutions and application to Benjamin Ono equation*

Wang Zhen(王 振)[a)†], Li De-Sheng(李德生)[a)b)],
Lu Hui-Fang(鲁慧芳)[a)], and Zhang Hong-Qing(张鸿庆)[a)]

[a)] *Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China*

[b)] *Department of Mathematics, Shenyang Normal University, Shenyang 110034, China*

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By using an improved projective Riccati equation method, this paper obtains several types of exact travelling wave solutions to the Benjamin Ono equation which include multiple soliton solutions, periodic soliton solutions and Weierstrass function solutions. Some of them are found for the first time. The method can be applied to other nonlinear evolution equations in mathematical physics.

Keywords: Benjamin Ono equation, nonlinear evolution equation, Weierstrass function solutions, exact solutions

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1. Introduction

Due to the wide applications of soliton theory in mathematics, physics, chemistry, biology, communications, astrophysics and geophysics, the study on integrable models has attracted much attention of many mathematicians and physicists. To find some exact explicit soliton solutions for integrable models is one of the most important and significant tasks. Various effective methods like the inverse scattering transformation, Darboux transformation, Bäcklund transformation,[1] Hirota's direct method,[2] mapping and deformation approach,[3] standard and extended truncated Painléve analysis[4] etc. have been well developed to find exact solutions for integrable models.

Conte and Musette[5] presented a projective Riccati equation method to seek more new solitary wave solutions to nonlinear evolution equations, which can be expressed as polynomial in two elementary functions satisfied a projective Riccati equations.[6] This method had been applied to find many solitary wave solutions of many equations.[5−10]

In this paper, we improve this method[5] and first apply it to the Benjamin Ono equation. It is shown that some new solutions to projective Riccati equations are brought up in Section 2. In Section 3, we obtain several families of solutions to Benjamin Ono equation which include Weierstrass elliptic functions solutions, kink-shaped soliton solutions, singular soliton solutions and new solitary waves. A short summary and discussion is presented in Section 4.

2.Some new solutions of projective Riccati equations

Now, we consider the projective Ricatti equations Refs.[5–8]

$$\begin{aligned} f'(\xi) &= pf(\xi)g(\xi), \\ g'(\xi) &= q + pg(\xi)^2 - rf(\xi), \end{aligned} \tag{1}$$

where $p^2 = 1$, q and r are two real constants. When $p = -1$ and $q = 1$, Eq.(1) reduces to the coupled equations given in the Refs.[5–7], and when $p = \pm 1$ and $q \geq 0$, Eq.(1) reduces to the coupled equations given in the Ref.[8].

For Eq.(1), under the relation between $f(\xi)$ and $g(\xi)$

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2+\delta}{q}f^2\right], \tag{2}$$

when $\delta = \pm 1$, Ref.[11] had discussed it. In this paper, we will discuss other cases.

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†E-mail: wangzhen8107@yahoo.com.cn

Case A when $\delta = -r^2$, Eq.(2) reduces to

$$g^2 = \frac{2r}{p}f - \frac{q}{p}. \tag{3}$$

Equations (1) with Eq.(3) have the following Weierstrass elliptic function solution:

$$f(\xi) = \frac{q}{6r} + \frac{2}{pr}\wp(\xi),$$
$$g(\xi) = \frac{12\wp'(\xi)}{q + 12p\wp(\xi)}, \tag{4}$$

where $p = \pm 1$, Weierstrass elliptic function $\wp(\xi) = \wp(\xi; g_2, g_3)$ satisfies

$$\wp'^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3.$$

and $g_2 = \frac{q^2}{12}$, $g_3 = \frac{p^3q^3}{216} = \frac{pq^3}{216}$.

Case B when $\delta = -\frac{r^2}{25}$, Eq.(2) reduces to

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{24r^2}{25q}f^2\right], \tag{5}$$

the projective Riccati equations have Weierstrass elliptic function solution:

$$f(\xi) = \frac{5q}{6r} + \frac{5pq^2}{72r\wp(\xi)},$$
$$g(\xi) = -\frac{q\wp'(\xi)}{\wp(\xi)(12\wp(\xi) + pq)}, \tag{6}$$

where $p = \pm 1$. q and r in Eq.(4) and Eq.(6) are arbitrary constants. There is also some other methods for constructing elliptic function solutions.[12−15]

Case C when $\delta = h^2 - s^2$ and $pq < 0$, Eq.(2) reduces to

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2 + h^2 - s^2}{q}f^2\right]. \tag{7}$$

Equations (1) with Eq.(7) admit a solitary solution

$$f(\xi) = \frac{q}{r + s\cosh\left(\sqrt{-pq}\xi\right) + h\sinh\left(\sqrt{-pq}\xi\right)},$$
$$g(\xi) = -\frac{\sqrt{-pq}}{p}\frac{s\sinh\left(\sqrt{-pq}\xi\right) + h\cosh\left(\sqrt{-pq}\xi\right)}{r + s\cosh\left(\sqrt{-pq}\xi\right) + h\sinh\left(\sqrt{-pq}\xi\right)}, \tag{8}$$

where s and h are arbitrary constants, $p = \pm 1$.

Case D when $\delta = -s^2 - h^2$ and $pq > 0$, Eq.(2) reduces to

$$g^2 = -\frac{1}{p}\left[q - 2rf + \frac{r^2 - h^2 - s^2}{q}f^2\right]. \tag{9}$$

Equations (1) with Eq.(9) have the following trigonometric function solution:

$$f(\xi) = \frac{q}{r + s\cos\left(\sqrt{pq}\xi\right) + h\sin\left(\sqrt{pq}\xi\right)},$$
$$g(\xi) = \frac{\sqrt{pq}}{p}\frac{s\sin\left(\sqrt{pq}\xi\right) - h\cos\left(\sqrt{pq}\xi\right)}{r + s\cos\left(\sqrt{pq}\xi\right) + h\sin\left(\sqrt{pq}\xi\right)}, \tag{10}$$

where s and h are arbitrary constants, $p = \pm 1$.

Case E when $q = 0$, the projective Riccati equations have a rational solution

$$f(\xi) = \frac{2}{pr\xi^2 + C_1\xi - C_2},$$
$$g(\xi) = -\frac{2pr\xi + C_1}{(pr\xi^2 + C_1\xi - C_2)p}. \tag{11}$$

Easily, we find that $f(\xi)$ and $g(\xi)$ satisfy

$$g^2 = \frac{2r}{p}f + \frac{C_1^2 + 4C_2pr}{4}f^2, \tag{12}$$

where C_1, C_2 and r are arbitrary constants, $p = \pm 1$.

Remark 1 In Case A and B, Weierstrass elliptic function can reduce into Jacobi elliptic function and hyperbolic functions in certain conditions. We will show it in next section.

Remark 2 In Case C, if we select $h = 1, s = 0$ and $h = 0, s = 1$, then Eq.(8) reduces into the first and the second type solution of projective Riccati equation given in Ref.[11] respectively.

Remark 3 In Case D, if we select $h = 1, s = 0$ and $h = 0, s = 1$, then Eq.(8) reduces into the third and the fourth type solution of projective Riccati equation given in Ref.[11] respectively.

Remark 4 Equations (11) had given in Ref.[11], but $f(\xi)$ and $g(\xi)$ don't satisfy Eq.(2). We give the relation Eq.(12) of $f(\xi)$ and $g(\xi)$ and obtain new rational solution to Benjamin Ono equation with the relation.

3. Exact solutions to Benjamin Ono equation

In the following, we discuss a soliton model, the Benjamin Ono equation,[16−18]

$$u_{tt} + \beta(u^2)_{xx} + \gamma u_{xxxx} = 0. \tag{13}$$

The travelling wave solutions of Eq.(13) take the following form

$$u(x,t) = u(\xi), \xi = kx + ct, \tag{14}$$

where c is wave speed. After substituting Eq.(14) into Eq.(13), we have

$$c^2u'' + 2\beta k^2(u'^2 + uu'') + k^4\gamma u'''' = 0. \tag{15}$$

In order to solve Eq.(13), a crucial ansatz

$$u(\xi) = A_0 + \sum_{i=1}^{n}(A_i f(\xi) + B_i g(\xi)) f^{i-1}(\xi) \qquad (16)$$

is introduced, and $f(\xi)$ and $g(\xi)$ are nonzero solutions to the projective Riccati equations.

The standard leading order analysis tells us that the truncated Painlevé expression of the Benjamin Ono equation has the following formal solution.

$$u = A_0 + A_1 f + B_1 g + A_2 f^2 + B_2 fg. \qquad (17)$$

Substituting Eqs.(1) and (3), along with Eq.(17) into Eq.(15) and collecting all terms with the same power in $f(\xi)^i g(\xi)^j$ (i=0, 1, 2, 3, 4, j=0, 1, 2, 3, 4), setting the coefficients of these terms $f(\xi)^i g(\xi)^j$ to zero, we obtain a nonlinear algebraic equation system with respect to the unknowns $A_0, A_1, B_1, c, k, p, q, r$.

We could determine the following solutions:

type 1.1

$$A_0 = \frac{-c^2 + k^4\gamma\, pq}{2\beta\, k^2},$$
$$A_1 = -3\,\frac{k^2\gamma\, rp}{\beta}, \quad A_2 = B_1 = B_2 = 0 \qquad (18)$$

and k, β, γ and c are arbitrary constants, $p = \pm 1$.

According to Eqs.(4), (17) and Eq.(18). we obtain the solution to Benjamin Ono equation Eq.(13):

family 1

$$u_1(x,t) = -\frac{c^2 + 12\,k^4\gamma\ \wp(\xi)}{2\beta k^2}; \qquad (19)$$

similarly, substituting Eqs.(1), (5) and (17) into Benjamin Ono equation, we have

type 2.1

$$A_0 = \frac{k^4\gamma\, qp - c^2}{2\beta\, k^2}, \quad A_1 = -3\,\frac{k^2\gamma\, pr}{\beta},$$
$$A_2 = 72\,\frac{k^2\gamma\, pr^2 q}{\beta}, \quad B_1 = 0,$$
$$B_2 = \pm\,6\,\frac{\gamma\,\sqrt{-6\,pq}k^2 rp}{\beta}, \qquad (20)$$

where $p = \pm 1$, obviously, there is a constraint $pq < 0$;

type 2.2 For $q = -\dfrac{\sqrt{6}}{12}, B_1 = B_2 = 0$, then

$$A_0 = -\frac{\sqrt{6}\left(k^4\gamma\, p + 2\sqrt{6}c^2\right)}{24k^2\beta}, \quad A_1 = -\frac{6k^2\gamma\, pr}{\beta},$$
$$A_2 = -\frac{12k^2\gamma\, pr^2\sqrt{6}}{\beta}. \qquad (21)$$

According to Eqs.(6), (17) and Eqs.(20), (21). we obtain two families Weierstrass elliptic function solutions to Benjamin Ono equation(13):

family 2

$$u_2(x,t) = \frac{-36\,c^2 - 144\,k^4\gamma\, qp + 3600\,k^4\gamma\, pq^3}{72\beta\, k^2} + \frac{600\,k^4\gamma q^4 - 15\,k^4\gamma q^2}{72\beta\, k^2\wp(\xi)} + \frac{\pm 30\,\gamma\,\sqrt{6}\sqrt{-pq}k^4 pq^2\wp'(\xi) + 25\,k^4\gamma\, pq^5}{72\beta\, k^2\wp(\xi)^2}, \qquad (22)$$

where $p = \pm 1$;

family 3

$$u_3(x,t) = -\frac{\sqrt{6}(k^4\gamma\, p + 2\sqrt{6}c^2)}{24k^2\beta} + \frac{k^2\gamma\, p}{\beta}\left(\frac{5\sqrt{6}}{12} - \frac{5p}{288\wp(\xi)}\right) - \frac{k^2\gamma p\sqrt{6}}{3\beta}\left(\frac{5\sqrt{6}}{12} - \frac{5p}{288\wp(\xi)}\right)^2, \qquad (23)$$

where $p = \pm 1$. As we know, there is a relation

$$\wp(\xi; g_2, g_3) = e_2 - (e_2 - e_3)\mathrm{cn}^2(\sqrt{(e_1 - e_3)}\xi; m) \qquad (24)$$

between the solution of

$$\wp'^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3 \qquad (25)$$

and the Jacobi elliptic functions. Where $m^2 = \dfrac{e_2 - e_3}{e_1 - e_3}$ is the modules of Jacobi elliptic functions. e_i ($i = 1, 2, 3; e_1 \geq e_2 \geq e_3$) are roots of $4z^3 - g_2 z - g_3 = 0$.

For simplicity, we use e_{i-j} to express $e_i - e_j$.

So, the Weierstrass elliptic function solutions u_1 of the Benjamin Ono equation can be rewritten as

$$u_{1'}(x,t) = -\frac{c^2 + 12\,k^4\gamma\ (e_2 - e_{2-3}\mathrm{cn}^2(\sqrt{e_{1-3}}\xi))}{2\beta k^2}$$

It is easy to find that $u_{1'}$ is equivalent to that in Ref.[18].

The Weierstrass elliptic function solutions u_2 of the Benjamin Ono equation can be rewritten as

$$u_{2'}(x,t)=\frac{-36\,c^2-144\,k^4\gamma\,qp+3600\,k^4\gamma\,pq^3}{72\beta\,k^2}+\frac{600\,k^4\gamma\,q^4-15\,k^4\gamma q^2}{72\beta\,k^2(e_2-e_{2-3}\mathrm{cn}^2(\sqrt{e_{1-3}}\xi))}$$
$$+\frac{\pm30\gamma\sqrt{6}\sqrt{-pq}k^4pq^2(2e_{2-3}\sqrt{e_{1-3}}\mathrm{sn}(\sqrt{e_{1-3}}\xi)\mathrm{cn}(\sqrt{e_{1-3}}\xi)\mathrm{dn}(\sqrt{e_{1-3}}\xi))+25\,k^4\gamma pq^5}{72\beta\,k^2(e_2-e_{2-3}\mathrm{cn}^2(\sqrt{e_{1-3}}\xi))^2},\qquad(26)$$

where $p=\pm1$.

As we know, when $m\to1$, Jacobi elliptic functions degenerate into hyperbolic functions, whereas when $m\to0$, Jacobi elliptic functions degenerate into trigonometric functions.

When $m\to1$, then $\mathrm{sn}(\xi;m)\to\tanh(\xi)$, $\mathrm{cn}(\xi;\mathrm{m})\to\mathrm{sech}(\xi)$, $\mathrm{dn}(\xi;m)\to\mathrm{sech}(\xi)$,

When $m\to0$, then $\mathrm{sn}(\xi;m)\to\sin(\xi)$, $\mathrm{cn}(\xi;\mathrm{m})\to\cos(\xi)$, $\mathrm{dn}(\xi;m)\to1$,

When $m\to1$, in other words, $e_2\to e_1$, then Eq.(26) degenerates into a soliton solution:

$$u_{2''}(x,t)=\frac{-36\,c^2-144\,k^4\gamma\,qp+3600\,k^4\gamma\,pq^3}{72\beta\,k^2}+\frac{600\,k^4\gamma\,q^4-15\,k^4\gamma q^2}{72\beta\,k^2(e_2-e_{2-3}\mathrm{sech}^2(\sqrt{e_{1-3}}\xi))}$$
$$+\frac{\pm60\gamma\sqrt{6}\sqrt{-pq}k^4pq^2(e_{1-3}{}^{\frac{3}{2}}\tanh(\sqrt{e_{1-3}}\xi)\mathrm{sech}^2(\sqrt{e_{1-3}}\xi))+25\,k^4\gamma\,pq^5}{72\beta\,k^2(e_1-e_{1-3}\mathrm{sech}^2(\sqrt{e_{1-3}}\xi))^2}.\qquad(27)$$

Substituting Eqs.(1), (7) and (17) into Benjamin Ono equation, we determine:

type 3.1 for $r=0$, then

$$B_1=0,\quad A_1=0,\quad B_2=0,$$
$$A_0=\frac{4\gamma\,pq-c^2}{2\beta\,k^2},\quad A_2=-6\,\frac{\gamma\,p\left(-h^2+s^2\right)}{\beta\,k^2q};\qquad(28)$$

type 3.2 for $r^2=s^2-h^2$, then

$$B_1=0,\quad B_2=0,$$
$$A_0=\frac{-c^2+\gamma\,pq}{2k^2\beta},\quad A_1=-3\,\frac{\gamma\,pr}{k^2\beta},\quad A_2=0;\qquad(29)$$

type 3.3 for $h=\pm s$, then

$$B_1=0,\quad B_2=0,\quad A_0=\frac{-c^2+\gamma\,pq}{2k^2\beta},$$
$$A_1=-6\,\frac{\gamma\,pr}{k^2\beta},\quad A_2=6\,\frac{\gamma\,pr^2}{k^2\beta\,q}\qquad(30)$$

type 3.4 for $B_1=0$,then

$$A_0=\frac{\gamma pq-c^2}{2\beta k^2},$$
$$A_1=-\frac{3\gamma pr}{\beta\,k^2},$$
$$A_2=\frac{3(r^2+h^2-s^2)\gamma p}{\beta\,k^2q},$$
$$B_2=\pm\frac{3\gamma p}{k^2\beta}\sqrt{\frac{p(r^2+h^2-s^2)}{-q}},\qquad(31)$$

where $p=\pm1$ in Eqs.(28)–(31).

According to Eqs.(8), (17) and Eqs.(28)–(31), we obtain five families soliton solutions to Benjamin Ono Eq.(13):

family 4

$$u_4(x,t)=\frac{4\gamma\,pq-c^2}{2\beta\,k^2}-\frac{6\gamma\,pq\left(-h^2+s^2\right)}{\beta\,k^2(s\cosh(\sqrt{-pq}\xi)+h\sinh(\sqrt{-pq}\xi))^2},\qquad(32)$$

where $p=\pm1$.

family 5 for $r^2=s^2-h^2$,

$$u_5(x,t)=\frac{-c^2+\gamma\,pq}{2k^2\beta}-\frac{1}{k^2\beta}\frac{3\gamma\,pqr}{r+s\cosh(\sqrt{-pq}\xi)+h\sinh(\sqrt{-pq}\xi)},\qquad(33)$$

where $p=\pm1$.

family 6

$$u_6(x,t)=\frac{\gamma\,pq-c^2}{2k^2\beta}-\frac{6\gamma\,pqr}{k^2\beta}\frac{s\cosh(\sqrt{-pq}\xi)\pm s\sinh(\sqrt{-pq}\xi)}{(r+s\cosh(\sqrt{-pq}\xi)\pm s\sinh(\sqrt{-pq}\xi))^2},\qquad(34)$$

where $p = \pm 1$.

family 7

$$u_7(x,t) = \frac{\gamma\, pq - c^2}{2\beta\, k^2} - \frac{3\gamma\, pqr}{\beta\, k^2(r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi))} + \frac{3\gamma pq(r^2+h^2-s^2) \pm 3\gamma pq\sqrt{(r^2+h^2-s^2)}(h\cosh(\sqrt{-pq}\xi) + s\sinh(\sqrt{-pq}\xi))}{\beta k^2(r + s\cosh(\sqrt{-pq}\xi) + h\sinh(\sqrt{-pq}\xi))^2}, \tag{35}$$

where $p = \pm 1$.

Substituting Eqs.(1), (9) and (17) into Benjamin Ono equation Eq.(15), we determined:

type 4.1 for $r = 0$, then

$$B_2 = 0, \quad B_1 = 0, \quad A_1 = 0,$$
$$A_0 = \frac{4\gamma\, pq - c^2}{2k^2\beta}, \quad A_2 = -6\,\frac{\gamma\, p\left(h^2+s^2\right)}{\beta\, k^2 q}; \tag{36}$$

type 4.2 for $r^2 = s^2 + h^2$,

$$B_1 = 0, \quad B_2 = 0, \quad A_2 = 0,$$
$$A_0 = \frac{\gamma\, pq - c^2}{2k^2\beta}, \quad A_1 = -3\,\frac{\gamma\, pr}{k^2\beta}; \tag{37}$$

type 4.3 for $r = 0, B_1 = A_1 = 0$, then

$$A_0 = \frac{\gamma^2 pq - c^2}{2k^2\beta}, \quad A_2 = -3\frac{(h^2+s^2)p}{\beta\, k^2 q},$$
$$B_2 = -3\,\frac{p\sqrt{\gamma}\sqrt{pq(h^2+s^2)}}{\beta\, k^2 q}; \tag{38}$$

type 4.4 for $h = \pm is$, then

$$B_1 = 0, \quad B_2 = 0, \quad A_1 = -6\,\frac{\gamma\, pr}{\beta\, k^2},$$
$$A_2 = 6\,\frac{\gamma\, pr^2}{\beta\, k^2 q}, \quad A_0 = \frac{\gamma\, pq - c^2}{2\beta\, k^2}, \tag{39}$$

where $\mathrm{i} = \sqrt{-1}$, $p = \pm 1$;

type 4.5 for $B_1 = 0$, then

$$A_0 = \frac{\gamma\, pq - c^2}{2\beta\, k^2}, \quad A_1 = -3\,\frac{\gamma\, pr}{\beta\, k^2},$$
$$A_2 = 3\,\frac{\left(h^2+s^2-r^2\right)\gamma\, p}{\beta\, k^2 q},$$
$$B_2 = \pm\frac{3\,\gamma p}{k^2\beta}\sqrt{\frac{p\left(h^2+s^2-r^2\right)}{q}}, \tag{40}$$

where $p = \pm 1$ in the above five types.

According to Eqs.(10), (17) and Eqs.(36)–(40), we obtain five families trigonometric function solutions to Benjamin Ono equation Eq.(13):

family 8

$$u_8(x,t) = \frac{4\gamma\, pq - c^2}{2k^2\beta} - \frac{6\gamma\, pq(h^2+s^2)}{\beta k^2(s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi))}; \tag{41}$$

where $p = \pm 1, pq > 0$ in this family and the following four families:

family 9 for $r^2 = s^2 + h^2$, then the solution to Benjamin Ono equation is

$$u_9(x,t) = \frac{\gamma\, pq - c^2}{2k^2\beta} - \frac{3\gamma\, pqr}{k^2\beta(r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi))}; \tag{42}$$

family 10

$$u_{10}(x,t) = \frac{\gamma^2 pq - c^2}{2k^2\beta} - \frac{3pq(h^2+s^2+\sqrt{\gamma(h^2+s^2)}(s\sin(\sqrt{pq}\xi) - h\cos(\sqrt{pq}\xi)))}{(s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi))^2}; \tag{43}$$

family 11

$$u_{11}(x,t) = \frac{\gamma\, pq - c^2}{2\beta\, k^2} - \frac{6\gamma pqr(s\cos(\sqrt{pq}\xi) \pm \mathrm{i}s\sin(\sqrt{pq}\xi))}{\beta k^2(r + s\cos(\sqrt{pq}\xi) \pm \mathrm{i}s\sin(\sqrt{pq}\xi))^2}; \tag{44}$$

family 12

$$u_{12}(x,t) = \frac{\gamma\, pq - c^2}{2k^2\beta} - \frac{3\gamma\, pqr}{k^2\beta(r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi))} + \frac{3\gamma pq(h^2+s^2-r^2 \pm \sqrt{h^2+s^2-r^2}(s\sin(\sqrt{pq}\xi) - h\cos(\sqrt{pq}\xi)))}{k^2\beta(r + s\cos(\sqrt{pq}\xi) + h\sin(\sqrt{pq}\xi))^2}. \tag{45}$$

Substituting Eqs.(1), (12) and (17) into Benjamin Ono equation Eq.(15), we determined:
type 5.1 for $B_1 = 0$, then

$$A_0 = -\frac{c^2}{2\beta k^2}, \quad A_1 = -3\,\frac{\gamma\, pr}{\beta k^2}, \quad A_2 = -\frac{3\gamma\,({C_1}^2 + 4\,C_2 pr)}{4\beta k^2}, \quad B_2 = \pm\frac{3\gamma\sqrt{4\,rC_2p + {C_1}^2}p}{2\beta k^2} \tag{46}$$

where $p = \pm 1$, c, k, C_1, C_2 are all arbitrary constants;
family 13

$$u_{13}(x,t) = -\frac{c^2}{2\beta k^2} - \frac{6\gamma\, pr + 3\gamma\,({C_1}^2 + 4\,C_2 pr)}{\beta k^2(pr\xi^2 + C_1\xi - C_2)^2} \pm \frac{(2\,pr\xi + C_1)3\gamma\sqrt{4\,rC_2p + {C_1}^2}}{\beta k^2(pr\xi^2 + C_1\xi - C_2)^2} \tag{47}$$

where $p = \pm 1$.

4. Conclusion

In this paper, we introduce five new relations about the solutions of the projective Riccati equations and apply them to solve Benjamin Ono equation. Many solutions are obtained for this Benjamin Ono equation, such as multiple solitary wave solutions constructed in terms of Weierstrass elliptic function, hyperbolic functions, periodic solutions expressed in terms of sin and cos functions, some solutions are not given in literatures to our knowledge. Of course, this method also can be applied to other nonlinear wave equations.

References

[1] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin, Heidelberg: Springer-Verlag)
[2] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[3] Lou S Y and Ni G J 1989 *J. Math. Phys.* **30** 1614
[4] Lou S Y 1998 *Z. Naturforsch* A **53**a 251
[5] Conte R and Musette M 1992 *J. Phys.* A: *Math. Gen.* **25** 5609
[6] Bountis T C, Papageorgiou V and Winternitz P 1986 *J. Math. Phys.* **27** 1215
[7] Zhang G X, Li Z B and Duan Y S 2000 *Sci. China* (Ser A) **30** 1103
[8] Yan Z Y 2003 *Chaos, Solitons and Fractals* **16** 759
[9] Xie F D and Gao X S 2004 *Chaos, Solitons and Fractals* **19** 1113
[10] Li D S and Zhang H Q 2004 *Chin. Phys.* **13** 1377
[11] Fu Z T, Liu S D and Liu S K 2004 *Chaos, Solitons and Fractals* **20** 301
[12] Liu S K, Fu Z T, Liu S D *et al* 2001 *Acta. Phys. Sin.* **50** 2068 (in Chinese)
[13] Liu S K, Fu Z T, Liu S D *et al* 2002 *Acta. Phys. Sin.* **51** 10 (in Chinese)
[14] Liu S D, Fu Z T, Liu S K *et al* 2002 *Acta. Phys. Sin.* **51** 718 (in Chinese)
[15] Liu S K, Fu Z T, Liu S D *et al* 2002 *Acta. Phys. Sin.* **51** 1927 (in Chinese)
[16] Gudkov V V 1997 *J. Math. Phys.* **38** 4794
[17] Olver P J 1977 *J. Math. Phys.* **18** 1212
[18] Fu Z T, Liu S K, Liu S D *et al* 2003 *Communications in Nonlinear Science and Numerical Simulation* **8** 67