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Quantum entanglement in the system of two two-level atoms interacting with a single-mode vacuum field*

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The entanglement properties of the system of two two-level atoms interacting with a single-mode vacuum field are explored. The quantum entanglement between two two-level atoms and a single-mode vacuum field is investigated by using the quantum reduced entropy; the quantum entanglement between two two-level atoms, and that between a single two-level atom and a single-mode vacuum field are studied in terms of the quantum relative entropy. The influences of the atomic dipole–dipole interaction on the quantum entanglement of the system are also discussed. Our results show that three entangled states of two atoms–field, atom–atom, and atom–field can be prepared via two two-level atoms interacting with a single-mode vacuum field.

Keywords: quantum entanglement, single mode vacuum field, reduced entropy, relative entropy
PACC: 4250

1. Introduction

Quantum entanglement (QE) is one of the most striking properties of quantum mechanics. It is a key problem in Einsten–Podolsky–Rosen (EPR) Paradox,[1] Bell's inequalities,[2] quantum cryptography,[3] quantum teleportation,[4] quantum computation[5] and so on, and has been viewed as an elementary resource for quantum information processing. It is essential to create and manipulate entangled states for quantum information application. Most experimental works of quantum entanglement have been implemented with photons. Although individual polarization states of photons are easily controlled and their quantum coherence can be preserved over many kilometers in an optical fibre,[6] photons cannot be stored for a long time, and the manipulation of collective entangled states demonstrates considerable difficulties even if photons are confined in the same cavity. The creation of long lived entangled pairs with atoms, on the other hand, is a relatively recent pursuit, which may provide reliable quantum information storage. The entangled state of a pair of two-level atoms by using pulse area technique in a microwave cavity has been realized by Hagley[7] based on the proposal of Cirac and Zoller.[8] The generation of atom-photon entanglement has also been proposed in Ref.[9] in a tripod-like laser-atom-cavity system, which sustains two cavity modes.

In this paper, we will explore the generation of atomic and atom–field entangled states via the system of two two-level atoms interacting with a single-mode vacuum field. We will investigate the degree of the entanglement (DE) between two two-level atoms and a single-mode vacuum field by using the quantum reduced entropy. In addition we will also study the DE between two two-level atoms, and that between a single two-level atom and a single-mode vacuum field through using the quantum relative entropy.[10]

2. Reduced density matrices of two two-level atoms and a single-mode vacuum field

The system considered here consists of two two-level atoms interacting with a single-mode vacuum field. We assume a two-level atom with an upper

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level $|+\rangle$ and a lower level $|-\rangle$. The upper level $|+\rangle$ is coupled to the lower level $|-\rangle$ by the single-mode vacuum field. In the interacting picture, the effective Hamiltonian [11] in the rotating wave approximation (RWA)[12,13] is ($\hbar = 1$)

$$H = H_0 + H_{\rm I}, \tag{1}$$

with

$$H_0 = \omega_0 \sum_{i=1}^{2} S_3^{(i)} + \omega a^+ a, \tag{2}$$

and

$$H_{\rm I} = \sum_{i=1}^{2} g(a^+ S_-^{(i)} + a S_+^{(i)}) + \Omega(S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)}). \tag{3}$$

Here a^+ and a are the creation and annihilation operators of the field mode of frequency ω; ω_0 is the atomic transition frequency; S_3, S_+ and S_- are the usual pseudo-spin operators of the two-level atom; g is the atom-field coupling constant, and Ω is the atomic dipole-dipole coupling constant, $i(=1,2)$ denotes ith atom. For simplicity, we consider only the resonant case ($\omega = \omega_0$).

We consider that at $t = 0$ the two atoms are in the excited state$|++\rangle$ and the field is in a single-mode vacuum state $|0\rangle$. The initial state of the system is a decoupled pure state, and the state vector is

$$|\Psi(0)\rangle = |\Psi_{a_1a_2}(0)\rangle \otimes |\Psi_f(0)\rangle = |++0\rangle. \tag{4}$$

In the interacting picture, at any time $t > 0$ the evolution of the state vector of the system obeys the Schrödinger equation ($\hbar = 1$)

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi_{\rm I}(t)\rangle = H_{\rm I}|\Psi_{\rm I}(t)\rangle. \tag{5}$$

It can be obtained by solving the Schrödinger equation

$$|\Psi_I(t)\rangle = C_0(t)|++0\rangle + C_1(t)|+-1\rangle + C_2(t)|-+1\rangle + C_3(t)|--2\rangle, \tag{6}$$

where the coefficients[11] are

$$\begin{aligned} C_0(t) &= \frac{2g^2}{\Omega_n}A + \frac{2}{3}, \\ C_1(t) &= C_2(t) = -\frac{g}{\Omega_n}B, \\ C_3(t) &= \frac{2\sqrt{2}g^2}{\Omega_n}A - \frac{\sqrt{2}}{3}, \end{aligned} \tag{7}$$

with

$$\begin{aligned} A &= \frac{\mathrm{e}^{\mathrm{i}at}}{a} - \frac{\mathrm{e}^{\mathrm{i}bt}}{b}, \\ B &= \mathrm{e}^{\mathrm{i}at} - \mathrm{e}^{\mathrm{i}bt}, \\ \Omega_n &= \sqrt{\Omega^2 + 24g^2}, \\ a &= \frac{(-\Omega + \sqrt{\Omega^2 + 24g^2})}{2}, \\ b &= \frac{(-\Omega - \sqrt{\Omega^2 + 24g^2})}{2}. \end{aligned} \tag{8}$$

The density matrix of the system is

$$\rho(t) = |\Psi_{\rm I}(t)\rangle\langle\Psi_{\rm I}(t)|. \tag{9}$$

The reduced density matrix of the subsystem composed of two atoms is

$$\rho_{a_1a_2}(t) = \mathrm{tr}_f(|\Psi_{\rm I}(t)\rangle\langle\Psi_{\rm I}(t)|) = \begin{bmatrix} C_0C_0^* & 0 & 0 & 0 \\ 0 & C_1C_1^* & C_1C_2^* & 0 \\ 0 & C_2C_1^* & C_2C_2^* & 0 \\ 0 & 0 & 0 & C_3C_3^* \end{bmatrix}. \tag{10}$$

Its eigenvalues can be obtained as follows,

$$\begin{aligned} \lambda_1(t) &= C_0C_0^*, \\ \lambda_2(t) &= C_1C_1^* + C_2C_2^*, \\ \lambda_3(t) &= C_3C_3^*, \\ \lambda_4(t) &= 0. \end{aligned} \tag{11}$$

Similarly, we can obtain the reduced density matrix of the single-mode vacuum field

$$\begin{aligned} \rho_f(t) =& \mathrm{tr}_{a_1a_2}\rho(t) \\ =& C_0C_0^*|0\rangle\langle 0| + C_1C_1^*|1\rangle\langle 1| \\ &+ C_2C_2^*|1\rangle\langle 1| + C_3C_3^*|2\rangle\langle 2| \\ =& \lambda_1(t)|0\rangle\langle 0| + \lambda_2(t)|1\rangle\langle 1| \\ &+ \lambda_3(t)|2\rangle\langle 2|. \end{aligned} \tag{12}$$

It turns out that the eigenvalues of $\rho_f(t)$ are identical to those of $\rho_{a_1a_2}(t)$. According to the Schmidt theorem,[14] the state vector of the system can be rewritten as

$$\begin{aligned} |\Psi_{\rm I}(t)\rangle =& \sqrt{\lambda_1(t)}|++\rangle|0\rangle + \sqrt{\lambda_2(t)/2}|+-\rangle|1\rangle \\ &+ \sqrt{\lambda_2(t)/2}|-+\rangle|1\rangle + \sqrt{\lambda_3(t)}|--\rangle|2\rangle. \end{aligned} \tag{13}$$

In what follows, we will utilize the above results to investigate the entanglement properties of the system and the subsystems.

3. Quantum entanglement between two two-level atoms and a single-mode vacuum field

We use the reduced quantum entropy as the measure of the DE between two two-level atoms and a single-mode vacuum field. The entropies of the subsystem can be defined through their respective reduced-density matrix as[15]

$$S_i(t) = -\mathrm{tr}[\rho_i(t)\ln\rho_i(t)], \quad (i = a_1a_2, f). \tag{14}$$

The entropies of a general two-component quantum system are linked to a remarkable theorem presented by Araki and Lieb,[16] which states that

$$\begin{aligned} |S_{a_1a_2}(t) - S_f(t)| \leq & S_{a_1a_2f}(t) \\ \leq & |S_{a_1a_2}(t) + S_f(t)|. \end{aligned} \tag{15}$$

Here $S_{a_1a_2f}(t) = -\mathrm{tr}[\rho(t)\ln\rho(t)]$ is the total entropy of the system of two two-level atoms and the single-mode vacuum field. It is worth noting that the $\rho(t)$ given by Eq.(9) is governed by a unitary time evolution and consequently the total entropy $S_{a_1a_2f}(t)$ is time independent. Since we have assumed that the two two-level atoms and the single-mode vacuum field are initially in a disentangled pure state, the total entropy $S_{a_1a_2f}(t)$ of the system is zero. One immediate consequence of inequality (15) is $S_{a_1a_2}(t) = S_f(t)$. As a result, we only need to calculate the atomic quantum entropy $S_{a_1a_2}(t)$. We can express the atomic quantum entropy in terms of the eigenvalues $\lambda_i(t)$ of the atomic reduced density matrix given by Eq.(10)

$$S_{a_1a_2}(t) = -\sum_{i=1}^{4}\lambda_i(t)\ln\lambda_i(t). \tag{16}$$

It reflects the DE between the two two-level atoms and the single-mode vacuum field: if $S_{a_1a_2}(t)$ takes its minimal value zero, the two atoms and the field are disentangled. If $S_{a_1a_2}(t)$ takes its nonzero value, the two atoms and the field are entangled.

Figure 1 displays the numerical results for the time evolution of the reduced entropy $S_{a_1a_2}(t)$ for $k = \Omega/g = 0,\ 10,\ 20$. In Fig.1 (a), k=0 corresponding to the time evolution of the reduced entropy $S_{a_1a_2}(t)$ in the absence of the dipole-dipole interaction between the two two-level atoms. It is observed that the reduced entropy $S_{a_1a_2}(t)$ evolves with a period of π. When $t = (n-1)\pi/g,\ (n = 1, 2, 3, \ldots)$, $S_{a_1a_2}(t)$ evolves to its zero values and the two two-level atoms are disentangled from the vacuum field, while at time $t = (2n-1)\pi/(4g)$, $S_{a_1a_2}(t)$ evolves to its maximal values, the two atoms are strongly entangled with the vacuum field, which means that the two atoms-field entangled state can be prepared by choosing the time $t = (2n-1)\pi/(4g)$.

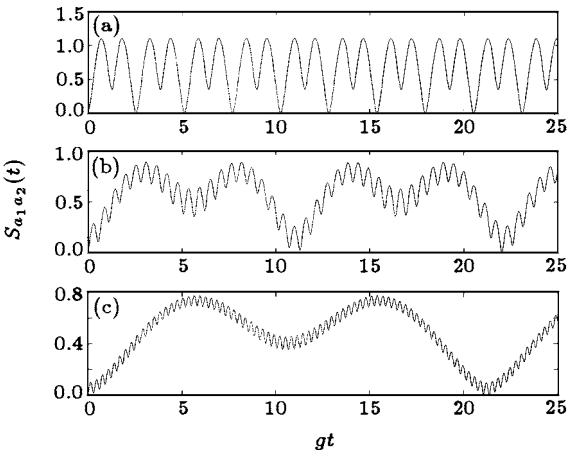


Fig.1. The time evolution of the reduced entropy $S_{a_1a_2}(t)$. (a) $k = 0$; (b) $k = 10$; (c) $k = 20$.

The results for the time evolution of reduced entropy $S_{a_1a_2}(t)$ in the presence of the dipole–dipole interaction are plotted in Figs.1 (b) and (c). From these figures, we can see the influences of the atomic dipole-dipole interaction on the atomic reduced entropy $S_{a_1a_2}(t)$. It is observed that the dipole-dipole interaction of the two atoms leads to the decrease of the maximal values of the atomic reduced entropy and the increase of its time evolution period, which means that in order to prepare the two atoms-field entangled state, the influence of the atomic dipole–dipole interaction should be reduced.

4. Quantum entanglement between two two-level atoms

The above results show that the effective Hamiltonian given by Eqs.(2) and (3) will lead to the entanglement between two two-level atoms and the single-mode vacuum field. Therefore, the states of atom-atom and a single atom-field may evolve into mixed states. In these cases, the reduced entropy cannot measure the DE between atom-atom and that between a single atom–field. However, the relative entropy of the entanglement is a good measure for the DE of two-atom mixed state and the DE of a single atom–field, which is defined as[17]

$$E_{\mathrm{R}} = \min_{\sigma\in D} S(\rho||\sigma), \tag{17}$$

where $S(\rho||\sigma) = \mathrm{tr}[\rho(\ln\rho - \ln\sigma)]$ is the quantum relative entropy, its minimum is taken over D, the set of all disentangled states. The relative entropy of the entanglement is viewed as the minimal 'distance' between the state ρ and the disentangled state σ. For a pure state, the relative entropy of the entanglement reduces to its reduced entropy, while for mixed states, it is usually difficult to calculate the relative entropy of the entanglement except for some specific states. Recently, the following theorem about the relative entropy of the entanglement has been proven,[18,19] and it is considerably suitable for our analysis.

If a bipartite quantum state can be expressed as

$$\rho = \sum_{n_1,n_2} a_{n_1,n_2} |\Psi_{n_1}\phi_{n_1}\rangle\langle\Psi_{n_2}\phi_{n_2}|. \tag{18}$$

Then the relative entropy of the entanglement is given by

$$E_j(\rho_j) = -\sum_n a_{n,n}\ln a_{n,n} + \mathrm{tr}(\rho_j \ln\rho_j) \quad (j = a_1a_2, af), \tag{19}$$

and the disentangled state σ that minimizes the quantum relative entropy $S(\rho||\sigma)$ is given by

$$\sigma = \sum_n a_{n,n} |\Psi_n\phi_n\rangle\langle\Psi_n\phi_n|, \tag{20}$$

where $|\Psi_n\rangle$ and $|\phi_n\rangle$ are orthonormal states of each subsystem.

It should be noted that the reduced density matrix given by Eq.(10) could take the form

$$\rho_{a_1a_2} = \sum_{n,m} a_{n,m}|n,n\rangle\langle m,m|, \tag{21}$$

where $a_{n,m} = \langle n,n|\rho_{a_1a_2}|m,m\rangle$. Hence, the relative entropy of entanglement for the state in Eq.(10) is written as

$$\begin{aligned} E_{a_1a_2}(\rho_{a_1a_2}) &= -\sum_n a_{n,n}\ln a_{n,n} - S(\rho_{a_1a_2}) \\ &= -\sum_n a_{n,n}\ln a_{n,n} \\ &\quad + \sum_{i=1}^{4}\lambda_i(t)\ln\lambda_i(t). \end{aligned} \tag{22}$$

Therefore, $E_{a_1a_2}(t)$ reflects the DE between two two-level atoms. The numerical results of Eq.(22) are shown in Fig.2 and the parameters used are the same as in Fig.1.

It is obvious that the relative entropy $E_{a_1a_2}(t)$ also evolves with a regular periodicity. In Fig.2 (a), we assume that the atomic dipole-dipole interaction is absent. At time $t = 2(n-1)\pi/(5g)$, $(n = 1, 2, 3, \cdots)$ the relative entropy achieves its minimum, which means that the two two-level atoms are disentangled.

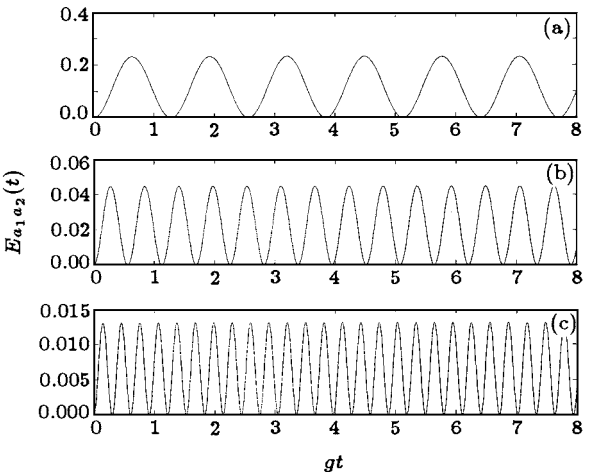


Fig.2. The time evolution of the relative entropy $E_{a_1a_2}(t)$. (a) $k = 0$; (b) $k = 10$; (c) $k = 20$.

At time $t = (2n-1)\pi/(5g)$, the relative entropy reaches its maximum, i.e. the two two-level atoms are strongly entangled. This result shows that at time $t = (2n-1)\pi/(5g)$, the atom-atom entangled state can be prepared. The results for the time evolution of relative entropy of the two atoms in the presence of the dipole-dipole interaction are shown in Figs.2(b) and (c). It is shown that the atomic dipole-dipole interaction leads to the decrease of the maximal values of the atomic relative entropy and the shortening of its time evolution period.

5. Quantum entanglement between a single two-level atom and a single-mode vacuum field

By tracing an atom for the density matrix of the system, the reduced density matrix of a single two-level atom and a single-mode vacuum field is obtained in the atom-field bases $|+0\rangle$, $|-1\rangle$, $|+1\rangle$ and $|-2\rangle$:

$$\rho_{af}(t) = \mathrm{tr}_a\rho(t) = \begin{bmatrix} C_0C_0^* & C_0C_1^* & 0 & 0 \\ C_1C_0^* & C_1C_1^* & 0 & 0 \\ 0 & 0 & C_2C_2^* & C_2C_3^* \\ 0 & 0 & C_3C_2^* & C_3C_3^* \end{bmatrix}. \tag{23}$$

Its eigenvalues $r_j(t)$, (j=1, 2, 3, 4) are given by

$$r_1(t) = C_0C_0^* + C_1C_1^*,$$
$$r_2(t) = C_2C_2^* + C_3C_3^*,$$
$$r_3(t) = r_4(t) = 0. \quad (24)$$

The relative entropy of the single atom-field is

$$E_{af}(t) = -\sum_{i=0}^{3} C_iC_i^* \ln(C_iC_i^*) + \sum_{j=1}^{4} r_j(t) \ln r_j(t). \quad (25)$$

The time evolution of $E_{af}(t)$ reflects the time evolution of the DE between a single two-level atom and a single-mode vacuum field. $E_{af}(t)$ is plotted in Fig.3. The parameters are the same as in Fig.1.

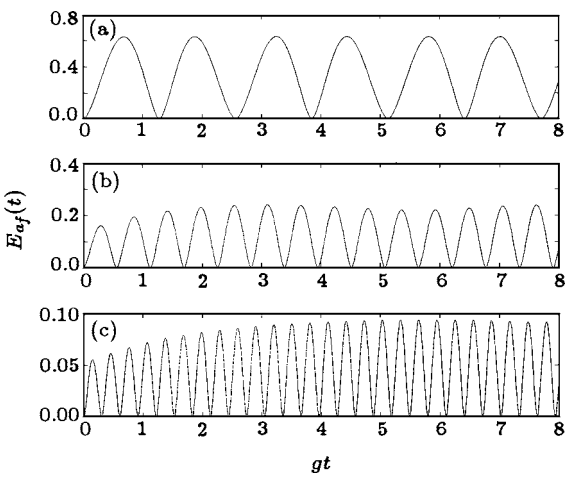


Fig.3. The time evolution of the relative entropy $E_{af}(t)$. (a) $k = 0$; (b) $k = 10$; (c) $k = 20$.

It is interesting to note that the entanglement dynamics of a single atom and a single-mode vacuum field is almost the same as that of the two atoms in section 4. This result shows that at time $t = (2n-1)\pi/(5g)$, the atom-field entangled state can be prepared.

6. Conclusion

In this paper, we explore the quantum entanglement of the system of two two-level atoms interacting with a single-mode vacuum field. We study the DE between two two-level atoms and a single-mode vacuum field by using the quantum reduced entropy, and investigate the DE between the two two-level atoms, and that between a single two-level atom and a single-mode vacuum field by using quantum relative entropy. We can conclude as follows: First, three entangled states (two atoms-field; atom-atom and atom-field) can be prepared via two two-level atoms interacting with a single-mode vacuum field. Second, three kinds of the QE evolve periodically with the same phase. Finally, the atomic dipole–dipole interaction leads to the decrease of the degrees of the entanglement. In order to prepare entangled states, the influence of the atomic dipole–dipole interaction should be reduced. Our results are important for the experimental realization of the preparation of entangled states.

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