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# Exact solutions for the coupled Klein–Gordon–Schrödinger equations using the extended F-expansion method*

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The extended F-expansion method or mapping method is used to construct exact solutions for the coupled Klein-Gordon Schrödinger equations (K-G-S equations) by the aid of the symbolic computation system Mathematica. More solutions in the Jacobi elliptic function form are obtained, including the single Jacobi elliptic function solutions, combined Jacobi elliptic function solutions, rational solutions, triangular solutions, soliton solutions and combined soliton solutions.

**Keywords:** extended F-expansion method, exact solutions, coupled K-G-S equations, Jacobi elliptic function

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## 1. Introduction

Seeking for exact solutions of nonlinear mathematical physics equations is an important and interesting subject. In the past decades, efforts have been made to search for powerful methods of obtaining exact solutions. There exit the following methods already: inverse scattering method,[1] Backlund transformation,[2] Darboux transformation,[3] truncated Painlevé expansion method,[4−7] Hirota's bilinear method,[8] sine–cosine method,[9] hyperbolic tangent function-series method,[10,11] homogeneous balance method,[12−15] improved homogeneous balance method[16] and so on. It is well known that the elliptic functions including Jacobi elliptic functions and Weierstrass elliptic functions, etc, are closely related to nonlinear differential equations.[17,18] Moreover, many nonlinear equations have been shown to possess the elliptic function solutions.[19−32] Recently, the F-method[33−36] has been developed to obtain Jacobi elliptic function solutions, soliton solutions and trigonometric function solutions.

In this paper we consider the coupled Klein–Gordon–Schrödinger (K-G-S) equations, which describe a classical model of the Yukawa interaction of conserved complex nucleon field with neutral real meson field.[37,38] The Cauchy problem and the initial boundary value problem of the coupled K-G-S equations have been quite extensively studied.[37,39−42] The Ref.[39] proved the existence of general solutions by using the $L^p - L^q$ estimates for the elementary solutions of the coupled K-G-S equations. The Ref.[40] studied the asymptotic behaviour of the solutions for the multi-dimensional K-G-S equations, and in Ref.[41] the authors considered the Cauchy problem of the equations in $R^3$ and proved the existence of the maximal attractor. In Ref.[37] the author discussed the initial boundary value problem of the coupled K-G-S equations and obtained the general existence of strong solutions in a three-dimensional case, and later the results were improved in Ref.[42]. Masahito.O studied the stability of stationary states for the coupled K-G-S equations in Ref.[43]. We extend the F-expansion method further for constructing exact solutions to the coupled K-G-S equations. More Jacobi elliptic function solutions are obtained including the combined Jacobi elliptic function solutions, rational

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solutions, triangular solutions, soliton solutions and combined soliton solutions.

## 2. Description of the extended F-expansion method

The brief description of the extended F-expansion method is given as follows. A nonlinear partial differential equation(PDE)

$$F(u, u_x, u_y, u_z, u_t, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}, u_{yz}, \ldots) = 0, \tag{1}$$

has travelling wave solutions in the following form:

$$u(x, y, z, t) = u(\xi), \quad \xi = kx + ly + mz - \omega t, \tag{2}$$

where $\boldsymbol{K} \equiv (k, l, m)$ and $\omega$ are the wave vector and angular frequency respectively, and Equation (1) becomes a nonlinear ordinary differential equation(ODE) for $u(\xi)$:

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \ldots) = 0. \tag{3}$$

$u(\xi)$ can be assumed to be expressed in the following form:

$$u(\xi) = a_0 + \sum_{i=1}^{n} f^{i-1}(\xi)(a_i f(\xi) + b_i g(\xi)), \tag{4}$$

where $f(\xi)$ and $g(\xi)$ are Jacobi elliptic functions and satisfy the following relations:

$$\begin{aligned} f^{'2}(\xi) =& P_1 f^4(\xi) + Q_1 f^2(\xi) + R_1, \\ g^{'2}(\xi) =& P_2 g^4(\xi) + Q_2 g^2(\xi) + R_2, \\ g^2(\xi) =& \varepsilon f^2 + \delta, \\ Q_1^2 - 3P_1R_1 =& Q_2^2 - 3P_2R_2, 2Q_2^2 \\ =& Q_1^2 + Q_1Q_2 + 9P_2R_2, \\ \varepsilon =& \frac{P_1}{P_2}, \delta = \frac{Q_1 - Q_2}{3P_2}, \delta \neq 0, \end{aligned} \tag{5}$$

$f' \equiv \frac{\mathrm{d}f}{\mathrm{d}\xi}$, $g' \equiv \frac{\mathrm{d}g}{\mathrm{d}\xi}$, $P_1$, $Q_1$, $R_1$, $P_2$, $Q_2$, and $R_2$ are constants, and $a_0$, $a_i$, $b_i$ are parameters to be determined, $n$ is the balance number determined by the balance mechanism in the nonlinear PDE which demands the balance between the nonlinear term and the highest order partial derivative term. Substituting the expansion (4) into the expression (3) with the relations (5), we obtain an equation for $f(\xi)$ and $g(\xi)$, and collect the similar terms. Setting the coefficient of each term to zero, a series of equations for the parameters is obtained. By solving these equations to determine the parameters and choosing the different kinds of Jacobi elliptic functions from the relations (5), in which different $(P_i, Q_i, R_i)(i = 1, 2)$ correspond to different Jacobi elliptic functions,[35,44,45] we obtain different kinds of Jacobi elliptic function solutions of Eq.(1) including the single Jacobi elliptic function solutions, combined Jacobi elliptic function solutions. In addition we see that when the modulus $\lambda \to 1$ or $\lambda \to 0$, Jacobi elliptic functions degenerate into hyperbolic functions or trigonometric functions respectively.

## 3. Generalized exact solutions for the coupled K-G-S equations

The coupled K-G-S equations read

$$\begin{aligned} \mathrm{i}\frac{\partial \Psi}{\partial t} + \alpha\nabla^2 \Psi + \rho \Phi \Psi = 0, \\ \frac{\partial^2 \Phi}{\partial t^2} - c_0^2 \nabla^2 \Phi + \mu^2 \Phi - \beta |\Psi|^2 = 0, \end{aligned} \tag{6}$$

which describe a system of conserved scalar nucleons interacting with neutral scalar mesons. Here, $\Psi$ represents a complex scalar neucleon field and $\Phi$ a real scalar meson field. The real constants $\mu$, $\rho$, and $\beta$ describe the mass of a meson and coupled constants respectively, the real constants $\alpha$ and $c_0^2$ are parameter constants, and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. In order to obtain the travelling wave solutions of K-G-S equations, we make the following transformation:

$$\Psi = U(\xi)\exp(\mathrm{i}\eta), \quad \Phi = V(\xi), \tag{7}$$

where $U(\xi)$ and $V(\xi)$ are real functions, $\xi = kx + ly + mz - \omega t$, $\eta = px + qy + rz - \sigma t$, and reduce the system(8) to the following ODEs:

$$\begin{aligned} K^2\alpha U^{''}(\xi) - \gamma U(\xi) + \rho U(\xi)V(\xi) = 0, \\ (\omega^2 - K^2c_0^2)V^{''}(\xi) + \mu^2 V(\xi) - \beta U^2(\xi) = 0, \\ 2\alpha(kp + lq + mr) - \omega = 0, \end{aligned} \tag{8}$$

where $\boldsymbol{K} = (k, l, m)$, $K^2 \equiv \boldsymbol{K} \cdot \boldsymbol{K} = k^2 + l^2 + m^2$, $\boldsymbol{P} \equiv (p, q, r)$, $P^2 \equiv \boldsymbol{P} \cdot \boldsymbol{P} = p^2 + q^2 + r^2$ and $\gamma = (p^2 + q^2 + r^2)\alpha - \sigma = P^2\alpha - \sigma$ .

The extended F-expansion method makes the following ansatz:

$$\begin{aligned} U(\xi) = a_0 + \sum_{n=i}^{n} f^{i-1}(\xi)(a_{2i-1}f(\xi) + a_{2i}g(\xi)), \\ V(\xi) = b_0 + \sum_{n=i}^{I} f^{i-1}(\xi)(b_{2i-1}f(\xi) + b_{2i}g(\xi)), \end{aligned} \tag{9}$$

where $n$ and $I$ can be determined by balancing the highest order of terms $U''$ and $V''$ with $UV$ and $U^2$ respectively. Thus we obtain $n = I = 2$. Therefore the

solutions of Eq.(8) can be expressed in the following forms:

$$\begin{aligned} U(\xi) =& a_0 + a_1 f(\xi) + a_2 g(\xi) \\ &+ a_3 f^2(\xi) + a_4 f(\xi) g(\xi), \\ V(\xi) =& b_0 + b_1 f(\xi) + b_2 g(\xi) \\ &+ b_3 f^2(\xi) + b_4 f(\xi) g(\xi), \end{aligned} \tag{10}$$

where $f(\xi)$ and $g(\xi)$ satisfy the relations (5). Substituting Eq.(10) into Eq.(8) along with the relations (5) yields a system of equations of power functions of $f(\xi)$ and $g(\xi)$ as follows:

$$\begin{aligned} & 9P_1^2(\rho a_3 b_4 + a_4(\rho b_3 + 6K^2\alpha P_1))f^5(\xi) \\ & + \cdots(\cdots) f^i g^j + \cdots = 0, \\ & - 18P_1^2(\beta a_3 a_4 - 3(\omega^2 - K^2 c_0^2) b_4 P_1) f^5(\xi) \\ & + \cdots(\cdots) f^i g^j + \cdots = 0. \end{aligned} \tag{11}$$

Setting the coefficients of $f^i g^j$ $(j = 0, 1)$ in the system of equations to zero, we can deduce the following set of algebraic polynomials with unknowns $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $b_0$, $b_1$, $b_2$, $b_3$ and $b_4$, namely,

$$\begin{aligned} & 9P_1^2(\rho a_3 b_4 + a_4(\rho b_3 + 6K^2\alpha P_1)) = 0, \\ & 9P_1^2(\rho a_4 b_1 + \rho a_3 b_2 + \rho a_2 b_3 + \rho a_1 b_4 + 2K^2\alpha a_2 P_1 = 0, \\ & 9P_1(\rho a_4 b_4 P_1 + a_3(\rho b_3 + 6K^2\alpha P_1) P_2) = 0, \\ & P_1(Q_1 - Q_2)(3\rho a_0 b_2 + a_2(-3\gamma + 3\rho b_0 + 2K^2\alpha Q_1 + K^2\alpha Q_2)) = 0, \\ & 3P_1(\rho(3a_2 b_1 P_1 + 3a_1 b_2 P_1 + b_4(3a_0 P_1 + a_3(Q_1 - Q_2))) \\ & + a_4(\rho b_3(Q_1 - Q_2) - 3P_1(\gamma - \rho b_0 - 5K^2\alpha Q_1 + K^2\alpha Q_2))) = 0, \\ & 9P_1(\rho a_4 b_2 P_1 + \rho a_2 b_4 P_1 + (\rho a_3 b_1 + a_1(\rho b_3 + 2K^2\alpha P_1)) P_2) = 0, \\ & 3P_1(\rho(3a_0 b_2 P_1 + (a_4 b_1 + a_3 b_2 + a_1 b_4)(Q_1 - Q_2)) \\ & + a_2(\rho b_3(Q_1 - Q_2) + P_1(-3\gamma + 3\rho b_0 + 4K^2\alpha Q_1 - K^2\alpha Q_2))) = 0, \\ & 3P_1(3\rho a_2 b_2 P_1 + 3a_3 P_2(-\gamma + \rho b_0 + 4K^2\alpha Q_1) + \rho(3a_1 b_1 P_2 + 3a_0 b_3 P_2 + a_4 b_4(Q_1 - Q_2))) = 0, \\ & P_1(3\rho(a_2 b_1 + a_1 b_2 + a_0 b_4)(Q_1 - Q_2) + a_4(11K^2\alpha Q_1^2 + 3(\gamma - \rho b_0) Q_2 \\ & - 7K^2\alpha Q_2^2 + Q_1(-3\gamma + 3\rho b_0 - 4K^2\alpha Q_2) + 18K^2\alpha P_2 R_2)) = 0, \\ & 3P_1(3a_1 P_2(-\gamma + \rho b_0 + K^2\alpha Q_1) + \rho(3a_0 b_1 P_2 + (a_4 b_2 + a_2 b_4)(Q_1 - Q_2))) = 0, \\ & 3(-3a_0(\gamma - \rho b_0) P_1 P_2 + \rho a_2 b_2 P_1(Q_1 - Q_2) + 2K^2\alpha a_3 P_2(Q_1^2 - Q_2^2 + 3P_2 R_2)) = 0, \\ & - 18P_1^2(\beta a_3 a_4 - 3(\omega^2 - K^2 c_0^2) b_4 P_1) = 0, \\ & - 18P_1^2(\beta a_2 a_3 + \beta a_1 a_4 - (\omega^2 - K^2 c_0^2) b_2 P_1) = 0, \\ & - 9P_1(\beta a_4^2 P_1 + (\beta a_3^2 - 6(\omega^2 - K^2 c_0^2) b_3 P_1) P_2) = 0, \\ & P_1(Q_1 - Q_2)(-6\beta a_0 a_2 + b_2(3\mu^2 + 2(\omega^2 - K^2 c_0^2) Q_1 + (\omega^2 - K^2 c_0^2) Q_2)) = 0, \\ & - 3P_1(6\beta a_1 a_2 P_1 + 6\beta a_0 a_4 P_1 - 3\mu^2 b_4 P_1 + 2\beta a_3 a_4 Q_1 - 15\omega^2 b_4 P_1 Q_1 \\ & + 15K^2 c_0^2 b_4 P_1 Q_1 - 2\beta a_3 a_4 Q_2 + 3\omega^2 b_4 P_1 Q_2 - 3K^2 c_0^2 b_4 P_1 Q_2) = 0, \\ & - 18P_1(\beta a_2 a_4 P_1 + (\beta a_1 a_3 - (\omega^2 - K^2 c_0^2) b_1 P_1) P_2) = 0, \\ & - 3P_1(6\beta a_0 a_2 P_1 + 2\beta(a_2 a_3 + a_1 a_4)(Q_1 - Q_2) - b_2 P_1(3\mu^2 + 4(\omega^2 - K^2 c_0^2) Q_1 - (\omega^2 - K^2 c_0^2) Q_2)) = 0, \\ & - 3P_1(3\beta a_2^2 P_1 + 3\beta a_1^2 P_2 + 6\beta a_0 a_3 P_2 - 3\mu^2 b_3 P_2 + \beta a_4^2 Q_1 - 12\omega^2 b_3 P_2 Q_1 + 12K^2 c_0^2 b_3 P_2 Q_1 - \beta a_4^2 Q_2) = 0, \\ & P_1(6\beta a_1 a_2(Q_2 - Q_1) + 6\beta a_0 a_4(Q_2 - Q_1) + b_4(11(\omega^2 - K^2 c_0^2) Q_1^2 - 3\mu^2 Q_2 \\ & - 7(\omega^2 - K^2 c_0^2) Q_2^2 + Q_1(3\mu^2 - 4(\omega^2 - K^2 c_0^2) Q_2) + 18(\omega^2 - K^2 c_0^2) P_2 R_2)) = 0, \\ & 3P_1(-6\beta a_0 a_1 P_2 + 3b_1 P_2(\mu^2 + (\omega^2 - K^2 c_0^2) Q_1) + 2\beta a_2 a_4(Q_2 - Q_1)) = 0, \\ & 3(-3\beta a_0^2 P_1 P_2 + 3\mu^2 b_0 P_1 P_2 - \beta a_2^2 P_1 Q_1 + 2\omega^2 b_3 P_2 Q_1^2 - 2K^2 c_0^2 b_3 P_2 Q_1^2 \\ & + \beta a_2^2 P_1 Q_2 - 2\omega^2 b_3 P_2 Q_2^2 + 2K^2 c_0^2 b_3 P_2 Q_2^2 + 6\omega^2 b_3 P_2^2 R_2 - 6K^2 c_0^2 b_3 P_2^2 R_2 = 0, \end{aligned}$$

$$2\alpha(kp + lq + mr) - \omega = 0. \tag{12}$$

We find the some particular solutions of the above mentioned equations in the following cases:

**Case 1**

$$a_4 = \pm\frac{3K\mu\sqrt{2\alpha P_1 P_2}}{\sqrt{\beta\rho(Q_1+Q_2)}},$$
$$a_3 = 0,\ a_2 = 0,\ a_1 = 0,\ a_0 = 0,\ b_4 = 0,$$
$$b_3 = -\frac{6K^2\alpha P_1}{\rho},\ b_2 = 0,\ b_1 = 0,$$
$$b_0 = -\frac{2K^2\alpha(Q_1-Q_2)(2Q_1+Q_2)}{3\rho(Q_1+Q_2)},$$
$$\gamma = \frac{K^2\alpha(5Q_1^2+14Q_1Q_2+5Q_2^2)}{3(Q_1+Q_2)},$$
$$2\alpha(kp+lq+mr)-\omega = 0,$$
$$\mu^2 + 2(\omega^2 - K^2c_0^2)(Q_1+Q_2) = 0.$$

**Case 2**

$$a_4 = 0,\ a_3 = \pm\frac{6P_1\sqrt{K^2\alpha(K^2c_0^2-\omega^2)}}{\sqrt{\beta\rho}},\ a_2 = 0,$$
$$a_1 = 0,\ a_0 = \mp\frac{\sqrt{K^2\alpha}(\mu^2+4(\omega^2-K^2c_0^2)Q_1)}{2\sqrt{\beta\rho(K^2c_0^2-\omega^2)}},$$
$$b_4 = 0,\ b_3 = -\frac{6K^2\alpha P_1}{\rho},\ b_2 = 0,$$
$$b_1 = 0,\ b_0 = \frac{K^2\alpha\left(\dfrac{\mu^2}{K^2c_0^2-\omega^2}-4Q_1\right)}{2\rho},$$
$$\gamma = \frac{K^2\alpha\mu^2}{K^2c_0^2-\omega^2},\ 2\alpha(kp+lq+mr)-\omega = 0,$$
$$\mu^4 - 16(\omega^2-K^2c_0^2)^2(Q_1^2-3P_1R_1) = 0.$$

This result appears in Refs.[24,35].

**Case 3**

$$a_4 = \pm\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1P_2}}{\sqrt{\beta\rho}},$$
$$a_3 = \pm\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1^2}}{\sqrt{\beta\rho}},\ a_2 = 0,\ a_1 = 0,$$
$$a_0 = \mp\frac{K^2\alpha P_1(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2\sqrt{K^2\alpha\beta\rho(K^2c_0^2-\omega^2)P_1^2}},$$
$$b_4 = -\frac{3K^2\alpha\sqrt{P_1P_2}}{\rho},\ b_3 = -\frac{3K^2\alpha P_1}{\rho},\ b_2 = 0,$$
$$b_1 = 0,\ b_0 = \frac{K^2\alpha(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2(K^2c_0^2-\omega^2)\rho},$$
$$\gamma = \frac{K^2\alpha\mu^2}{K^2c_0^2-\omega^2},\ 2\alpha(kp+lq+mr)-\omega = 0,$$
$$(\omega^2-K^2c_0^2)^2(11Q_1^2+26Q_1Q_2+11Q_2^2)-3\mu^4 = 0.$$

**Case 3'**

$$a_4 = \pm\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1P_2}}{\sqrt{\beta\rho}},$$
$$a_3 = \mp\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1^2}}{\sqrt{\beta\rho}},\ a_2 = 0,\ a_1 = 0,$$
$$a_0 = \mp\frac{K^2\alpha P_1(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2\sqrt{K^2\alpha\beta\rho(K^2c_0^2-\omega^2)P_1^2}},$$
$$b_4 = \frac{3K^2\alpha\sqrt{P_1P_2}}{\rho},\ b_3 = -\frac{3K^2\alpha P_1}{\rho},\ b_2 = 0,$$
$$b_1 = 0,\ b_0 = \frac{K^2\alpha(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2(K^2c_0^2-\omega^2)\rho},$$
$$\gamma = \frac{K^2\alpha\mu^2}{K^2c_0^2-\omega^2},\ 2\alpha(kp+lq+mr)-\omega = 0,$$
$$(\omega^2-K^2c_0^2)^2(11Q_1^2+26Q_1Q_2+11Q_2^2)-3\mu^4 = 0.$$

Substituting the results in Cases 1–3' into the transformation (7) with the expressions (12) yields general formal solutions of the equations (6) in the following different kinds:

**Case 1**

$$\begin{aligned}&\Psi(x,y,z,t)\\ &= \pm\frac{3K\mu\sqrt{2\alpha P_1P_2}}{\sqrt{\beta\rho(Q_1+Q_2)}}f(\xi)g(\xi)\\ &\quad\times \mathrm{e}^{\mathrm{i}\left(px+qy+rz-\left(P^2\alpha-\frac{K^2\alpha(5Q_1^2+14Q_1Q_2+5Q_2^2)}{3(Q_1+Q_2)}\right)t\right)},\\ &\Phi(x,y,z,t)\\ &= -\frac{2K^2\alpha(Q_1-Q_2)(2Q_1+Q_2)}{3\rho(Q_1+Q_2)}\\ &\quad-\frac{6K^2\alpha P_1}{\rho}f^2(\xi),\end{aligned}\tag{13}$$

where $\xi = kx+ly+mz-\omega t$, $K^2 = k^2+l^2+m^2$, $P^2 = p^2+q^2+r^2$, $\omega = 2\alpha(kp+lq+mr)$, $K^2c_0^2-\omega^2 > 0$, and $Q_1+Q_2 > 0$, and they satisfy the following relation:

$$\mu^2 + 2(\omega^2-K^2c_0^2)(Q_1+Q_2) = 0.\tag{14}$$

**Case 2**

$$\begin{aligned}\Psi(x,y,z,t) =&\left(\mp\frac{\sqrt{K^2\alpha}(\mu^2+4(\omega^2-K^2c_0^2)Q_1)}{2\sqrt{\beta\rho(K^2c_0^2-\omega^2)}}\right.\\ &\left.\pm\frac{6P_1\sqrt{K^2\alpha(K^2c_0^2-\omega^2)}}{\sqrt{\beta\rho}}f^2(\xi)\right)\mathrm{e}^{\mathrm{i}(\eta)},\\ \Phi(x,y,z,t) =&\frac{K^2\alpha\left(\dfrac{\mu^2}{K^2c_0^2-\omega^2}-4Q_1\right)}{2\rho}\\ &-\frac{6K^2\alpha P_1}{\rho}\ f^2(\xi),\end{aligned}\tag{15}$$

where $\xi = kx+ly+mz-\omega t$, $\eta = px+qy+rz-\left(P^2\alpha-\dfrac{K^2\alpha\mu^2}{K^2c_0^2-\omega^2}\right)t$, $K^2 = k^2+l^2+m^2$, $P^2 = p^2+q^2+r^2$,

$\omega = 2\alpha(kp + lq + mr)$, and $K^2c_0^2 - \omega^2 > 0$, and they satisfy the following relation:

$$\mu^4 - 16(\omega^2 - K^2c_0^2)^2(Q_1^2 - 3P_1R_1) = 0. \tag{16}$$

**Case 3**

$$\begin{aligned}
&\Psi(x,y,z,t)\\
=&\Bigg(\mp\frac{K^2\alpha P_1(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2\sqrt{K^2\alpha\beta\rho(K^2c_0^2-\omega^2)P_1^2}}\\
&\pm\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1^2}}{\sqrt{\beta\rho}}f^2(\xi)\\
&\pm\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1P_2}}{\sqrt{\beta\rho}}\,f(\xi)g(\xi)\Bigg)\mathrm{e}^{\mathrm{i}(\eta)},\\
&\Phi(x,y,z,t)\\
=&\frac{K^2\alpha(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2(K^2c_0^2-\omega^2)\rho}\\
&-\frac{3K^2\alpha P_1}{\rho}\,f^2(\xi)-\frac{3K^2\alpha\sqrt{P_1P_2}}{\rho}f(\xi)g(\xi),
\end{aligned}\tag{17}$$

where $\xi = kx+ly+mz-\omega t$, $\eta = px+qy+rz-\left(P^2\alpha-\dfrac{K^2\alpha\mu^2}{K^2c_0^2-\omega^2}\right)t$, $K^2 = k^2+l^2+m^2$, $P^2 = p^2+q^2+r^2$, $\omega = 2\alpha(kp+lq+mr)$, and $K^2c_0^2-\omega^2 > 0$, and they satisfy the following relation:

$$(\omega^2-K^2c_0^2)^2(11Q_1^2+26Q_1Q_2+11Q_2^2)-3\mu^4 = 0. \tag{18}$$

**Case 3'**

$$\begin{aligned}
&\Psi(x,y,z,t)\\
=&\Bigg(\mp\frac{K^2\alpha P_1(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2\sqrt{K^2\alpha\beta\rho(K^2c_0^2-\omega^2)P_1^2}}\\
&\mp\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1^2}}{\sqrt{\beta\rho}}f^2(\xi)\\
&\pm\frac{3K\sqrt{(K^2c_0^2-\omega^2)\alpha P_1P_2}}{\sqrt{\beta\rho}}f(\xi)g(\xi)\Bigg)\mathrm{e}^{\mathrm{i}(\eta)},\\
&\Phi(x,y,z,t)\\
=&\frac{K^2\alpha(\mu^2+(\omega^2-K^2c_0^2)(3Q_1+Q_2))}{2(K^2c_0^2-\omega^2)\rho}\\
&-\frac{3K^2\alpha P_1}{\rho}f^2(\xi)+\frac{3K^2\alpha\sqrt{P_1P_2}}{\rho}f(\xi)g(\xi),
\end{aligned}\tag{19}$$

where $\xi = kx+ly+mz-\omega t$, $\eta = px+qy+rz-\left(P^2\alpha-\dfrac{K^2\alpha\mu^2}{K^2c_0^2-\omega^2}\right)t$, $K^2 = k^2+l^2+m^2$, $P^2 = p^2+q^2+r^2$, $\omega = 2\alpha(kp+lq+mr)$, and $K^2c_0^2-\omega^2 > 0$, and they satisfy the following relation:

$$(\omega^2-K^2c_0^2)^2(11Q_1^2+26Q_1Q_2+11Q_2^2)-3\mu^4 = 0. \tag{20}$$

In some special choices of $P_1$, $Q_1$, $R_1$ and $P_2$, $Q_2$, $R_2$ (see Refs.[35, 44, 45]), if we take $P_1 = 1$, $Q_1 = -1-\lambda^2$, $R_1 = \lambda^2$ and $P_2 = 1$, $Q_2 = 2-\lambda^2$, $R_2 = 1-\lambda^2$, then $f = \mathrm{ns}(\xi)$ and $g = \mathrm{cs}(\xi)$. Because Case 2 is the same as that in Refs.[24, 35] and Case 3 is similar to Case 3', we consider Case 1 and Case 3 only. Therefore some particular solutions of Eq.(6) can be obtained as follows:

**Case 1**

$$\begin{aligned}
&\Psi(x,y,z,t)\\
=&\pm\frac{6K\sqrt{\alpha(K^2c_0^2-\omega^2)}}{\sqrt{\beta\rho}}\mathrm{ns}(\xi)\mathrm{cs}(\xi)\\
&\times\mathrm{e}^{\mathrm{i}\left(px+qy+rz-\left(P^2\alpha+\frac{K^2\alpha(6(K^2c_0^2-\omega^2)^2-\mu^4)}{\mu^2(K^2c_0^2-\omega^2)}\right)t\right)},\\
&\Phi(x,y,z,t)\\
=&-\frac{3K^2\alpha(2(K^2c_0^2-\omega^2)-\mu^2)}{\rho\mu^2}\\
&-\frac{6K^2\alpha}{\rho}\mathrm{ns}^2(\xi),
\end{aligned}\tag{21}$$

where $\xi = kx+ly+mz-\omega t$, $K^2 = k^2+l^2+m^2$, $P^2 = p^2+q^2+r^2$, $\omega = 2\alpha(kp+lq+mr)$, $K^2c_0^2-\omega^2 > 0$, and $\lambda < \dfrac{\sqrt{2}}{2}$, and they satisfy the following relation:

$$\mu^2 + 2(\omega^2 - K^2c_0^2)(1-2\lambda^2) = 0. \tag{22}$$

**Case 3**

$$\begin{aligned}
&\Psi(x,y,z,t)\\
=&\Bigg(\mp\frac{K\alpha(\mu^2+(1+4\lambda^2)(K^2c_0^2-\omega^2))}{2\sqrt{\alpha\beta\rho(K^2c_0^2-\omega^2)}}\\
&\pm\frac{3K\sqrt{\alpha(K^2c_0^2-\omega^2)}}{\sqrt{\beta\rho}}\mathrm{ns}^2(\xi)\\
&\pm\frac{3K\sqrt{\alpha(K^2c_0^2-\omega^2)}}{\sqrt{\beta\rho}}\,\mathrm{ns}(\xi)\mathrm{cs}(\xi)\Bigg)\mathrm{e}^{\mathrm{i}(\eta)},\\
&\Phi(x,y,z,t)\\
=&\frac{K^2\alpha(\mu^2+(1+4\lambda^2)(K^2c_0^2-\omega^2))}{2(K^2c_0^2-\omega^2)\rho}\\
&-\frac{3K^2\alpha}{\rho}\mathrm{ns}^2(\xi)-\frac{3K^2\alpha}{\rho}\mathrm{ns}(\xi)\mathrm{cs}(\xi),
\end{aligned}\tag{23}$$

where $\xi = kx+ly+mz-\omega t$, $\eta = px+qy+rz-\left(P^2\alpha-\dfrac{K^2\alpha\mu^2}{K^2c_0^2-\omega^2}\right)t$, $K^2 = k^2+l^2+m^2$, $P^2 = p^2+q^2+r^2$, $\omega = 2\alpha(kp+lq+mr)$, $K^2c_0^2-\omega^2 > 0$, and $\lambda < \dfrac{\sqrt{2-\sqrt{3}}}{2}$ or $\lambda > \sqrt{\dfrac{1}{2}+\dfrac{\sqrt{3}}{4}}$, and they satisfy the following relation:

$$(\omega^2-K^2c_0^2)^2(16\lambda^4-16\lambda^2+1)-\mu^4 = 0. \tag{24}$$

If we choose other $(P_1, Q_1, R_1)$ and $(P_2, Q_2, R_2)$, more different solutions of Eq.(6) can be obtained, but we here omit them for simplicity.

## 4. Discussion

In short, we have obtained three different kinds of solutions of the K-G-S equations in this paper and the second kind of solutions(Case 2) are the same as the solutions in Refs.[24,35]. The extended F-expansion or mapping method is presented for finding more Jacobi elliptic function solutions of nonlinear PDEs. A series of solutions is obtained including the single and combined Jacobi elliptic function solutions. When $m \to 0$ or $m \to 1$, their degenerative solutions can be acquired respectively.

It is easy to see that the above-mentioned method is only applied to these nonlinear ODEs with constant coefficients or nonlinear partial differential equations, which can be reduced to the corresponding nonlinear ODEs with constant coefficients by using some transformations, otherwise the method will not work. In order to overcome the disadvantage of the method, we change the method into a general form as follows.

If we do not reduce Eq.(1) to a nonlinear ODE with constant coefficients, then we directly assume that Eq.(1) has the formal solution

$$\begin{aligned} u(x,y,z,t) =& a_0(x,y,z,t) \\ &+ \sum_{i=1}^{n} f^{i-1}(x,y,z,t)(a_i f(x,y,z,t) \\ &+ b_i g(x,y,z,t)) \end{aligned} \tag{25}$$

The other process is the same as that in this method. About the applications of the more general form, we shall give some examples in future.

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