Vol 14 No 10, October 2005
1009-1963/2005/14(10)/1919-07

Chinese Physics

© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

Exact invariants and adiabatic invariants of Raitzin's canonical equations of motion for a nonlinear nonholonomic mechanical system*

Qiao Yong-Fen(乔永芬)[a)b)c)], Li Ren-Jie(李仁杰)[c)], and Sun Dan-Na(孙丹娜)[c)]

[a)] *Department of Mechanical Engineering and Automation, Zhejiang Institute of Science and Technology, Hangzhou 310018, China*

[b)] *Engineering College of Northeast Agricultural University, Harbin 150030, China*

[c)] *Faculty of Science Laiyang Agricultural College, Laiyang 265200, China*

(Received 11 November 2004; revised manuscript received 17 May 2005)

The exact invariants and the adiabatic invariants of Raitzin's canonical equations of motion for a nonlinear nonholonomic mechanical system are studied. The relations between the invariants and the symmetries of the system are established. Based on the concept of higher-order adiabatic invariant of a mechanical system under the action of a small perturbation, the forms of the exact invariants and adiabatic invariants and the conditions for their existence are proved. Finally, the inverse problem of the perturbation to symmetries of the system is studied and an example is also given to illustrate the application of the results.

Keywords: nonholonomic system, Raitzin's canonical equation, symmetry, perturbation, exact invariant, adiabatic invariant

PACC: 0320, 0200

1. Introduction

The conservation laws for a dynamical system has great mathematical significance and very deep physical background. Furthermore, their study has become one of most important directions in modern analytical mechanics. There are five main methods used to seek the conserved quantities of a mechanical system, namely, the Noether theory,[1] the Lie method,[2] the theory of integrating factors,[3] the Mei's form invariance,[4] (or Mei symmetry) and the Hojman method.[5] Significant achievements have been obtained in the study of these methods in recent years.[6−17]

In 1917, Burgers first proposed adiabatic invariants which pointed to a particular kind of Hamilton system.[18] The adiabatic invariants mean that they are almost not changed when the parameter varies very slowly and they play a very important role in the research on quasi-integrability of a mechanical system. In fact, that the parameter varies very slowly is equivalent to the action of a small perturbation, so we perform research using the latter model in this paper.

The study of adiabatic invariants has become a popular subject in mechanics, atomic and molecular physics,[19] and some important results were obtained.[20−25]

In 1961, a Spanish scholar Raitzin proposed new forms of canonical variables r and s and established new differential equations of motion, called Raitzin's canonical equations.[26] In practice, compared with the Hamilton canonical equations, some dynamical problems were solved just simply by applying the equations. Therefore, in analytical mechanics, Raitzin's canonical equations are significant theoretically and practically.[27,28]

In this paper, the perturbation to symmetries and adiabatic invariants of Raitzin's canonical equations for a nonlinear nonholonomic mechanical system are studied.

2. Raitzin's canonical equations of a nonlinear nonholonomic system

*Project supported by the Heilongjiang Natural Science Foundation of China (Grant No 9507).

We consider a mechanical system composed of N particles, and its configuration is determined by n generalized coordinates $q_1, q_2, \cdots, q_n$. The motion of the system is subjected to the following g ideal bilateral nonholonomic constraints of Chetaev type:

$$f_\beta(t,q,\dot{q})=0, \qquad \frac{\partial f_\beta}{\partial \dot{q}_j}\delta q_j=0, \tag{1}$$

$$(\beta=1,2,\cdots,g; \quad j=1,2,\cdots,n).$$

The equations of motion of the system can be written in the Routh form[7]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=Q_j+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_j}, \quad (j=1,2,\cdots,n), \tag{2}$$

where L is the Lagrangian of the system, Q_j are the non-potential generalized forces and λ_β are the constrained multipliers.

Now, we introduce the Raitzin canonical variables and function[26−28] as follows:

$$r_j=\frac{\partial L}{\partial q_j}, \qquad s_j=\dot{q}_j, \tag{3}$$

$$R(t,r,s)=L-r_jq_j. \tag{4}$$

According to the Raitzin variables (3), the nonholonomic constraints (1) and constrained multipliers λ_β can be written as

$$\tilde{f}_\beta(t,r,s)=f_\beta(t,q(t,r,s),s)=0, \tag{5}$$

$$\tilde{\lambda}_\beta(t,r,s)=\lambda_\beta(t,q(t,r,s),s), \tag{6}$$

and then from the nonholonomic system given by Eqs.(2), (5) and (6), we obtain the Raitzin canonical equations of the nonlinear nonholonomic system as follows:

$$s_j=-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j}, \qquad (q_j=-\frac{\partial R}{\partial r_j}), \tag{7}$$

$$r_j=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}-\tilde{Q}_j-\tilde{\lambda}_\beta\frac{\partial \tilde{f}_\beta}{\partial s_j}, \tag{8}$$

$$(j=1,2,\cdots,n; \ \beta=1,2,\cdots,g).$$

where the symbol "$\sim$" is the expression which is obtained by substituting t,r,s for $q,\dot{q}$.

3. Infinitesimal transformations and exact invariants (Lie symmetrical conserved quantity)

We study the variation of the Raitzin canonical action.

We introduce the Raitzin canonical action

$$I=\int_{t_0}^{t_1}L\mathrm{d}t=\int_{t_0}^{t_1}(R+r_jq_j)\mathrm{d}t, \tag{9}$$

and the infinitesimal transformations

$$\begin{aligned}\bar{t}&=t+\varepsilon\tau^0(t,r,s),\\ \bar{q}_j&=q_j+\varepsilon\xi_j^0(t,r,s),\\ \bar{r}_j&=r_j+\varepsilon\eta_j^0(t,r,s),\end{aligned} \tag{10}$$

where ε is an infinitesimal parameter, τ^0, ξ_j^0 and η_j^0 are infinitesimal generators.

Under the infinitesimal transformations (10), the variation of Raitzin's canonical action is

$$\begin{aligned}\Delta I&=\delta I+\dot{I}\Delta t\\ &=\int_{t_0}^{t_1}\left[\frac{\partial R}{\partial s_j}\delta s_j+\frac{\partial R}{\partial r_j}\delta r_j+q_j\delta r_j+r_j\delta q_j\right]\mathrm{d}t\\ &\quad+(R+r_jq_j)\Delta t\Big|_{t_0}^{t_1}=\int_{t_0}^{t_1}\left\{\left(\frac{\partial R}{\partial r_j}+q_j\right)\delta r_j\right.\\ &\quad+\left(r_j-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}\right)\delta q_j+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial R}{\partial s_j}\delta q_j\right.\\ &\quad\left.\left.+(R+r_jq_j)\Delta t\right]\right\}\mathrm{d}t.\end{aligned} \tag{11}$$

Noticing

$$\begin{aligned}\Delta t=\varepsilon\tau^0, \quad \delta q_j=\Delta q_j-s_j\Delta t=\varepsilon(\xi_j^0-s_j\tau^0),\\ \delta r_j=\Delta r_j-\dot{r}_j\Delta t=\varepsilon(\eta_j^0-\dot{r}_j\tau^0),\end{aligned} \tag{12}$$

the formula (11) can be expressed as

$$\begin{aligned}\Delta I=\int_{t_0}^{t_1}\varepsilon\Bigg\{&\left(\frac{\partial R}{\partial r_j}+q_j\right)(\eta_j^0-\dot{r}_j\tau^0)\\ &+\left(r_j-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}\right)\\ &\times(\xi_j^0-s_j\tau^0)+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial R}{\partial s_j}(\xi_j^0-s_j\tau^0)\right.\\ &\left.+(R+r_jq_j)\tau^0\right]\Bigg\}\mathrm{d}t.\end{aligned} \tag{13}$$

This is the fundamental formula of the variation of the Raitzin canonical action (9).

Definition 1

For the infinitesimal transformations (10), if the variation of the Raitzin canonical action (9) satisfies

$$\Delta I=-\int_{t_0}^{t_1}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G^0)+\tilde{Q}_j\delta q_j\right]\mathrm{d}t, \tag{14}$$

where $\tilde{Q}_j=\tilde{Q}_j(t,r,s)$ are generalized non-potential forces, $G^0=G^0(t,r,s)$ is the gauge function, then the transformations (10) are called the generalized quasi-symmetrical transformations of the given system.

Theorem 1

For the nonlinear nonholonomic system given by Eqs.(5), (7) and (8), if infinitesimal transformations (10) are generalized quasi-symmetrical transformations and they satisfy the Appell–Chetaev conditions

$$\frac{\partial \tilde{f}_\beta}{\partial s_j}(\xi_j^0 - s_j\tau^0) = 0, \quad (\beta = 1, 2, \cdots, g; j = 1, 2, \cdots, n), \tag{15}$$

then the system possesses the following exact invariant:

$$I_0 = \frac{\partial R}{\partial s_j}(\xi_j^0 - s_j\tau^0) + (R + r_j q_j)\tau^0 + G^0 = \text{const.} \tag{16}$$

Theorem 2

For the nonlinear nonholonomic system given by Eqs.(5), (7) and (8), if the generators τ^0, ξ_j^0 and η_j^0 of the infinitesimal transformations (10) and the gauge function G^0 satisfy the following conditions:

$$\tilde{\lambda}_\beta \frac{\partial \tilde{f}_\beta}{\partial s_j}(\xi_j^0 - s_j\tau^0) = 0, \quad (\beta = 1, 2, \cdots, g), \tag{17}$$

$$\begin{aligned}&\left(\frac{\partial R}{\partial r_j} + q_j\right)(\eta_j^0 - \dot{r}_j\tau^0)\\ &+ \left(r_j - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} + \tilde{Q}_j\right)(\xi_j^0 - s_j\tau^0)\\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial R}{\partial s_j}(\xi_j^0 - s_j\tau^0) + (R + r_j q_j)\tau^0\right] = -\dot{G}^0.\end{aligned} \tag{18}$$

then the system possesses the same exact invariants as Eq.(16).

4. Perturbation to the symmetries and adiabatic invariants

First we give the concept of higher-order adiabatic invariants.

Definition 2

If $I_z(t, r, s, \varepsilon)$ is a physical quantity including ε in which the highest power is z in a mechanical system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a zth-order adiabatic invariant of the mechanical system.

Suppose that the nonlinear nonholonomic system given by Eqs.(5),(7) and (8) is perturbed by small quantities εW_j, then the equations of motion of the system become

$$\begin{aligned}&s_j = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j}, \quad (q_j = -\frac{\partial R}{\partial r_j}),\\ &r_j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} - \tilde{Q}_j - \tilde{\lambda}_\beta\frac{\partial \tilde{f}_\beta}{\partial s_j} - \varepsilon \mathrm{W}_j.\end{aligned} \tag{19}$$

Due to the action of εW_j, the primary symmetries and invariants of the system may vary. Suppose that the variation is a small perturbation based on the symmetrical transformations of the system without perturbation, and $\tau(t, r, s)$, $\xi_j(t, r, s)$ and $\eta_j(t, r, s)$ express the new generators after being perturbed, then

$$\begin{aligned}\tau &= \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots,\\ \xi_j &= \xi_j^0 + \varepsilon\xi_j^1 + \varepsilon^2\xi_j^2 + \cdots,\\ \eta_j &= \eta_j^0 + \varepsilon\eta_j^1 + \varepsilon^2\eta_j^2 + \cdots,\end{aligned} \tag{20}$$

and they satisfy

$$\begin{aligned}&\left(\frac{\partial R}{\partial r_j} + q_j\right)(\eta_j - \dot{r}_j\tau)\\ &+ \left(r_j - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} + \tilde{Q}_j\right)(\xi_j - s_j\tau)\\ &+ \varepsilon \mathrm{W}_j(\xi_j - s_j\tau) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial R}{\partial s_j}(\xi_j - s_j\tau)\right.\\ &\left.+ (R + r_j q_j)\tau\right] = -\dot{G},\end{aligned} \tag{21}$$

$$\frac{\partial \tilde{f}_\beta}{\partial s_j}(\xi_j - s_j\tau) = 0, \qquad (\beta = 1, 2, \cdots, g), \tag{22}$$

with G in Eq.(21) being a gauge function. Let

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots. \tag{23}$$

Substituting Eqs.(20) and (23) into Eqs.(21) and (22) yields

$$\begin{aligned}&\left(\frac{\partial R}{\partial r_j} + q_j\right)(\eta_j^k - \dot{r}_j\tau^k)\\ &+ \left(r_j - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} + \tilde{Q}_j\right)(\xi_j^k - s_j\tau^k)\\ &+ W_j(\xi_j^{k-1} - s_j\tau^{k-1}) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial R}{\partial s_j}(\xi_j^k - s_j\tau^k)\right.\\ &\left.+ (R + r_j q_j)\tau^k\right] = -\dot{G}^k,\\ &(k = 0, 1, 2, \cdots, z).\end{aligned} \tag{24}$$

When $k = 0$, the condition $W_j = 0$ holds.

By supposing that the new generators ε^k, τ^k under infinitesimal transformations still satisfy the Appell–Chetaev conditions (1) after being perturbed and by introducing the Lagrangian multipliers $\tilde{\lambda}_\beta$, we have

$$\tilde{\lambda}_\beta\frac{\partial \tilde{f}_\beta}{\partial s_j}(\xi_j^k - s_j\tau^k) = 0. \tag{25}$$

Theorem 3

For the nonlinear nonholonomic system given by Eqs.(5) and (19), which is perturbed by small quantities εW_j, if the generators $\tau^k(t, r, s)$, $\xi_j^k(t, r, s)$ and

$\eta_j^k(t,r,s)$ under infinitesimal transformations and the gauge function $G^k(t,r,s)$ satisfy Eqs.(24) and (25), then

$$I_z = \varepsilon^k\left[\frac{\partial R}{\partial s_j}(\xi_j^k - s_j\tau^k) + (R + r_j q_j)\tau^k + G^k\right]. \quad (26)$$

is a zth-order adiabatic invariant of the mechanical system.

Proof

By differentiating I_z with respect to time t, we have

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t} =&\varepsilon^k\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}(\xi_j^k - s_j\tau^k)\right.\\ &+\frac{\partial R}{\partial s_j}(\dot{\xi}_j^k - \dot{s}_j\tau^k - s_j\dot{\tau}^k)\\ &+\dot{R}\tau^k + R\dot{\tau}^k + r_j\dot{q}_j\tau^k\\ &\left.+ q_j\dot{r}_j\tau^k + q_j r_j\dot{\tau}^k + \dot{G}^k\right]. \end{aligned} \quad (27)$$

Substituting Eqs.(24) and (25) into the above formula gives

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t} =&\varepsilon^k\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}(\xi_j^k - s_j\tau^k)\right.\\ &+\frac{\partial R}{\partial s_j}(\dot{\xi}_j^k - \dot{s}_j\tau^k - s_j\dot{\tau}^k)\\ &+\dot{R}\tau^k + R\dot{\tau}^k + r_j\dot{q}_j\tau^k + q_j\dot{r}_j\tau^k\\ &+ q_j r_j\dot{\tau}^k - \left(\frac{\partial R}{\partial r_j} + q_j\right)\\ &\times(\eta_j^k - \dot{r}_j\tau^k) - \left(r_j - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}\right.\\ &\left.+\tilde{Q}_j + \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j}\right)(\xi_j^k - s_j\tau^k)\\ &- W_j(\xi_j^{k-1} - s_j\tau^{k-1}) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}(\xi_j^k - s_j\tau^k)\\ &-\frac{\partial R}{\partial s_j}(\dot{\xi}_j^k - \dot{s}_j\tau^k - s_j\dot{\tau}^k) - \dot{R}\tau^k - R\dot{\tau}^k\\ &\left.- r_j\dot{q}_j\tau^k - q_j\dot{r}_j\tau^k - r_j q_j\dot{\tau}^k\right]. \end{aligned} \quad (28)$$

Using Eqs.(19) yields

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \varepsilon^k[\varepsilon W_j(\xi_j^k - s_j\tau^k) - W_j(\xi_j^{k-1} - s_j\tau^{k-1})], \quad (29)$$

$$(k = 0, 1, 2, \cdots, z).$$

By expanding the above formula and making summation, we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \varepsilon^{z+1}W_j(\xi_j^z - s_j\tau^z). \quad (30)$$

This shows that the $\dfrac{\mathrm{d}I_z}{\mathrm{d}t}$ is in direct proportion to ε^{z+1}, so I_z is a zth-order adiabatic invariant of the mechanical system.

5. Inverse theorem of the perturbation to the symmetries of system

Suppose that the nonlinear nonholonomic system described by Eqs.(5) and (19) has a first-order adiabatic invariant as follows:

$$I_1 = \tilde{\varphi}_0(t,r,s) + \varepsilon\tilde{\varphi}_1(t,r,s). \quad (31)$$

In order to make the calculation simple, using canonical variables (3) and the formula $p_j = \dfrac{\partial R}{\partial s_j}$, the above formula can be written as

$$I_1 = \varphi_0(t,q,p) + \varepsilon\varphi_1(t,q,p). \quad (32)$$

So we have

$$\begin{aligned}\frac{\mathrm{d}I_1}{\mathrm{d}t} =&\frac{\partial\varphi_0}{\partial t} + \frac{\partial\varphi_0}{\partial q_j}\dot{q}_j + \frac{\partial\varphi_0}{\partial p_j}\dot{p}_j\\ &+\varepsilon\left(\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_1}{\partial q_j}\dot{q}_j + \frac{\partial\varphi_1}{\partial p_j}\dot{p}_j\right). \end{aligned} \quad (33)$$

Because

$$\dot{p}_j = r_j + \tilde{Q}_j + \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j} + \varepsilon W_j, \quad (34)$$

we have

$$\begin{aligned}\frac{\mathrm{d}I_1}{\mathrm{d}t} =&\frac{\partial\varphi_0}{\partial t} + \frac{\partial\varphi_0}{\partial q_j}\dot{q}_j + \frac{\partial\varphi_0}{\partial p_j}\\ &\times\left(r_j + \tilde{Q}_j + \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j} + \varepsilon W_j\right)\\ &+\varepsilon\left[\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_1}{\partial q_j}\dot{q}_j + \frac{\partial\varphi_1}{\partial p_j}\right.\\ &\left.\times\left(r_j + \tilde{Q}_j + \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j} + \varepsilon W_j\right)\right]. \end{aligned} \quad (35)$$

From Eqs.(19), we obtain

$$\begin{aligned}&\left(-r_j + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} - \tilde{Q}_j - \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j} - \varepsilon W_j\right)(\xi_j - s_j\tau)\\ &+\left(\frac{\partial R}{\partial r_j} + q_j\right)(\eta_j - \dot{r}_j\tau) = 0. \end{aligned} \quad (36)$$

By using Eq.(30) and adding Eqs.(35) and (36), we have

$$\begin{aligned}&\frac{\partial\varphi_0}{\partial t} + \frac{\partial\varphi_0}{\partial q_j}\dot{q}_j + \frac{\partial\varphi_0}{\partial p_j}\left(r_j + \tilde{Q}_j + \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j} + \varepsilon W_j\right)\\ &+\varepsilon\left[\frac{\partial\varphi_1}{\partial t} + \frac{\partial\varphi_1}{\partial q_j}\dot{q}_j + \frac{\partial\varphi_1}{\partial p_j}\left(r_j + \tilde{Q}_j + \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j} + \varepsilon W_j\right)\right]\\ &+\left(-r_j + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} - \tilde{Q}_j - \tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j} - \varepsilon W_j\right)(\xi_j - s_j\tau)\\ &+\left(\frac{\partial R}{\partial r_j} + q_j\right)(\eta_j - \dot{r}_j\tau) = \varepsilon^2 W_j(\xi_j^1 - s_j\tau^1). \end{aligned} \quad (37)$$

Considering Eqs.(20) and rearranging the above equation, we have

$$\begin{aligned}
&\frac{\partial\varphi_0}{\partial t}+\frac{\partial\varphi_0}{\partial q_j}\dot{q}_j+\frac{\partial\varphi_0}{\partial p_j}\left(r_j+\tilde{Q}_j+\tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j}+\varepsilon W_j\right)\\
&-r_j(\xi_j^0-s_j\tau^0)+\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}-\tilde{Q}_j\right.\\
&\left.-\tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j}-\varepsilon W_j\right)(\xi_j^0-s_j\tau^0)\\
&+\varepsilon\left(-r_j+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j}-\tilde{Q}_j-\tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j}\right)(\xi_j^1-s_j\tau^1)\\
&+\varepsilon\left[\frac{\partial\varphi_1}{\partial t}+\frac{\partial\varphi_1}{\partial q_j}\dot{q}_j+\frac{\partial\varphi_1}{\partial p_j}\right.\\
&\left.\times\left(r_j+\tilde{Q}_j+\tilde{\lambda}_\beta\frac{\partial\tilde{f}_\beta}{\partial s_j}+\varepsilon W_j\right)\right]\\
&+\left(\frac{\partial R}{\partial r_j}+q_j\right)(\eta_j-\dot{r}_j\tau)+O(\varepsilon)=0. \qquad (38)
\end{aligned}$$

Now, we seek the generators τ^0, ξ_j^0 and η_j^0 of the infinitesimal transformations without perturbation. Firstly, by separating the terms not containing ε in Eq.(38), and then by separating the terms containing r_j and taking their coefficients as zeros, we obtain

$$\frac{\partial\varphi_0}{\partial p_j}-(\xi_j^0-s_j\tau^0)=0. \qquad (39)$$

Further let

$$\tilde{\varphi}_0=\frac{\partial R}{\partial s}(\xi_j^0-s_j\tau^0)+(R+r_jq_j)\tau^0+G^0. \qquad (40)$$

Then from Eqs.(39) and (40), we can obtain

$$\tau^0=\frac{\tilde{\varphi}_0-\dfrac{\partial R}{\partial s_j}\dfrac{\partial\varphi_0}{\partial p_j}-G^0}{R+r_jq_j}, \qquad (41)$$

$$\xi_j^0=\frac{s_j\left(\tilde{\varphi}_0-\dfrac{\partial R}{\partial s_j}\dfrac{\partial\varphi_0}{\partial p_j}-G^0\right)}{R+r_jq_j}+\frac{\partial\varphi_0}{\partial p_j}. \qquad (42)$$

Because

$$\begin{aligned}
\Delta r_j=&\varepsilon\eta_j^0,\\
\Delta r_j=&\frac{\partial r_j}{\partial t}\Delta t+\frac{\partial r_j}{\partial q_i}\Delta q_i+\frac{\partial r_j}{\partial\dot{q}_i}\Delta\dot{q}_j\\
=&\varepsilon\left[\frac{\partial r_j}{\partial t}\tau^0+\frac{\partial r_j}{\partial q_i}\xi_i^0+\frac{\partial r_j}{\partial\dot{q}_i}(\dot{\xi}_i^0-\dot{q}_i\dot{\tau}^0)\right],
\end{aligned}$$

we have

$$\eta_j^0=\frac{\partial r_j}{\partial t}\tau^0+\frac{\partial r_j}{\partial q_i}\xi_i^0+\frac{\partial r_j}{\partial\dot{q}_i}(\dot{\xi}_i^0-\dot{q}_i\dot{\tau}^0). \qquad (43)$$

Further, by separating the terms containing ε in Eq.(38), and then by separating the terms containing r_j and taking their coefficients as zeros, we have

$$\tau^1=\frac{\tilde{\varphi}_1-\dfrac{\partial R}{\partial s_j}\dfrac{\partial\varphi_1}{\partial p_j}-G^1}{R+r_jq_j}, \qquad (44)$$

$$\xi_j^1=\frac{s_j\left(\tilde{\varphi}_1-\dfrac{\partial R}{\partial s_j}\dfrac{\partial\varphi_1}{\partial p_j}-G^1\right)}{R+r_jq_j}+\frac{\partial\varphi_1}{\partial p_j}. \qquad (45)$$

Next, in the same way as that of calculating the above formula (43), we have

$$\eta_j^1=\frac{\partial r_j}{\partial t}\tau^1+\frac{\partial r_j}{\partial q_i}\xi_i^1+\frac{\partial r_j}{\partial\dot{q}_i}(\dot{\xi}_i^1-\dot{q}_i\dot{\tau}^1). \qquad (46)$$

Then we have the following theorem.

Inverse theorem 1

For the nonlinear nonholonomic system given by Eqs.(5) and (19), which is perturbed by small quantities εW_j, if it has a first order adiabatic invariant, then in the system there exists an infinitesimal symmetrical transformation, and the items without perturbation and the first-order perturbation items of the infinitesimal generators of the transformations are determined by Eqs.(41)–(43) and (44)–(46) respectively, when the gauge functions G^0 and G^1 are given.

6. Example

Suppose that the Lagrangian of a mechanical system is

$$L=\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-mgq_3. \qquad (47)$$

The nonlinear nonholonomic constraint acting on the system is

$$f=\dot{q}_1^2+\dot{q}_2^2-\dot{q}_3^2=0. \qquad (48)$$

We study the perturbation to symmetries and adiabatic invariants of the system.

The first step is to seek the exact invariants of the system.

Equations (2) give

$$m\ddot{q}_1=2\lambda\dot{q}_1,\quad m\ddot{q}_2=2\lambda\dot{q}_2, m\ddot{q}_3+mg=-2\lambda\dot{q}_3. \qquad (49)$$

$$p_j=\frac{\partial L}{\partial\dot{q}_j}=m\dot{q}_j=ms_j,\qquad (j=1,2,3). \qquad (50)$$

From Eqs.(48) and (49), we obtain

$$\lambda=-\frac{mg}{4\dot{q}_3}. \qquad (51)$$

Introducing the Raitzin canonical variables and function as

$$r_1=\frac{\partial L}{\partial q_1}=0, r_2=\frac{\partial L}{\partial q_2}=0, r_3=\frac{\partial L}{\partial q_3}=-mg, \qquad (52)$$

$$s_j=\dot{q}_j,\qquad (j=1,2,3),$$

$$R = L - r_j q_j = \frac{1}{2}m(s_1^2 + s_2^2 + s_3^2), \tag{53}$$

then the nonholonomic constraint (48) and the constrained multiplier (51) will be written as

$$\tilde{f} = s_1^2 + s_2^2 - s_3^2 = 0, \quad \tilde{\lambda} = -\frac{mg}{4s_3}. \tag{54}$$

Because

$$\begin{aligned}
&\tilde{\lambda}\frac{\partial \tilde{f}}{\partial s_1}(\xi_1^0 - s_1\tau^0) = -\frac{mgs_1}{2s_3}(\xi_1^0 - s_1\tau^0),\\
&\tilde{\lambda}\frac{\partial \tilde{f}}{\partial s_2}(\xi_2^0 - s_2\tau^0) = -\frac{mgs_2}{2s_3}(\xi_2^0 - s_2\tau^0),\\
&\tilde{\lambda}\frac{\partial \tilde{f}}{\partial s_3}(\xi_3^0 - s_3\tau^0) = \frac{mg}{2}(\xi_3^0 - s_3\tau^0).
\end{aligned} \tag{55}$$

Then from Eqs.(17) and (18) we obtain

$$\begin{aligned}
&-\frac{mgs_1}{2s_3}(\xi_1^0 - s_1\tau^0) - \frac{mgs_2}{2s_3}(\xi_2^0 - s_2\tau^0)\\
&+\frac{mg}{2}(\xi_3^0 - s_3\tau^0) = 0,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&q_1\eta_1^0 + q_2\eta_2^0 + q_3\eta_3^0 - m\dot{s}_1(\xi_1^0 - s_1\tau^0) - m\dot{s}_2(\xi_2^0 - s_2\tau^0)\\
&+(-mg - m\dot{s}_3)(\xi_3^0 - s_3\tau^0) + \frac{\mathrm{d}}{\mathrm{d}t}\Big\{ms_1(\xi_1^0 - s_1\tau^0)\\
&+ms_2(\xi_2^0 - s_2\tau^0) + ms_3(\xi_3^0 - s_3\tau^0)\\
&+\left[\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3\right]\tau^0\Big\} = -\dot{G}^0.
\end{aligned} \tag{57}$$

Equations (56) and (57) have solutions

$$\begin{aligned}
&\eta_1^0 = \eta_2^0 = \eta_3^0 = 0, \quad \xi_1^0 = \xi_2^0 = \xi_3^0 = 0,\\
&\tau^0 = 1, \quad G^0 = 0.
\end{aligned} \tag{58}$$

So we can obtain the following exact invariant by Theorem 2

$$I_0 = -\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3 = \text{const.} \tag{59}$$

The second step is to study the adiabatic invariant of the system.

We suppose that the system is perturbed by small quantities as follows:

$$\varepsilon W_j = \varepsilon q_j, \qquad (j = 1, 2, 3). \tag{60}$$

Then from Eqs.(24) and (25), we have

$$\begin{aligned}
&q_1\eta_1^1 + q_2\eta_2^1 + q_3\eta_3^1 - m\dot{s}_1(\xi_1^1 - s_1\tau^1) - m\dot{s}_2(\xi_2^1 - s_2\tau^1)\\
&+(-mg - m\dot{s}_3)(\xi_3^1 - s_3\tau^1) + \frac{\mathrm{d}}{\mathrm{d}t}\Big\{ms_1(\xi_1^1 - s_1\tau^1)\\
&+ms_2(\xi_2^1 - s_2\tau^1) + ms_3(\xi_3^1 - s_3\tau^1)\\
&+\left[\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3\right]\tau^1\Big\}\\
&+q_1(\xi_1^0 - s_1\tau^0) + q_2(\xi_2^0 - s_2\tau^0)\\
&+q_3(\xi_3^0 - s_3\tau^0) = -\dot{G}^1,
\end{aligned} \tag{61}$$

$$\begin{aligned}
&-\frac{mgs_1}{2s_3}(\xi_1^1 - s_1\tau^1) - \frac{mgs_2}{2s_3}(\xi_2^1 - s_2\tau^1)\\
&+\frac{mg}{2}(\xi_3^1 - s_3\tau^1) = 0.
\end{aligned} \tag{62}$$

The solutions of Eqs.(61) and (62) are

$$\begin{aligned}
&\eta_1^1 = \eta_2^1 = \eta_3^1 = 0, \quad \xi_1^1 = \xi_2^1 = \xi_3^1 = 0,\\
&\tau^1 = 1, \quad G^1 = \frac{1}{2}(q_1^2 + q_2^2 + q_3^2).
\end{aligned} \tag{63}$$

So we can obtain the following first-order adiabatic invariant by Theorem 3:

$$\begin{aligned}
I_0 = &-\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3\\
&+\varepsilon\left[-\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3\right.\\
&\left.+\frac{1}{2}(q_1^2 + q_2^2 + q_3^2)\right] = \text{const.}
\end{aligned} \tag{64}$$

Furthermore we can obtain more higher-order adiabatic invariants.

Third step is to study the inverse problem.

Suppose that the system is perturbed by the small quantities (60), and there exists a first order adiabatic invariant (64).

Because

$$\begin{aligned}
&\tilde{\varphi}_0 = -\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3,\\
&\varphi_0 = -\frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - mgq_3,\\
&\frac{\partial R}{\partial s_j} = ms_j, \qquad (j = 1, 2, 3),\\
&\frac{\partial \varphi_0}{\partial p_j} = -\frac{p_j}{m} = -s_j, \qquad (j = 1, 2, 3),\\
&R + r_j q_j = \frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3.
\end{aligned}$$

By substituting the above formulas into Eqs.(41), (42), when $G^0 = 0$, we have

$$\tau^0 = 1, \quad \xi_1^0 = \xi_2^0 = \xi_3^0 = 0. \tag{65}$$

Further, by substituting formulas (65) into Eq.(43), and considering the Eqs.(52), we obtain

$$\eta_1^0 = \eta_2^0 = \eta_3^0 = 0. \tag{66}$$

In the same method, we have

$$\begin{aligned}
&\tilde{\varphi}_1 = -\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3 + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2),\\
&\varphi_1 = -\frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - mgq_3 + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2),\\
&\frac{\partial R}{\partial s_j} = ms_j, \qquad (j = 1, 2, 3),\\
&\frac{\partial \varphi_1}{\partial p_j} = -\frac{p_j}{m} = -s_j, \qquad (j = 1, 2, 3),\\
&R + r_j q_j = \frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3.
\end{aligned}$$

When $G^1 = \frac{1}{2}(q_1^2 + q_2^2 + q_3^2)$, from the formulas (44)–(46) and by considering the Eqs.(52), we have

$$\tau^1 = 1, \quad \xi_1^1 = \xi_2^1 = \xi_3^1 = 0, \eta_1^1 = \eta_2^1 = \eta_3^1 = 0. \quad (67)$$

References

[1] Noether A E 1918 *Nachr. Akad. Wiss. Göett.* II. *Math. Phys.* **KI**, II 235

[2] Lutzky M 1979 *J. Phys. A: Math. Gen.* **19** 973

[3] Djukić D S and Sutela T 1984 *Int. Nonlin. Math.* **19** 331

[4] Mei F X 2001 *Chin. Phys.* **10** 177

[5] Hojman S A 1992 *J. Phys. A: Math. Gen.* **25L** 291

[6] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) p178 (in Chinese)

[7] Qiao Y F, Meng J and Zhao S H 2002 *Chin. Phys.* **11** 859

[8] Fang J H 2002 *Chin. Phys.* **11** 312

[9] Luo S K 2002 *Chin. Phys.* **11** 429

[10] Qiao Y F and Zhao S H 2001 *Acta Phys. Sin.* **50** 1 (in Chinese)

[11] Qiao Y F, Zhang Y L and Han G C 2002 *Chin. Phys.* **11** 998

[12] Qiao Y F, Li R J and Zhao S H 2001 *Acta Phys. Sin.* **50** 811 (in Chinese)

[13] Qiao Y F, Yu Q W and Dong Y A 1994 *Appl. Math. Mech.* **15** 877

[14] Qiao Y F, Zhang Y L and Han G C 2003 *Acta Phys. Sin.* **52** 1051 (in Chinese)

[15] Qiao Y F, Zhao S H and Li R J 2004 *Chin. Phys.* **13** 292

[16] Fu J L, Chen L Q and Yang X D 2004 *Chin. Phys.* **13** 285

[17] Qiao Y F, Zhao S H and Li R J 2004 *Acta Phys. Sin.* **53** 2035 (in Chinese)

[18] Burgers J M 1917 *Ann Phys. Lpz* **52** 195

[19] Ostrovsky V N and Prudov N V 1995 *J. Phys. B: At. Mol. Opt. Phys.* **20** 4435

[20] Zhao Y Y and Mei F X 1996 *Acta Mech. Sin.* **28** 207 (in Chinese)

[21] Chen X W, Zhang R C and Mei F X 2002 *Acta Mech. Sin.* **16** 282 (in Chinese)

[22] Chen X W and Mei F X 2000 *Chin. Phys.* **9** 721

[23] Chen X W, Shang M and Mei F X 20021 *Chin. Phys.* **10** 991

[24] Chen X W and Li Y M 2003 *Chin Phys.* **12** 1349

[25] Chen X W and Li Y M 2003 *Chin Phys.* **12** 936

[26] Raitzin C M 1961 *An. Soc. Cient. Argent.* **171** 50

[27] Qiao Y F 1990 *Acta Mech. Sin.* **22** 99 (in Chinese)

[28] Qiao Y F, Zhang Y L and Zhao S H 2002 *Acta Phys. Sin.* **51** 1661 (in Chinese)