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Synchronization of an uncertain chaotic system via recurrent neural networks*

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Incorporating distributed recurrent networks with high-order connections between neurons, the identification and synchronization problem of an unknown chaotic system in the presence of unmodelled dynamics is investigated. Based on the Lyapunov stability theory, the weights learning algorithm for the recurrent high-order neural network model is presented. Also, analytical results concerning the stability properties of the scheme are obtained. Then adaptive control law for eliminating synchronization error of uncertain chaotic plant is developed via Lyapunov methodology. The proposed scheme is applied to model and synchronize an unknown Rossler system.

Keywords: chaos, recurrent neural networks, adaptive control, synchronization, nonlinear system
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1. Introduction

Chaos is an apparently irregular motion that is, in reality, nonlinear but deterministic. Controlling this kind of complex dynamical systems has recently attracted a great deal of attention from various disciplines. Different strategies have been proposed to achieve chaos control.[1−5] A chaotic system exhibits extremely sensitive dependence on initial conditions, and two identical chaotic systems starting at nearly the same point follow trajectories that divert rapidly from each other and become uncorrelated quickly. Thus, it has originally been considered impossible to construct equivalent synchronized chaos in physical systems. Nevertheless, since the idea of synchronizing two same chaotic systems with different initial conditions was introduced by Pecora and Carrol,[6] synchronization in chaotic systems has been extensively investigated.

For unknown chaotic systems, the uncertainties caused by inaccurate measurements, modelling error and so on, have to be carefully dealt with for safe and improved operations. It has been proven that, even with one hidden layer, neural networks can uniformly approximate any continuous function over a compact set, provided that the network has a sufficient number of neurons.[7] Therefore, interest has recently been increased towards the application of neural networks for modelling and identification of dynamical systems.

In this paper, we incorporate distributed recurrent networks with higher-order connections between neurons, the chaos synchronization problem in the presence of uncertainties is studied. Firstly, a recurrent high-order neural network (RHONN) is used as identifier to build an on-line model for the unknown chaotic plant, then an adaptive controller is derived to synchronize the uncertain system with different initial conditions.

The rest of this paper is organized as follows. In Section 2 we outline the RHONN model. The learning laws in the cases both with no modelling error and with modelling error for RHONN as the model of an uncertain nonlinear dynamical system are derived in Section 3. We design an adaptive synchronization controller for trajectory tracking by Lyapunov synthesis method in Section 4. Simulation results are offered in Section 5 to support our theoretic findings. Finally, concluding remarks are given in Section 6.

2. High-order network model

Now consider a RHONN consisting of n neurons

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and m inputs. The state of each neuron is governed by a differential equation of the form

$$\dot{x}_i = -a_i x_i + \sum_{k=1}^{L} w_{ik} \prod_{j \in I_k} y_j^{d_j(k)}, \tag{1}$$

where x_i is the ith neuron state, $\{I_1, I_2, \cdots, I_L\}$ is a collection of L non-ordered subsets of $\{1, 2, \cdots, m+n\}$, a_i is real coefficients, w_{ik} are the adjustable synapse weights of the neural network, and $d_j(k)$ are non-negative integers, and y is the vector consisting of inputs to each neuron, defined as

$$\begin{aligned} y =& [y_1, \cdots, y_n, y_{n+1}, \cdots, y_{n+m}]^{\mathrm{T}} \\ =& [S(x_1), \cdots, S(x_n), S(u_1), \cdots, S(u_m)]^{\mathrm{T}}. \end{aligned} \tag{2}$$

Here $u = [u_1, u_2, \cdots, u_m]^{\mathrm{T}}$ is the external input vector to the network. The function $S(\cdot)$ is a monotonic increasing, differentiable sigmoidal function formulated by $S(x) = \dfrac{\alpha}{1 + \mathrm{e}^{-\beta x}} + \varepsilon$, with α and β being positive real numbers, and ε a small real number.

There we introduce the L-dimensional vector z defined by $z = \left[\prod_{j \in I_1} y_j^{d_j(1)}, \cdots, \prod_{j \in I_L} y_j^{d_j(L)} \right]^{\mathrm{T}}$, hence Eq.(1) is rewritten as

$$\dot{x}_i = -a_i x_i + \sum_{k=1}^{L} w_{ik} z_k, \quad (i = 1, \cdots, n)$$

or

$$\dot{x}_i = -a_i x_i + W_i z, (i = 1, \cdots, n), \tag{3}$$

where $W_i = [w_{i,1}, \cdots, w_{i,L}]$ represents the adjustable weights of the network, while the coefficients $\{a_i : i = 1, \cdots, n\}$ are part of the underlying network architecture and fixed during training. The dynamic behaviour of the overall network is described by expressing Eq.(3) in vector notation as

$$\dot{x} = Ax + Wz, (i = 1, \cdots, n), \tag{4}$$

where $x \in \Re^n$, $W \in \Re^{n \times L}$, and $A = -\kappa I, \kappa > 0$ is an $n \times n$ diagonal stable matrix.

3. Plant identification

Consider the unknown nonlinear chaotic plant

$$\dot{\chi} = f(\chi, u), \tag{5}$$

where $\chi \in \Re^n$ is the state of the system, $u \in \Re^m$ is the input to the system and $f : \Re^{n+m} \mapsto \Re^n$ is a smooth vector field. Taking into account that f is unknown, one can model Eq.(5) by a recurrent neural network as in Eq.(4). Hence we will consider the plant model as

$$\dot{x} = Ax + Wz(\chi) + \delta, \tag{6}$$

where δ represents the modelling error. In order to derive adaptive learning law which minimizes the identification error, we first consider the case without modelling error. We assume that there exist weight vectors W^* such that each state of the unknown dynamical system (5) is completely described by the neural network of the form

$$\dot{\chi} = A\chi + W^* z(\chi), \tag{7}$$

where all the elements are defined as earlier. Thus the state error between the identifier and the real system is expressed as

$$e_i = x - \chi, \tag{8}$$

whose error dynamics is given by

$$\dot{e}_i = \dot{x} - \dot{\chi}, \dot{e}_i = Ae_i + \tilde{W} z(\chi), \tag{9}$$

with $\tilde{W} = W - W^*$. We consider the Lyapunov function to be a candidate

$$V = \frac{1}{2} \|e_i\|^2 + \frac{1}{2\gamma} \mathrm{tr}\{\tilde{W}^{\mathrm{T}} \tilde{W}\}, \quad \gamma > 0; \tag{10}$$

differentiating Eq.(10) along the solutions of Eq.(9), we have

$$\dot{V} = \frac{1}{2}(\dot{e}_i^{\mathrm{T}} e_i + \dot{e}_i^{\mathrm{T}} e_i) + \frac{1}{2\gamma}(\mathrm{tr}\{\dot{\tilde{W}}^{\mathrm{T}} \tilde{W}\} + \mathrm{tr}\{\tilde{W}^{\mathrm{T}} \dot{\tilde{W}}\}). \tag{11}$$

Substituting Eq.(9) into Eq.(11), we obtain

$$\dot{V} = -\kappa \|e_i\|^2 + \frac{1}{\gamma} \mathrm{tr}\{\dot{\tilde{W}}^{\mathrm{T}} \tilde{W}\} + e_i^{\mathrm{T}} \tilde{W} z(\chi). \tag{12}$$

If we select

$$\mathrm{tr}\{\dot{\tilde{W}}^{\mathrm{T}} \tilde{W}\} = -\gamma e_i^{\mathrm{T}} \tilde{W} z(\chi), \tag{13}$$

which is formulated in the form of element as

$$\dot{w}_{i,j} = -\gamma e_i z(\chi_j), \quad i = 1, 2, \cdots, n, j = 1, 2, \cdots, L, \tag{14}$$

then $\dot{V} = -\kappa \|e_i\|^2 \leq 0$. Thus the asymptotic stability of the identification error is achieved, and we have

$$\lim_{t \to \infty} e_i(t) = 0, \lim_{t \to \infty} \dot{\tilde{W}}(t) = 0.$$

Following the above analysis, we consider the case where the modelling error is present. It is noted that the behaviour of each state of system (5) can be expressed by a differential equation of the form

$$\dot{\chi}_i = -a_i \chi_i + W_i^* z(\chi, u) + \delta_i, \tag{15}$$

where the modelling error δ_i is given by

$$\delta_i = f_i(\chi, u) + a_i\chi - W_i^* z(\chi, u). \tag{16}$$

The function $f_i(\chi, u)$ denotes the ith component of the vector field $f(\chi, u)$, while the unknown optimal weight vector W_i^* is defined as the value of the weight vector W_i that minimizes the L_∞-norm difference between $f_i(\chi, u) + a_i\chi$ and $W_i z(\chi, u)$ for all $(\chi, u) \in \Omega \subset \Re^{m+n}$, subjected to the constraint of $|W_i| \leq M_i$, where M_i is a large design constant. The region Ω denotes the small compact subset of $\Re^{m+n}$.

In summary, for $i = 1, \cdots, n$, the optimal weight vector W_i^* is defined as

$$W_i^* = \arg \min_{|W_i| \leq M_i} \{\sup |f_i(\chi, u) + a_i\chi - W_i z(\chi, u)|\}. \tag{17}$$

The state error $e_i = x_i - \chi_i$ satisfies

$$\dot{e}_i = -a_i e_i + \phi_i z - \delta_i, \tag{18}$$

where $\phi_i = W_i - W_i^*$, W_i being the estimate of the unknown optimal weight vector W_i^*. Due to the presence of the modelling error δ_i, the learning law given by Eq.(14) is modified as follows:

$$\dot{w}_{i,j} = \begin{cases} -\lambda_i z e_i & |w_{i,j}| \leq M_i, \\ -\lambda_i z e_i - \sigma_i \lambda_i w_{i,j} & |w_{i,j}| > M_i, \end{cases} \tag{19}$$

where the gain $\lambda_i > 0$, and σ_i is a positive constant chosen by the designer. It is clear that the above weight adjustment is the same as in Eq.(14) if $w_{i,j}$ belongs to a ball of radius M_i. In the case that the weights leave this ball, the weight learning law is modified by the addition of a leakage term $\sigma_i \lambda_i w_{i,j}$, whose objective is to prevent the weight values from drifting to infinity.

Theorem 1 Consider the RHONN model given by Eq.(3) whose weights are updated according to Eq.(19). Then for $i = 1, \cdots, n$, we have

• $e_i, \phi_i \in L_\infty$;

also, there exist constants η and ρ such that

• $\int_0^t |e(\tau)|^2 \mathrm{d}\tau \leq \eta + \rho \int_0^t |\delta(\tau)|^2 \mathrm{d}\tau$,

where $\delta = [\delta_1, \cdots, \delta_n]^{\mathrm{T}}, e = [e_1, \cdots, e_n]^{\mathrm{T}}$. The proof can be found in Ref.[8].

The above theorem states the weight learning law (19) guarantees that e_i and ϕ_i remain bounded; furthermore, if the modelling error is a square integral, i.e., $\delta \in L_2$, then $e(t)$ converges to zero asymptotically.

Once the identification error is minimized, we can consider the synchronization control problem for the neural network, including the modelling error.

4. Synchronization controller design

Consider a response system (5) modelled by a RHONN identifier with modelling error term as follows:

$$\dot{x} = Ax + Wz(\chi) + \delta + u; \tag{20}$$

the drive system is represented by

$$\dot{x}_{\mathrm{r}} = f_{\mathrm{r}}(x_{\mathrm{r}}, u_{\mathrm{r}}), x_{\mathrm{r}} \in \Re^n. \tag{21}$$

In order to realize synchronization of two systems, namely, the states of the response system track asymptotically that of the drive system, we should determine a suitable control u.

Let the tracking error between the drive and response systems be

$$e_t = \chi - x_{\mathrm{r}}. \tag{22}$$

The time derivative of the error dynamic equation is

$$\begin{aligned} \dot{e}_t =& \dot{\chi} - \dot{x}_{\mathrm{r}} = Ax + Wz(\chi) + \delta + u - f_{\mathrm{r}}(x_{\mathrm{r}}, u_{\mathrm{r}}) \\ =& Ae_t + Wz(\chi) + \Delta_{\mathrm{r}}(x_{\mathrm{r}}) + \delta + u, \end{aligned} \tag{23}$$

where $\Delta_{\mathrm{r}}(x_{\mathrm{r}}) = A(x + x_{\mathrm{r}} - \chi) - f_{\mathrm{r}}(x_{\mathrm{r}}, u_{\mathrm{r}})$. The tracking problem can be analysed as a stabilization problem for the error dynamics (Eq.(23)). To solve this problem, we have employed the Lyapunov synthesis method.

Consider the Lyapunov function

$$V = \frac{1}{2}\|e_i\|^2 + \frac{1}{2}\|e_t\|^2 + \frac{1}{2\gamma}\mathrm{tr}\{\tilde{W}^{\mathrm{T}}\tilde{W}\}. \tag{24}$$

Differentiate Eq.(24) along the solution of Eq.(23), and after some simple manipulation we obtain

$$\begin{aligned} \dot{V} =& -\kappa\|e_i\|^2 + e_i^{\mathrm{T}}\tilde{W}z(\chi) + \tfrac{1}{\gamma}\mathrm{tr}\{\dot{\tilde{W}}^{\mathrm{T}}\tilde{W}\} \\ & -\kappa\|e_t\|^2 + e_t^{\mathrm{T}}Wz(\chi) + e_t^{\mathrm{T}}(\Delta_{\mathrm{r}}(x_{\mathrm{r}}) + \delta) + e_t^{\mathrm{T}}u. \end{aligned} \tag{25}$$

Substituting the learning law (14) into Eq.(25), and choosing control input u

$$u = -(Wz(\chi) + \Delta_{\mathrm{r}}(x_{\mathrm{r}}) + \delta), \tag{26}$$

then $\dot{V}$ is formulated as follows:

$$\begin{aligned} \dot{V} =& -\kappa\|e_i\|^2 - \kappa\|e_t\|^2 + e_t^{\mathrm{T}}Wz(\chi) \\ & + e_t^{\mathrm{T}}(\Delta_{\mathrm{r}}(x_{\mathrm{r}}) + \delta) + e_t^{\mathrm{T}}u \\ =& -\kappa(\|e_i\|^2 + \|e_t\|^2) \leq 0, \end{aligned} \tag{27}$$

where we assume the uncertain term δ is bounded by the above, as $|\delta_i| \leq \upsilon, i = 1, \cdots, n$. In applications, we take the upper bound of δ_i in Eq.(26).

Therefore, we always maintain the boundedness for V by using the control (26). Because e_i is bounded, the bound of V in turn implies the bound of e_t. Consequently, we have a suitable tracking error.

5. Simulation results

In order to demonstrate the applicability of the proposed control scheme, an unknown system of Rossler chaotic attractor is tested. The dynamic equations are

$$\begin{cases} \dot{\chi}_1 = -\chi_2 - \chi_3 + u_1, \\ \dot{\chi}_2 = \chi_1 + a\chi_2 + u_2, \\ \dot{\chi}_3 = -b\chi_3 + c + \chi_1\chi_3 + u_3, \end{cases} \tag{28}$$

where a, b, and c are physical parameters of the system. It can be shown that without control, i.e. $u = 0$, the system is chaotic. Figure 1 is the chaotic attractor for $a = 0.4$, $b = 4$, $c = 2$ and initial conditions $\chi(0) = [-4, -1, 6]^{\mathrm{T}}$. In the simulations, the recurrent neural network (4) is used to model the system (Eq.(28)), the parameter values are taken as κ=6.0, α=4, β=1, ε=0.1; and Eq.(19) is applied for the weights updating, in which we choose σ_i=30, λ_i=50, M_i=40, i=1, 2, 3.

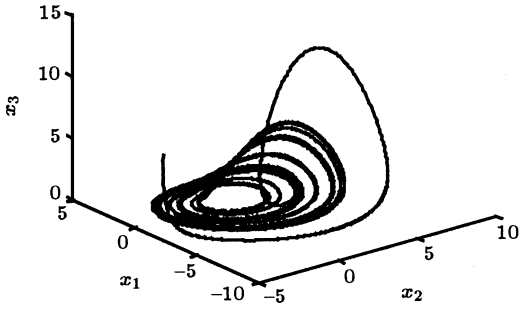


Fig.1. Phase portrait of Rossler's system.

Based on the above identification model, we utilize control law (26) to synchronize the Rossler attractor under different initial conditions. For control input (Eq.(26)), we select $\upsilon = 0.3$. Let the Rossler attractor evolve for 20s, then we activate the controller to synchronize the chaotic plants. Figure 2 shows the synchronization of the drive and response positions. The solid lines correspond to the states of response system and the dashed lines are the states of the drive system. Figure 3 shows the tracking error between the plant and the reference signal. As is seen, the unknown system follows the reference trajectory with a very small error.

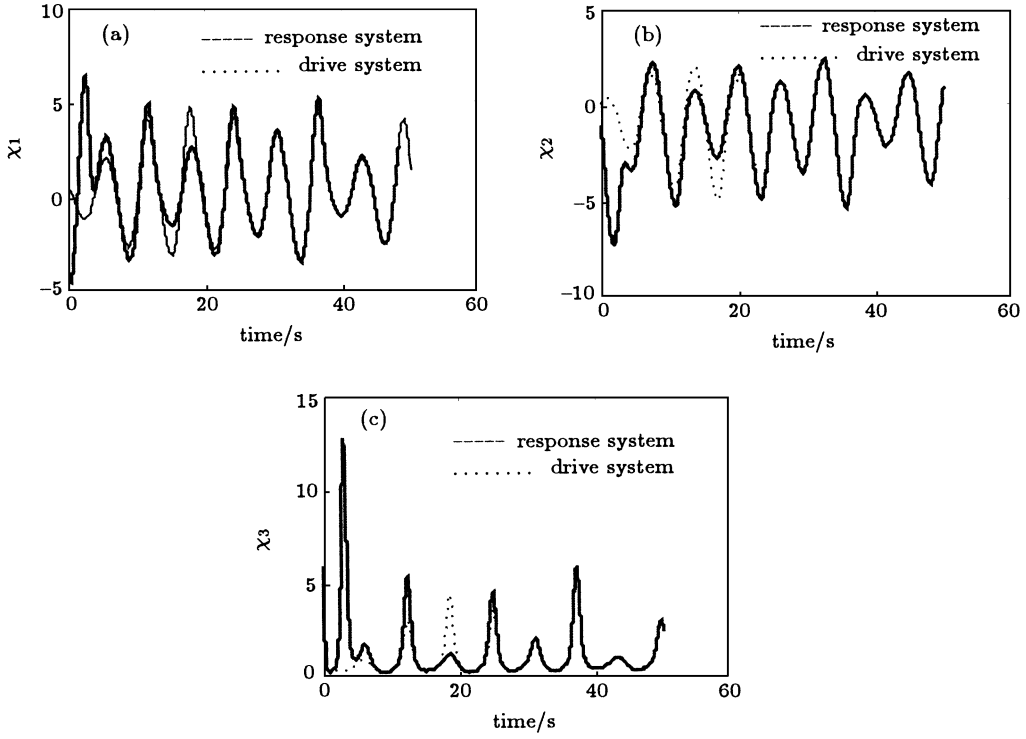


Fig.2. The synchronization of the uncertain Rossler chaotic attractor.

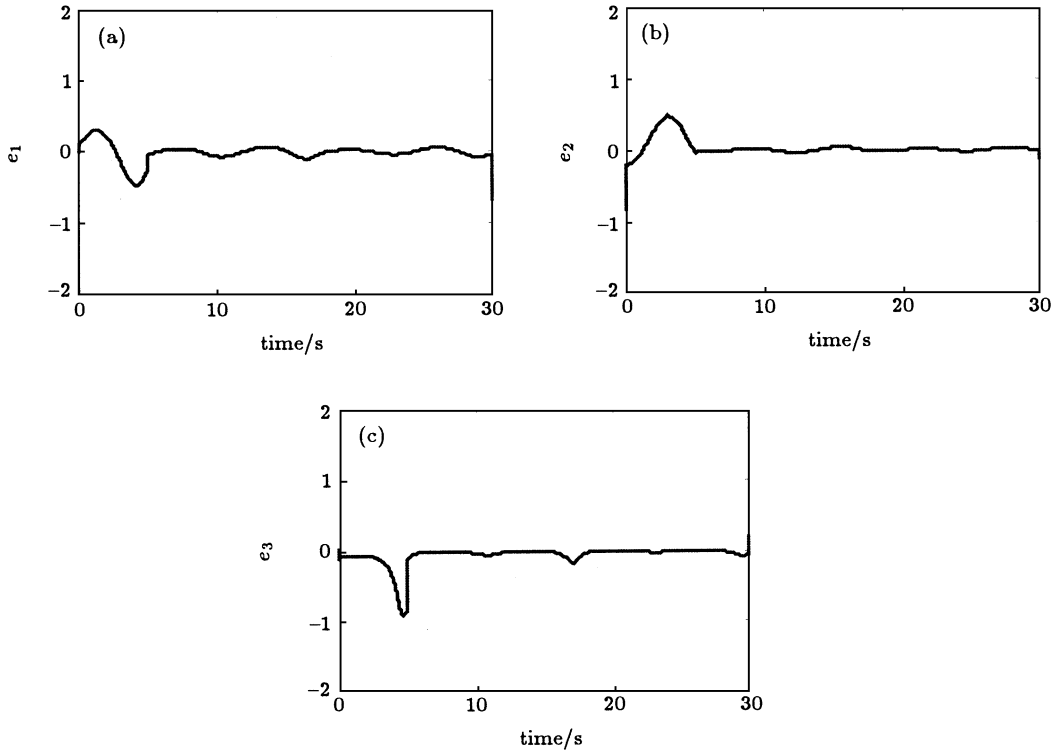


Fig.3. Synchronization error for the controlled Rossler chaotic system.

6. Conclusions

In this paper, a new adaptive control technique, based on RHONN as a model of uncertain nonlinear dynamical system, is presented. The identification scheme and weight updating laws for RHONN is developed. And, the convergence and stability properties of these weight adaptive laws are analysed, also in the case that modelling errors are present. A modified learning algorithm of guaranteeing the stability of the overall scheme is proposed. Based on the identification model, an adaptive synchronization controller for trajectory tracking is derived by means of the Lyapunov synthesis method. The feasibility of these techniques is demonstrated by considering the modelling and synchronizing of an unknown Rossler chaotic system.

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