Vol 14 No 1, January 2005
1009-1963/2005/14(01)/0045-04

Chinese Physics

© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

Mass defect effect in the gravitational field*

Gong Tian-Xi(龚添喜) and Wang Yong-Jiu(王永久)†

Institute of Physics, Hunan Normal University, Changsha 410081, *China*

(Received 13 April 2004; revised manuscript received 13 May 2004)

Using the Hellings–Nordtvedt theory, some expressions are obtained of the mass defect effect for a type of charged celestial body in a gravitational field with a nonzero cosmological constant λ. And the mass defects are obtained in several other types of fields and under different conditions by discussing the parameters in the expressions. These are meaningful calculations of the radiated energies in the processes of forming those kinds of celestial bodies in astrophysics.

Keywords: mass defect, gravitational effect, Reissner–Nordstrom–de Sitter field
PACC: 0420F

1. Introduction

In 1961, Brans and Dicke proposed a theory in which gravitation was produced by two fields, i.e. a tensor metric field and an auxiliary scalar field. The field equations can be derived from the Lagrangian action[1]

$$A=\int\sqrt{-g}\left(16\pi L_m+\Phi R+\frac{\omega}{\Phi}g^{\mu\nu}\Phi_{,\,\mu}\Phi_{,\,\nu}\right)\mathrm{d}^4x, \quad (1)$$

where $L_m = L_m$ ($g_{\mu\nu}$, matter variables) is the matter Lagrangian, Φ is the scalar field, ω is a dimensionless parameter. Matter "sees" only the metric field, and free test bodies will follow geodesics. If the quantity L_{m} contains additional gravitational scalar, vector or tensor fields, the matter will not in general follow the geodesics. An analysis by Dicke[2] has shown that the high-precision null experiments, such as Eotvös experiments and Hughs–Drevor experiments, do not rule out the existence of vector or additional second-rank-tensor fields coupling directly to matter. As pointed out by Will and Nordtvedt,[3] however, such vector and additional fields may exist along with the metric field as long as the additional fields do not couple directly to matter.

In this paper, we discuss a metric theory of gravity containing a vector field and cosmological constant λ in addition to the metric field. We require a Lagrangian

$$\begin{aligned}A=\int\sqrt{-g}(&16\pi G_0L_m+R\\&-H_{\mu\nu}H^{\mu\nu}+2\lambda+F_{\mu\nu}F^{\mu\nu}\\&+\omega K_\mu K^\mu R+\eta K^\mu K^\nu R_{\mu\nu})\mathrm{d}^4x,\end{aligned} \quad (2)$$

where $L_m = L_m$ ($g_{\mu\nu}$, matter variables) is the matter Lagrangian, $H_{\mu\nu}=K_{\mu;\nu}-K_{\nu;\mu}$ in analogy with electrodynamics, ω and η are dimensionless parameters, $F_{\mu\nu}$ the Maxwell tensor, λ is a cosmological constant and G_0 is the gravitational constant. Variation of $g_{\mu\nu}$ in Eq.(2) reads

$$\begin{aligned}&R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\lambda g_{\mu\nu}\\&+\omega\left(K_\mu K_\nu R+\Phi R_{\mu\nu}\right.\\&\left.-\frac{1}{2}g_{\mu\nu}\Phi R+\Phi_{;\mu\nu}-g_{\mu\nu}\Phi_{;\alpha\beta}g^{\alpha\beta}\right)\\&+\eta k^\alpha k^\beta\left(g_{\mu\alpha}R_{\nu\beta}+g_{\nu\beta}R_{\mu\alpha}-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}\right.\\&-\frac{1}{2}\eta[g_{\mu\nu}(K^\alpha K^\beta)_{;\alpha\beta}+(K_\mu K_\nu)_{;\alpha\beta}g^{\alpha\beta}\\&-(K_\mu K^\alpha)_{;\nu\alpha}-(K_\nu K^\alpha)_{;\mu\alpha}]\\&+2F_{\mu\alpha}F^\alpha_\nu+\frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}=-8\pi G_0T,\end{aligned} \quad (3)$$

where $\Phi = K_\alpha K^\alpha$.

The contraction of Eq.(3) gives

$$\begin{aligned}&R+(3\omega+\frac{1}{2}\eta)\Phi_{;\alpha\beta}g^{\alpha\beta}\\&+\eta(K^\alpha K^\beta)_{;\alpha\beta}-4\lambda=8\pi G_0T.\end{aligned} \quad (4)$$

*Project supported by the National Basic Research Programme of China (Grant No 2003CB716300).
†Correspondence author, E-mail: wyj@hunnu.edu.cn

Variation of K_ν in Eq.(3) is

$$\omega R K^\mu + \eta R^{\mu\nu} K_\nu + 2F^{\mu\nu}_{;\nu} = 0. \tag{5}$$

Letting $\omega = \eta = 0$, $c = G = 1$, we obtain the metric of the gravitational field,[4] which can be described by metrics of the stationary, spherically symmetric forms

$$\mathrm{d}s^2 = V(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{V(R)} - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2), \tag{6}$$

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}, \tag{7}$$

where Q and q are, respectively, the electric charge and the magnetic charge of the celestial body.

2. Mass defect effect in the gravitational field

First, we discuss the mass defect effect of a test particle in the gravitational field with Eq.(6). The nonzero components of $g_{\mu\nu}$ and $g^{\mu\nu}$ of the gravitational metric are found as follows:

$$\begin{aligned}
g_{00} &= \frac{1}{g^{00}} = 1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}, \\
g_{11} &= \frac{1}{g^{11}} = -\left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{-1}, \\
g_{22} &= \frac{1}{g^{22}} = -r^2, \\
g_{33} &= \frac{1}{g^{33}} = -r^2\sin^2\theta.
\end{aligned} \tag{8}$$

In this field, the equation of motion for the test particle with mass m_0 and electric charge e is

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = F^\mu_\alpha \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{e}{m_0}. \tag{9}$$

Taking $\mu = 0$, and considering Eqs.(8) and (9),we find

$$\begin{aligned}
\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2} &= F^0_\alpha \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{e}{m_0} \\
&= -\frac{e}{m_0}\left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{-1}\frac{\mathrm{d}\Phi}{\mathrm{d}s},
\end{aligned} \tag{10}$$

where Φ is the electrostatic potential. Equation (10) can be rewritten as

$$m_0\left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2} + e\frac{\mathrm{d}\Phi}{\mathrm{d}s} = 0. \tag{11}$$

Integrating Eq.(10) and substituting $x^0 = t$, we have

$$m_0\left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)\frac{\mathrm{d}t}{\mathrm{d}s} + e\Phi = A, \tag{12}$$

where A is an integral constant, which denotes the total energy m_0' of the test particle in the gravity field.

On the other hand, from Eq.(6), we obtain

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}s} =& \left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{-\frac{1}{2}} \\
&\times \left[1 + \left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 \right. \\
&\left. + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 + r^2\sin^2\theta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{13}$$

Letting $\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{\mathrm{d}\varphi}{\mathrm{d}s} = 0$, in Eq.(13), we have

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{-\frac{1}{2}}. \tag{14}$$

Substituting Eq.(14) into Eq.(12), we have

$$m_0' = m_0\left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{\frac{1}{2}} + e\Phi. \tag{15}$$

Thus the mass defect when the test particle is displaced from infinity to the point (t, r, θ, φ) in the field is

$$\begin{aligned}
\Delta m =& m_0' - m_0 \\
=& m_0\left[\left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{\frac{1}{2}} \right. \\
&\left. - 1 + \frac{e}{m_0}\Phi\right].
\end{aligned} \tag{16}$$

The lost mass is radiated out in the form of thermal energy or other types of energies.

Second, we discuss the mass defect of a homogeneous spherical charged shell. In order to simplify the calculation, we make an assumption $\frac{Q}{M} = \frac{e}{m} = \alpha = \text{const}$, and $\frac{q}{M} = \frac{q_0}{m_0} = \beta = \text{const}$, which means that the defects of electric and magnetic charges are, respectively, proportional to the mass defects of both the spherical shell and the celestial body.[5,6−9]

We suppose a and M are, respectively, the radius and mass of the spherical shell, whose thickness is neglected. In the gravitational field induced by the spherical shell, we consider a mass unit $\mathrm{d}M$ on the surface of it, the mass of the mass unit is $\mathrm{d}M_0$ being in infinity. Considering the magnetic charge of the mass unit and Eq.(15), we obtain

$$\begin{aligned}
\mathrm{d}M_0 =& \left[1 - \frac{M\alpha}{a}(\alpha+\beta)\right] \\
&\times \left[1 - \frac{2M}{a} + \frac{(\alpha^2+\beta^2)}{a^2}M^2 - \frac{\lambda a^2}{3}\right]^{-\frac{1}{2}}\mathrm{d}M,
\end{aligned} \tag{17}$$

where $\alpha = \frac{e}{\mathrm{d}M} = \frac{Q}{M}$, $\beta = \frac{q}{M} = \frac{q_0}{\mathrm{d}M}$, and $\Phi = \frac{Q}{a}$. Integrating Eq.(17), we obtain the relationship between M_0 and M

$$M_0 = \int_0^M \left[1 - \frac{M\alpha}{a}(\alpha+\beta)\right] \times \left[1 - \frac{2M}{a} + \frac{(\alpha^2+\beta^2)}{a^2}M^2 - \frac{\lambda a^2}{3}\right]^{-\frac{1}{2}} \mathrm{d}M. \tag{18}$$

The mass defect is

$$\Delta M_0 = M_0 - M = \int_0^M \left[1 - \frac{M\alpha}{a}(\alpha+\beta)\right]\left[1 - \frac{2M}{a} + \frac{(\alpha^2+\beta^2)}{a^2}M^2 - \frac{\lambda a^2}{3}\right]^{-\frac{1}{2}} \mathrm{d}M - M. \tag{19}$$

Finally, we discuss the mass defect of a spherically-symmetrical celestial body with charges. We assume that r, M, ρ_0 are, respectively, the radius, the mass and the density of the celestial body, they are related by $M = \frac{4\pi\rho_0 r^3}{3}$. On the surface of the solid sphere, we suppose that there is a shell of mass unit, whose mass and thickness are $\mathrm{d}M$ and $\mathrm{d}r$ respectively, its mass is $\mathrm{d}M_0$ in infinity

$$\mathrm{d}M_0 = \left[1 - \frac{4\pi\rho_0\alpha(\alpha+\beta)r^2}{3}\right]\left[1 - \frac{8\pi\rho_0 r^2}{3} + \frac{16\pi^2(\alpha^2+\beta^2)\rho_0^2 r^4}{9} - \frac{\lambda r^2}{3}\right]^{-\frac{1}{2}} 4\pi\rho_0 r^2 \mathrm{d}r. \tag{20}$$

Integrating Eq.(20), we obtain the relationship between M_0 and M:

$$M_0 = \int_0^a 4\pi\rho_0\left[1 - \frac{4\pi\rho_0\alpha(\alpha+\beta)r^2}{3}\right]\left[1 - \frac{8\pi\rho_0 r^2}{3} + \frac{16\pi^2(\alpha^2+\beta^2)\rho_0^2 r^4}{9} - \frac{\lambda r^2}{3}\right]^{-\frac{1}{2}} r^2\mathrm{d}r. \tag{21}$$

The mass defect is

$$\Delta M = M_0 - M = \int_0^a 4\pi\rho_0\left[1 - \frac{4\pi\rho_0\alpha(\alpha+\beta)r^2}{3}\right]\left[1 - \frac{8\pi\rho_0 r^2}{3} + \frac{16\pi^2(\alpha^2+\beta^2)\rho_0^2 r^4}{9} - \frac{\lambda r^2}{3}\right]^{-\frac{1}{2}} r^2\mathrm{d}r - M. \tag{22}$$

3. Discussion

3.1. For neutral particle

Taking $e = 0$ in Eqs.(9)–(16), such as in Eq.(16), when the electric charge e vanishes, we obtain the mass defect when the neutral particle moves from infinity to some point in the gravitational field

$$(\Delta m)_{m,k} = m_0\left[\left(1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{\frac{1}{2}} - 1\right]. \tag{23}$$

3.2. For the RN field

Taking $q_0 = q = 0$, $\beta = q_0/m = q/M = 0$ in all of the above equations, the mass defects in three cases are found as follows:

(1) The mass defect of the test particle is

$$\Delta m = m_0\left[\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\lambda r^2}{3}\right)^{\frac{1}{2}} - 1\right] + e\Phi. \tag{24}$$

(2) The mass defect of the homogeneous spherical shell with charge is

$$\Delta M = \int_0^M \left[1 - \frac{M}{a}\alpha^2\right] \times \left[1 - \frac{2M}{a} + \frac{\alpha^2}{a^2}M^2 - \frac{\lambda a^2}{3}\right]^{-\frac{1}{2}} \mathrm{d}M - M. \tag{25}$$

(3) The mass defect of the homogeneous celestial body with charge is

$$\Delta M = \int_0^a 4\pi\rho_0\left[1 - \frac{4\pi\rho_0\alpha^2 r^2}{3}\right]\left[1 - \frac{8\pi\rho_0 r^2}{3} + \frac{16\pi^2\alpha^2\rho_0^2 r^4}{9} - \frac{\lambda r^2}{3}\right]^{-\frac{1}{2}} r^2\mathrm{d}r - M. \tag{26}$$

3.3. For the Schwarzschild field

Taking $e = Q = q = q_0 = \alpha = \beta = 0$ in all of the above equations, the mass defects in three cases are found as follows:

(1) The mass defect of the test particle is

$$\Delta m = m_0\left[\left(1 - \frac{2M}{r} - \frac{\lambda r^2}{3}\right)^{\frac{1}{2}} - 1\right]. \tag{27}$$

(2) The mass defect of the homogeneous spherical shell is

$$\Delta M = \int_0^M \left(1 - \frac{2M}{a} - \frac{\lambda a^2}{3}\right)^{-\frac{1}{2}} \mathrm{d}M - M = a\left(\sqrt{1 - \frac{\lambda a^2}{3}} - \sqrt{1 - \frac{2M}{a} - \frac{\lambda a^2}{3}}\right) - M. \tag{28}$$

(3) The mass defect of the homogeneous celestial body is

$$\begin{aligned}\Delta M =& \int_0^a 4\pi\rho_0\left(1-\frac{8\pi\rho_0 r^2}{3}-\frac{\lambda r^2}{3}\right)^{-\frac{1}{2}} r^2 \mathrm{d}r - M \\ =& 2\pi\rho_0\left(\frac{3}{8\pi\rho_0+\lambda}\right)^{\frac{2}{3}}\left\{\sin^{-1}\left(\frac{8\pi\rho_0+\lambda}{3}a\right)\right. \\ & \left.-\frac{a}{3}\sqrt{(8\pi\rho_0+\lambda)[3-(8\pi\rho_0+\lambda)a^2]}\right\} - M. \end{aligned} \tag{29}$$

3.4. For zero cosmological constant λ

The metric function in Eq.(7) reads $V(r) = 1 - \frac{2M}{r} + \frac{Q^2+q^2}{r^2}$. Hence, in all of the above expressions of the mass defects, we take $\lambda = 0$, so the mass defect of the test particle in the gravitational field reads

$$\Delta m = m_0\left[\left(1-\frac{2M}{r}+\frac{Q^2+q^2}{r^2}\right)^{\frac{1}{2}}-1\right]+e\Phi. \tag{30}$$

In the same way, we can easily obtain the remaining expressions of the mass defects in other fields respectively.

Thus we see that the gravitational field solution (6) contains a variety of formally interesting gravitational fields. We obtain the mass defects in several types of fields and under different conditions by discussing the parameters in the expressions of the mass defect effects in the gravitational field. From this point of view the work of this paper is more interesting.

References

[1] Hellings W and Noedtvedt K 1973 *Phys. Rev.* D **7** 3593

[2] Dicke R H 1965 *Theoretical Significance of Experimental Relativity* (New York: Gordon and Breach) p30

[3] Will C M and Nordtvedt K Jr. 1972 *Astrophys. J.* **177** 757

[4] Fukuda Y *et al* 1998 *Phys. Rev. Lett.* **81** 1158

[5] Romans L J 1992 *Nucl. Phys.* B **383** 397

[6] Wang Y J and Tang Zh M 1990 *Theory and Effects of Gravitation* (Changsha: Hunan Science Technology Press) p315 (in Chinese)

[7] Wang Y J and Tang Zh M 2002 *Astrophys. Space. Sci.* **281** 689

[8] Wang Y J and Tang Zh M 2001 *Chin. Phys.* **10** 679

[9] Wang Y J and Cao D 2004 *Chin. Phys.* **13** 579