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Bound states of Klein–Gordon equation and Dirac equation for ring-shaped non-spherical oscillator scalar and vector potentials

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Solving the Klein–Gordon equation and Dirac equation with ring-shaped non-spherical oscillator gives the exact bound state wavefunction and energy equation, and the relations between non-relativistic Schrödinger equation, Klein–Gordon equation and Dirac equation with equal scalar and vector potentials.

Keywords: ring-shaped non-spherical oscillator, Klein–Gordon equation, Dirac equation, bound state
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1. Introduction

When the relativistic effect is considered, the behaviour of a particle in a potential field must be described by the Klein–Gordon equation or Dirac equation. Except for a few examples, such as the hydrogen atom and free electron, in relativistic quantum mechanics there are few problems that can be exactly solved using the Klein–Gordon or Dirac equation perhaps due to the difficulties in mathematics. In recent years, Dominguez–Adame and Talukdar have obtained the s-wave bound state and scattering state solutions of Klein–Gordon equation for the Hulthén potential,[1,2] and some other investigators have obtained the s-wave bound state solutions of particles in some typical central potential fields.[3−5] Qiang[6] obtained the exact solution of the ring-shaped harmonic oscillator only in relation to the Klein–Gordon equation. In this paper we solve the Klein–Gordon equation as well as the Dirac equation with a ring-shaped non-spherical oscillator for equal scalar and vector potentials, and give a complete wavefunction and energy equation. Especially, by analogue, we find that there exist mathematical similarities between the non-relativistic Schrödinger equation, Klein–Gordon equation and Dirac equation with equal scalar and vector potentials. A ring-shaped non-spherical oscillator potential is given by[7]

$$V(r,\theta) = \frac{1}{2}\mu\omega^2 r^2 + \frac{\hbar^2 A}{2\mu r^2} + \frac{\hbar^2 b}{2\mu r^2 \sin^2\theta}. \quad (1)$$

In this paper, $\hbar = C = \omega = 1$ will be adopted.

2. Klein–Gordon equation

The Klein–Gordon equation with equal scalar end vector potentials is[8]

$$[P^2 + (\mu + V)^2 - (E - V)^2]\Psi = 0, \quad (2)$$

i.e.

$$[P^2 + (\mu^2 - E^2) + 2(\mu + E)V]\Psi = 0, \quad (3)$$

where P is the momentum operator, E and μ are the energy and rest mass of the harmonic oscillator respectively.

In spherical coordinates, let

$$\Psi = \frac{u(r)}{r} H(\theta)\Phi(\varphi). \quad (4)$$

Substituting Eqs.(1) and Eq.(4) into Eq.(2), and then separating variables, we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \left(B + c\mu r^2 + \frac{\frac{cA}{\mu} + \lambda}{r^2}\right)\right] u = 0, \quad (5)$$

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$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\lambda-\frac{\frac{bc}{\mu}+m^2}{\sin^2\theta}\right)H(\theta)=0, \quad (6)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\Phi+m^2\Phi=0, \quad (7)$$

where $B=\mu^2-E^2$, $c=\mu+E$, and m are separated variable constants.

2.1. The solution of angular equation

Obviously, the solution of Eq.(7) satisfying periodic boundary condition is

$$\Phi_m=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\varphi}, \quad (m=0,\pm1,\pm2,\cdots). \quad (8)$$

Letting $\beta=\sqrt{\frac{bc}{\mu}+m^2}$, $\lambda=l(l+1)$, $x=\cos\theta$, Eq.(6) becomes

$$\left\{(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}-2x\frac{\mathrm{d}}{\mathrm{d}x}+\left[l(l+1)-\frac{\beta^2}{1-x^2}\right]\right\}H=0, \quad (9)$$

which is a general associated-Legendre equation. The solution of Eq.(9) satisfying boundary conditions $H|_{x=\pm1}=$ finity is a general associated-Legendre polynomial[9]

$$p_l^\beta(x)=\frac{2^\beta}{\Gamma(1-\beta)}(x^2-1)^{\frac{-\beta}{2}}\times\sum_{k=0}^{\left(\frac{l-\beta}{2}\right)}\frac{(-1)^k\Gamma(2l-2k+1)}{2^l k!(l-\beta-2k)\Gamma(l-k+1)}x^{l-\beta-2k}, \quad (10)$$

where l and β must satisfy the relation

$$l=\beta+s,(s=0,1,2\cdots). \quad (11)$$

So the normalized μ-dimensional wavefunction is

$$H(\theta)=A_{l\beta}P_l^\beta(\cos\theta), \quad (12)$$

where $A_{l\beta}$ is normalization constant

$$A_{l\beta}=\sqrt{\frac{2l+1}{2}\frac{(l-\beta)!}{\Gamma(l+\beta+1)}}.$$

2.2. The solution of radial equation

Letting $u(r)=r^{L+1}\mathrm{e}^{-\frac{\sqrt{c\mu}}{2}r^2}f(r)$, we can convert the radial equation Eq.(5) into

$$f''+\left[\frac{2(L+1)}{r}-2\sqrt{c\mu}r\right]f'-\left[B+\sqrt{c\mu}(2L+3)\right]f=0, \quad (13)$$

where

$$L(L+1)=A+l(l+1). \quad (14)$$

Then we let $y=\sqrt{c\mu}r^2$, and Eq.(5) becomes

$$yf''+\left(L+\frac{3}{2}-y\right)f'-\frac{1}{2}\left(L+\frac{3}{2}+\frac{B}{2\sqrt{c\mu}}\right)f=0, \quad (15)$$

which is a confluent hypergeometric equation and its solution is a confluent hypergeometric function

$$F\left(\frac{L}{2}+\frac{3}{4}+\frac{B}{4\sqrt{c\mu}},L+\frac{3}{2},y\right). \quad (16)$$

For bound states, the confluent hypergeometric function must be terminated with a polynomial, which demands

$$\frac{L}{2}+\frac{3}{4}+\frac{B}{4\sqrt{c\mu}}=-n,n=0,1,2\ldots. \quad (17)$$

This condition, together with Eq.(14) and $B=\mu^2-E^2$, gives the energy equation of a relativistic ring-shaped non-spherical oscillator

$$\frac{\mu^2-E^2}{\sqrt{\mu^2+\mu E}}+4n+2+\sqrt{1+4\left[A+\left(\sqrt{\frac{b(\mu+E)}{\mu}+m^2}+s+1\right)\left(\sqrt{\frac{b(\mu+E)}{\mu}+m^2}+s\right)\right]}=0. \quad (18)$$

The radial function (unnormalized) becomes

$$u(r)=r^{L+1}\mathrm{e}^{-\frac{\sqrt{c\mu}}{2}r^2}\times F\left(\frac{L}{2}+\frac{3}{4}+\frac{B}{4\sqrt{c\mu}},L+\frac{3}{2},\sqrt{c\mu}r^2\right). \quad (19)$$

3. Dirac equation

The Dirac equation with scalar and vector potentials is given by[8]

$$[\hat{\alpha}\cdot\hat{P}+\hat{\beta}(\mu+s)-(E-V)]\Psi=0. \quad (20)$$

Let

$$\Psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}. \quad (21)$$

When $s=V$, Eq.(20) becomes

$$(\mu+2V-E)\varphi+\sigma P\chi=0, \quad (22)$$

$$\sigma P\varphi-(\mu+E)\chi=0. \quad (23)$$

Substituting Eq.(23) into Eq.(22),we obtain

$$[P^2 + (\mu^2 - E^2) + 2(\mu + E)V]\varphi = 0. \qquad (24)$$

Now we can see that Eq.(24) is mathematically similar to Eq.(3), So the solution of Eq.(24) is

$$\varphi = r^{L+1} \mathrm{e}^{-\frac{\sqrt{c\mu}}{2} r^2} F\left(\frac{L}{2} + \frac{3}{4} + \frac{B}{4\sqrt{c\mu}}, L + \frac{3}{2}, \sqrt{c\mu} r^2\right) Y_{lm}(\theta, \varphi), \qquad (25)$$

and x can be easily obtained by inserting the above expression into Eq.(23)

$$\chi = \frac{1}{(\mu + E)} \sigma P \varphi.$$

The energy equation is

$$\frac{\mu^2 - E^2}{\sqrt{\mu^2 + \mu E}} + 4n + 2\sqrt{+ \quad 1 + 4\left[A + \left(\sqrt{\frac{b(\mu + E)}{\mu} + m^2} + s + 1\right)\left(\sqrt{\frac{b(\mu + E)}{\mu} + m^2} + s\right)\right]} = 0. \qquad (26)$$

4. Discussion

After solving the Klein–Gordon and Dirac equations for scalar plus vector potentials, we should make some remarks.

(a) When solving the Dirac equation with equal scalar and vector potentials, we find that, by making some transformation, the component differential equation of the Dirac equation is mathematically similar to the Klein–Gordon equation. So we do not have to use the usual approaches[3−5] to solve the Dirac equation. That is to say, what we should do is just to solve the Klein–Gordon equation.

(b) Analysing Eq.(3), we find that the Klein–Gordon equation with equal scalar and vector potentials is mathematically similar to the non-relativistic Schrödinger equation

$$[P^2 + (\mu^2 - E^2) + 2(\mu + E)V]\Psi = 0, \qquad (27)$$

$$[P^2 - 2\mu E + 2\mu V]\Psi = 0. \qquad (28)$$

From the above equations we can see that, analogously,

$$-2\mu E \leftrightarrow \mu^2 - E^2$$

$$\mu \leftrightarrow (\mu + E),$$

Equations (27) and (28) are mathematically equivalent to each other.

It can be seen that, if the solution of non-relativistic Schrödinger equation with a certain vector potential can be obtained, the solutions of Klein–Gordon and Dirac equations with equal scalar and vector potentials can be easily obtained just by using the corresponding relations given above. In addition, traditional approaches are just able to obtain some s-wave solutions,[3−5] while the above corresponding relations can be widely used to solve the problems whose solutions of Schrödinger equations have been obtained.

5. Conclusion

From the above analyses, it can be seen that Klein–Gordon equation and Dirac equation with ring-shaped non-spherical oscillator with equal scalar and vector potentials can be exactly solved. Especially, by analogue, we find that there exist mathematical similarities between the non-relativistic Schrödinger equation, Klein–Gordon equation and Dirac equation with equal scalar and vector potentials.

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