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The adiabatic approximation solutions of cylindrical and spherical dust ion-acoustic solitary waves*

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By using the equivalent particle theory, the adiabatic approximation solutions of the Korteweg–de Vries type equation (including KdV equation, cylindrical KdV equation and spherical KdV equation) in dust ion-acoustic solitary waves were obtained. The method can be extended to other nonlinear evolution equations.

Keywords: KdV type equation, equivalent particle theory, adiabatic approximation solutions
PACC: 0340K, 0290, 5235S

1. Introduction

The study of nonlinear wave in dusty plasma in nonplanar (cylindrical or spherical) geometry has attracted a great deal of interest in recent years.[1–7] Recently Mamun and Shukla derived the cylindrical and spherical Korteweg–de Vries (KdV) equations for dust acoustic (DA) waves and dust ion acoustic (DIA) waves, and obtained the numerical solution of the cylindrical KdV equation and spherical KdV equation.[8,9] To derive the cylindrical and spherical KdV equations they considered a nonplanar cylindrical or spherical geometry and studied the nonlinear propagation of dust ion acoustic solitary (DIAS) waves in an unmagnetized dusty plasma whose constituents are inertial ions, Boltzmann electrons and stationary dust particles. The nonlinear dynamics of the DIAS waves whose phase speed is much smaller (larger) than the electron (ion) thermal speed, in nonplanar cylindrical and spherical geometries, is governed by

$$\frac{\partial n_{\rm i}}{\partial \tau}+\frac{1}{r^{\delta}}\frac{\partial}{\partial r}(r^{\delta}n_{\rm i}u_{\rm i})=0, \tag{1}$$

$$\frac{\partial u_{\rm i}}{\partial \tau}+u_{\rm i}\frac{\partial u_{\rm i}}{\partial r}=-\frac{\partial \phi}{\partial r}, \tag{2}$$

$$\frac{1}{r^{\delta}}\frac{\partial}{\partial r}\left(r^{\delta}\frac{\partial \phi}{\partial r}\right)=\mu\exp(\phi)-n_{\rm i}+(1-\mu), \tag{3}$$

where $\delta=0$ for one-dimensional geometry, and $\delta=1,2$ for cylindrical and spherical geometry, respectively; $n_{\rm i}$ is the ion number density normalized by its equilibrium value $n_{\rm i0}$; $u_{\rm i}$ is the ion fluid speed normalized by $C_{\rm i}=(k_{\rm B}T_{\rm e}/m_{\rm i})^{1/2}$, and ϕ is the electrostatic potential normalized by $k_{\rm B}T_{\rm e}/e$. The time and space variables are in units of the ion plasma period $\omega_{pi}^{-1}=(m_{\rm i}/4\pi n_{\rm i0}e^2)^{1/2}$ and the Debye radius $\lambda_{Dm}=(k_{\rm B}T_{\rm e}/4\pi n_{\rm i0}e^2)^{1/2}$, respectively. We have denoted $\mu=n_{\rm e0}/n_{\rm i0}$. The KdV type equation is derived by using the reductive perturbation method (the stretched coordinates $x=-\epsilon^{1/2}(r+v_0\tau), t=\epsilon^{3/2}\tau$, where ϵ is a small parameter),[9] as follows:

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\frac{\delta}{2t}u=0. \tag{4}$$

Here $u=\phi^{(1)}, \alpha=\sqrt{\mu}(3-1/\mu)/2, \beta=1/2\mu^{3/2}$. It is obvious that the term $(\delta/2t)u$ in Eq.(4) is due to the effect of the cylindrical or spherical geometry.

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Equation (4) contains the following important cases:

(1) If $\delta = 0$, Eq.(4) becomes the well known KdV equation

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0. \tag{5}$$

(2) If $\delta = 1$, Eq.(4) becomes the cylindrical KdV equation

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \frac{1}{2t} u = 0. \tag{6}$$

(3) If $\delta = 2$, Eq.(4) becomes the spherical KdV equation[10]

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \frac{1}{t} u = 0. \tag{7}$$

Equation (6) is completely integrable; Sahu and Roychoudhury[11] have obtained its exact solution by group analysis method. Wang *et al*[12] have also derived the exact solution of Eq.(6) by using homogeneous balance method. Maxon and Viecelli[10] have obtained the numerical solution of the spherical KdV equation.

As is well known, the standard KdV equation has a solitary wave solution

$$u = 2k^2 \mathrm{sech}^2(kx - 4k^3 t - x_0), \tag{8}$$

where k and x_0 are constants. Zabusky and Kruskal[13] discovered that these solitary wave solutions have the remarkable property that the interaction of two solitary wave solutions is elastic, and the solutions are called solitons. Furthermore, it is found that KdV equation had an infinite number of conservation laws. Therefore, the soliton can be regarded as an equivalent particle which has "mass", "momentum", and "energy".[14] Kosevich[15] studied the particle and wave properties of solitons, resonant and non-resonant soliton scatterings by impurities. Aceves *et al*[16] have studied the reflection and transmission characteristics of a self-focused channel by using equivalent particle theory. The equivalent particle theory of soliton also was applied to KdV equation and NLS equation by Chen and Liang[17] and Robert Knapp.[18]

In this article, the adiabatic approximation solutions of Eq.(4) is derived by using the equivalent particle theory and the perturbation method. In Section 2, we introduce the equivalent particle theory of the KdV equation. In Section 3, we apply the method to Eq.(4) and obtain its adiabatic approximation solutions. Discussion will be presented in the last section.

2. The equivalent particle theory of the KdV equation

The idea of the equivalent particle method[17,18] is to define quantities for a solution of the perturbed KdV equation which correspond to the position, mass, momentum and energy of an equivalent particle. Our model is based on the one-dimensional KdV equation (perturbed KdV equation)

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = Vu, \tag{9}$$

where V is the perturbed term of the KdV equation. Considering a function $\varphi(x,t)$ which is a solution of the KdV equation (9) within $(-\infty, +\infty)$, we define the quantities of the equivalent particle as follows. The mass M is given by

$$M = \int_{-\infty}^{\infty} \varphi \mathrm{d}x. \tag{10}$$

The position $\bar{x}$ is given by the centre of mass

$$\bar{x}(t) = \frac{1}{M} \int_{-\infty}^{\infty} x\varphi \mathrm{d}x. \tag{11}$$

The velocity v is defined by

$$\begin{aligned} v(t) =& \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} \\ =& -\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}t}\bar{x} + \frac{1}{M}\int_{-\infty}^{\infty} xV\varphi \mathrm{d}x \\ & - \frac{3}{M}\int_{-\infty}^{\infty} \varphi^2 \mathrm{d}x. \end{aligned} \tag{12}$$

The momentum P is given by

$$P = -3\int_{-\infty}^{\infty} \varphi^2 \mathrm{d}x - \bar{x}\frac{\mathrm{d}M}{\mathrm{d}t} + \int_{-\infty}^{\infty} xV\varphi \mathrm{d}x. \tag{13}$$

This definition is natural, since momentum P of a body equals its mass M times its motion velocity: $P = Mv$.

When V is zero, M and P are conserved quantities. In fact, they are just conserved quantities of the unperturbed KdV equation.

3. The adiabatic approximation solutions of Eq.(4)

Now we use the equivalent particle method to obtain approximate solutions of Eq.(4).

Let

$$z = \beta^{-\frac{1}{3}} x, \quad \varphi = -\frac{\alpha}{6}\beta^{-\frac{1}{3}} u, \tag{14}$$

then Eq.(4) is reduced to

$$\frac{\partial \varphi}{\partial t} - 6\varphi \frac{\partial \varphi}{\partial z} + \frac{\partial^3 \varphi}{\partial z^3} = -\frac{\delta}{2t}\varphi = V(t)\varphi, \tag{15}$$

where $V(t) = -\delta/2t$, which is regarded as a dissipation term. When Eq.(15) satisfies the adiabatic approximation condition, i.e. when $|z| \to 0$, $\varphi \to 0, \partial\varphi/\partial z \to 0$ and $\partial^2\varphi/\partial z^2 \to 0$, then Eqs.(12) and (13) are reduced to[15]

$$v(t) = \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = -\frac{3}{M}\int_{-\infty}^{\infty}\varphi^2\mathrm{d}x \tag{16}$$

and

$$P = -3\int_{-\infty}^{\infty}\varphi^2\mathrm{d}x; \tag{17}$$

and we suppose that the solution of Eq.(15) is of the following form:

$$\varphi(z,t) = -2K^2(t)\mathrm{sech}^2[K(t)(z-\xi(t)t)], \tag{18}$$

where both $K(t)$ and $\xi(t)$ are functions of variable t only. Substituting Eq.(18) into Eqs.(10), (11), (16) and (17), we can determine the mass, position and momentum of the soliton

$$\begin{aligned} &M(t) = -4K(t),\\ &\bar{z}(t) = \xi(t),\\ &P(t) = -16K^3(t). \end{aligned} \tag{19}$$

From Eq.(17), we can obtain

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 2V(t)P. \tag{20}$$

Integrating Eq.(20) once with respect to t and taking the integration constants as P_0, we obtain

$$P = P_0\exp\left[2\int_{t_0}^{t}V(t)\mathrm{d}t\right]. \tag{21}$$

Then, from Eqs.(19) and (21), we obtain $K(t)$ as follows:

$$K(t) = K_0\left(\frac{t_0}{t}\right)^{\frac{\delta}{3}}, \tag{22}$$

where $K_0 = K(t)\mid_{t=t_0}$. Furthermore, from Eq.(19) and $P = Mv$ we can derive

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = v = 4K^2(t) = 4K_0^2\left(\frac{t_0}{t}\right)^{\frac{2\delta}{3}}, \tag{23}$$

thus we have

$$\xi(t) = \frac{12K_0^2t_0^{\frac{2\delta}{3}}}{3-2\delta}t^{1-\frac{2\delta}{3}} + \xi_0, \tag{24}$$

where $\xi_0 = 12K_0^2t_0/(3-2\delta)$. Substituting Eqs.(22) and (24) into Eq.(18), we can obtain the solution of Eq.(15) as follows:

$$\varphi(z,t) = -2K_0^2\left(\frac{t_0}{t}\right)^{\frac{2\delta}{3}}\mathrm{sech}^2\left[K_0\left(\frac{t_0}{t}\right)^{\frac{\delta}{3}}\left(z - \frac{12K_0^2t_0^{\frac{2\delta}{3}}}{3-2\delta}t^{1-\frac{2\delta}{3}} + \xi_0\right)\right]. \tag{25}$$

Substituting Eq.(14) into Eq.(25), we can obtain the solution of Eq.(4):

$$u(x,t) = \frac{12\beta^{\frac{1}{3}}K_0^2}{\alpha}\left(\frac{t_0}{t}\right)^{\frac{2\delta}{3}}\mathrm{sech}^2\left[K_0\left(\frac{t_0}{t}\right)^{\frac{\delta}{3}}\beta^{\frac{1}{3}}\left(x - \frac{12K_0^2\beta^{-\frac{1}{3}}t_0^{\frac{2\delta}{3}}}{3-2\delta}t^{1-\frac{2\delta}{3}} + \xi_0\right)\right]. \tag{26}$$

It is seen from Eq.(26) that the amplitude of a single solitary wave is $A = \frac{12\beta^{\frac{1}{3}}K_0^2}{\alpha}\left(\frac{t_0}{t}\right)^{\frac{2\delta}{3}}$, which depends upon the time variable t when $\delta \neq 0$. It seems that the amplitude decreases as time t increases. Meanwhile, the width of solitary wave $\left(L = 1/K_0\left(\frac{t_0}{t}\right)^{\frac{\delta}{3}}\beta^{\frac{1}{3}}\right)$ increases (AL^2=const) as t increases, and the solitary wave will be attenuated.

4. Discussion

Now we discuss several special situations:

case a: If $\delta = 0$, Eq.(26) reduces to

$$\begin{aligned} u(x,t) =& \frac{12\beta^{\frac{1}{3}}K_0^2}{\alpha}\\ &\times\mathrm{sech}^2\left[K_0\beta^{\frac{1}{3}}\left(x - 4K_0^2\beta^{-\frac{1}{3}}t + \xi_0'\right)\right], \end{aligned} \tag{27}$$

where $\xi_0' = 4K_0^2t_0$. It is obvious that Eq.(27) is the well known solitary wave solution of the KdV equation.

case b: If $\delta = 1$, Eq.(26) reduces to

$$\begin{aligned} u(x,t) =& \frac{12\beta^{\frac{1}{3}}K_0^2}{\alpha}\left(\frac{t_0}{t}\right)^{\frac{2}{3}}\\ &\times\mathrm{sech}^2\left[K_0\left(\frac{t_0}{t}\right)^{\frac{1}{3}}\beta^{\frac{1}{3}}\right.\\ &\left.\times\left(x - 12K_0^2\beta^{-\frac{1}{3}}t_0^{\frac{2}{3}}t^{\frac{1}{3}} + \xi_0'\right)\right], \end{aligned} \tag{28}$$

where $\xi_0' = 12K_0^2t_0$. This is the solitary wave solution of the cylindrical KdV equation. It is shown that its amplitude will increase and its width will decrease with decreasing value of t, and the results are depicted in Fig.1.

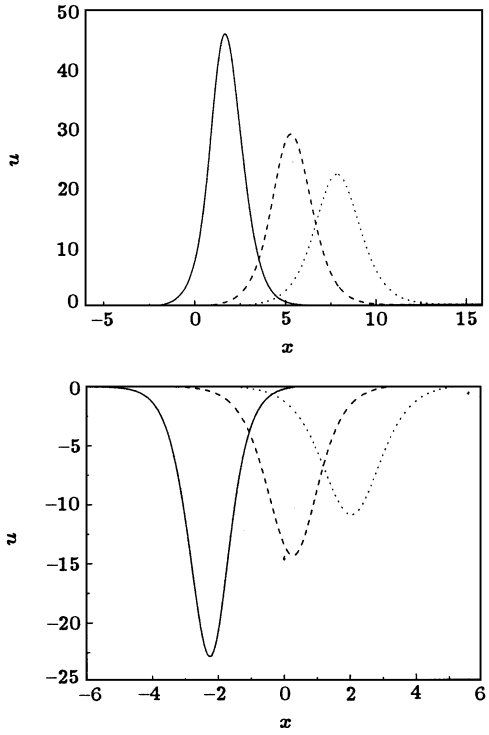


Fig.1. Time evolution of cylindrical KdV solitary waves: u versus spatial coordinate x at times $t = 3$ (solid), $t = 6$ (dashed), and $t = 9$ (dotted) for $\mu = 0.4$, $K_0 = 1$, $t_0 = 1$ (upper plot) and $\mu = 0.2$, $K_0 = 1$, $t_0 = 1$ (lower plot).

case c: If $\delta = 2$, Eq.(26) reduces to

$$u(x,t) = \frac{12\beta^{\frac{1}{3}}K_0^2}{\alpha}\left(\frac{t_0}{t}\right)^{\frac{4}{3}} \times \mathrm{sech}^2\left[K_0\left(\frac{t_0}{t}\right)^{\frac{2}{3}}\beta^{\frac{1}{3}} \times \left(x - 12K_0^2\beta^{-\frac{1}{3}}t_0^{\frac{4}{3}}t^{-\frac{1}{3}} + \xi_0'\right)\right], \quad (29)$$

where $\xi_0' = -12K_0^2t_0$. This is the solitary wave solution of the spherical KdV equation. The time evolution of the spherical KdV equation is shown in Fig.2.

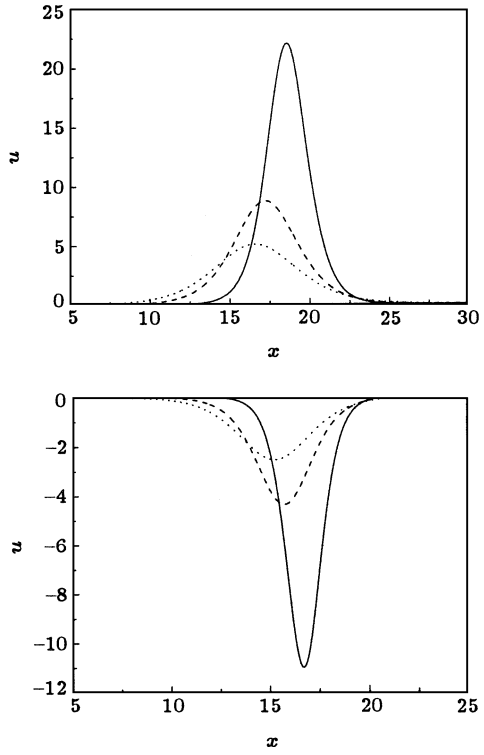


Fig.2. Time evolution of spherical KdV solitary waves: u versus spatial coordinate x at times $t = 3$ (solid), $t = 6$ (dashed), and $t = 9$ (dotted) for $\mu = 0.4$, $K_0 = 1$, $t_0 = 1$ (upper plot) and $\mu = 0.2$, $K_0 = 1$, $t_0 = 1$ (lower plot).

It is obvious from Figs.1 and 2 that for $\mu = 0.4 > 1/3$, a dusty plasma supports compressive DIAS waves which are associated with a positive potential. For $\mu = 0.2 < 1/3$, a dusty plasma supports rarefactive DIAS waves which are associated with a negative potential.

References

[1] Hershkowitz N and Romesser T 1974 *Phys. Rev. Lett.* **32** 581

[2] Hirota R 1979 *Phys. Lett. A* **71** 393

[3] Johnson R S 1979 *Phys. Lett.* A **72** 197

[4] Okutsu E and Schott L 1980 *Plasma Phys.* **22** 477

[5] Zhang J F and Chen F Y 2001 *Acta Phys. Sin.* **50** 1648 (in Chinese)

[6] Liu S K *et al* 2002 *Acta Phys. Sin.* **51** 1923 (in Chinese)

[7] Yan Z Y and Zhang H Q 1999 *Chin. Phys.* **8** 889

[8] Mamuna A A and Shukla P K 2001 *Phys. Lett.* A **290** 173

[9] Mamuna A A and Shukla P K 2002 *Phys. Plasmas* **9** 1468

[10] Maxon S and Viecelli J 1974 *Phys. Rev. Lett.* **32** 4

[11] Sahu B and Roychoudhury R 2003 *Phys. Plasmas* **10** 4162

[12] Wang M L, Wang Y M and Zhou Y B 2002 *Phys. Lett.* A **303** 45

[13] Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240

[14] Kaup D J and Newell A C 1978 *Proc. R. Soc. London, Ser.* A **361** 413

[15] Kosevich A M 1990 *Physica* D **41** 253

[16] Aceves A B, Molney J V and Newell A C 1989 *Phys. Rev.* A **39** 1809

[17] Chen L Y and Liang C H 1997 *Theory and Applications of Soliton—Theory of Optical Solitons and Communications of Optical Solitons* (Xi'an: Xi'an Electronics Technology University Press) p65 (in Chinese)

[18] Knapp R 1995 *Physica* D **85** 496