Vol 14 No 1, January 2005
1009-1963/2005/14(01)/0017-04

Chinese Physics

© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

An extended Jacobian elliptic function method for the discrete mKdV lattice*

Zhu Jia-Min(朱加民)† and Ma Zheng-Yi(马正义)

Department of Physics, Lishui University, Lishui 323000, *China*

(Received 2 June 2004; revised manuscript received 16 July 2004)

In this paper, the discrete mKdV lattice is solved by using the modified Jacobian elliptic function expansion method. As a consequence, abundant families of Jacobian elliptic function solutions are obtained. When the modulus $m \to 1$, these periodic solutions degenerate to the corresponding solitary wave solutions, including bell-type and kink-type excitations.

Keywords: discrete mKdV lattice, Jacobian elliptic function expansion method, bell-type solitary, kink-type solitary

PACC: 0340, 0365, 0545

1. Introduction

Seeking exact solutions of nonlinear evolution equations is of important significance in mathematical physics. In the past decades, several powerful methods have been developed to construct many types of exact solutions of nonlinear partial differential equations (PDEs), such as inverse scattering theory,[1], Bäcklund transformation,[2] the tanh function method,[3−5] homogeneous balance method,[6−8] multilinear variable separation approach[9−11] and Jacobian elliptic function method,[12−15] etc. In recent years, the direct search for exact solutions of PDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allows us to perform the complicated and tedious algebraic calculations on computer. In particular, one of the most effective direct methods to construct exact solutions of PDEs is the Jacobian elliptic function method.

Since the work of Fermi, Pasta and Ulam in the 1950s,[16] the investigation of exact solutions of the nonlinear differential-different equations (DDEs) has played a crucial role in the modelling of many phenomena in different fields, which include condensed matter physics, biophysics and mechanical engineering. One also encounters such systems in numerical simulation of soliton dynamics in high energy physics where they arise as approximations of continuum models. Unlike difference equations which are fully discretized, DDEs are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous.

However, different from the considerable works done on finding exact solutions to PDEs just mentioned above, to our knowledge, only a few works have investigated exact solutions for DDEs. Recently, Dai *et al*[17] extended this approach to the Jacobian elliptic function expansion method. From this, one can obtain some types of excitations in many spatially discrete nonlinear models. In this paper, we further extend the Jacobian elliptic function method presented by Dai and derive some new Jacobian doubly periodic solutions for the discrete mKdV lattice. When the modulus $m \to 1$, we can obtain the kink-type solitary solution and the bell-type excitation.

2. The extended Jacobian elliptic function method for nonlinear DDEs

First, we recall the properties of Jacobian elliptic

*Project supported by the National Natural Science Foundation of China (Grant No 10172056) and the Youth Foundation of Lishui University (Grant No QN04008).

†E-mail: zjm64@163.com

function,

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \quad \mathrm{dn}^2\xi = 1 - m^2\mathrm{sn}^2\xi,$$

$$(\mathrm{sn}\xi)' = \mathrm{cn}\xi\mathrm{dn}\xi, \ (\mathrm{cn}\xi)' = -\mathrm{sn}\xi\mathrm{dn}\xi,$$
$$(\mathrm{dn}\xi)' = -m^2\mathrm{sn}\xi\mathrm{cn}\xi. \tag{1}$$

$$\mathrm{sn}(\xi_1+\xi_2) = \frac{\mathrm{sn}\xi_1\mathrm{cn}\xi_2\mathrm{dn}\xi_2 + \mathrm{sn}\xi_2\mathrm{cn}\xi_1\mathrm{dn}\xi_1}{1 - m^2\mathrm{sn}^2\xi_1\mathrm{sn}^2\xi_2},$$

$$\mathrm{cn}(\xi_1+\xi_2) = \frac{\mathrm{cn}\xi_1\mathrm{cn}\xi_2 - \mathrm{sn}\xi_1\mathrm{sn}\xi_2\mathrm{dn}\xi_1\mathrm{dn}\xi_2}{1 - m^2\mathrm{sn}^2\xi_1\mathrm{sn}^2\xi_2},$$

$$\mathrm{dn}(\xi_1+\xi_2) = \frac{\mathrm{dn}\xi_1\mathrm{dn}\xi_2 - m^2\mathrm{sn}\xi_1\mathrm{sn}\xi_2\mathrm{cn}\xi_1\mathrm{dn}\xi_1}{1 - m^2\mathrm{sn}^2\xi_1\mathrm{sn}^2\xi_2}, \tag{2}$$

where m is a modulus of Jacobian elliptic function. When $m \to 1$, the Jacobian elliptic functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}\xi \to \tanh\xi, \ \mathrm{cn}\xi \to \mathrm{sech}\xi, \ \mathrm{dn}\xi \to \mathrm{sech}\xi. \tag{3}$$

When $m \to 0$, the Jacobian elliptic functions degenerate to the triangular functions, i.e.

$$\mathrm{sn}\xi \to \sin\xi, \ \mathrm{cn}\xi \to \cos\xi, \ \mathrm{dn}\xi \to 1. \tag{4}$$

Given a system of M polynomial DDEs,

$$\Delta(u_{n+p_1}(x), \cdots, u_{n+p_k}(x), u'_{n+p_1}(x), \cdots, u'_{n+p_k}(x), \cdots, u^{(r)}_{n+p_1}(x), \cdots, u^{(r)}_{n+p_k}(x)) = 0, \tag{5}$$

where the dependent variable u has M components u_i, the continuous variable x has N components x_i, the discrete variable n has Q components n_j, the k shift vectors p_i, and $u^{(r)}(x)$ denotes the collection of mixed derivative terms of order r.

The main steps of the modified Jacobian elliptic function method are outlined as follows:

Step 1 When we seek the travelling wave solutions of Eq.(5), the first step is to introduce the wave transformation

$$\xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} k_j x_j + \delta, \tag{6}$$

where the coefficients d_i, k_j and the phase δ are constants to be determined. In this way, Eq.(5) becomes

$$\Delta(u_{n+p_1}(\xi_n), \cdots, u_{n+p_k}(\xi_n), u'_{n+p_1}(\xi), \cdots, u'_{n+p_k}(\xi), u^{(r)}_{n+p_1}(x), \cdots, u^{(r)}_{n+p_k}(x)) = 0. \tag{7}$$

Step 2 We propose the following series expansion which is extended from the corresponding formula in Ref.(18) as a solution of Eq.(7):

$$u_n \equiv u_n(\xi_n) = \sum_{j=0}^{l} a_j \mathrm{sn}^j\xi_n + \sum_{j=1}^{l} b_j \mathrm{cn}^j\xi_n + \sum_{j=1}^{l} c_j \mathrm{dn}^j\xi_n, \tag{8}$$

where $a_j, b_j, c_j (j = 0, \cdots, l)$ are constants to be determined and l is given according to the homogeneous balance principle. From identity (2), we have

$$u_{n+p_s}(\xi_n) = \sum_{j=0}^{l} a_j \left(\frac{\mathrm{sn}\xi_n\mathrm{cn}(\varphi_s)\mathrm{dn}(\varphi_s) + \mathrm{sn}(\varphi_s)\mathrm{cn}\xi_n\mathrm{dn}\xi_n}{1 - m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right)^j + \sum_{j=1}^{l} b_j \left(\frac{\mathrm{cn}\xi_n\mathrm{cn}(\varphi_s) - \mathrm{sn}\xi_n\mathrm{sn}(\varphi_s)\mathrm{dn}\xi_n\mathrm{dn}(\varphi_s)}{1 - m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right)^j + \sum_{j=1}^{l} c_j \left(\frac{\mathrm{dn}\xi_n\mathrm{dn}(\varphi_s) - m^2\mathrm{sn}\xi_n\mathrm{sn}(\varphi_s)\mathrm{cn}\xi_n\mathrm{dn}\xi_n}{1 - m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right)^j, \tag{9}$$

with φ_s satisfying

$$\varphi_s = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q. \tag{10}$$

Step 3 Substituting ansatzes (8) and (9) into Eq.(7), clearing the denominator and setting the coefficients of power terms in $\mathrm{sn}^i\xi_n$ and $\mathrm{cn}\xi_n\mathrm{dn}\xi_n\mathrm{sn}^i\xi_n (i = 0, \cdots, l)$ to zero, we obtain a system of nonlinear algebraic equations with $a_j, b_j, c_j (j = 0, \cdots, l)$ and k. From these equations, we can obtain the corresponding undetermined coefficients.

Finally, substituting the obtained constants a_j, b_j, c_j $(j = 0, \cdots, l)$ and k into ansatz (8), a series of explicit exact solutions can be obtained, including periodic wave solutions and solitary wave solutions.

3. Solutions of discrete mKdV lattice

The discrete mKdV lattice reads[18]

$$\dot{u}_n(t) = (\alpha - u_n^2)(u_{n+1} - u_{n-1}), \tag{11}$$

where α is a constant.

Let $u_n = v_n(\xi_n), \xi_n = d_1 n + kt + \delta$, then Eq.(11) becomes

$$kv'_n = (\alpha - v_n^2)(v_{n+1} - v_{n-1}). \tag{12}$$

We expand the solution of Eq.(12) in the form of Eqs.(8) and (9). In this case, $p_s = 1, -1$. According

to the homogeneous balance principle, we can choose the solution of Eq.(12) in the form

$$\begin{aligned}
v_n =& a_0 + a_1\mathrm{sn}\xi_n + b_1\mathrm{cn}\xi_n + c_1\mathrm{dn}\xi_n,\\
v_{n+1} =& a_0 + a_1\left(\frac{\mathrm{sn}\xi_n\mathrm{cn}(\varphi_s)\mathrm{dn}(\varphi_s)+\mathrm{sn}(\varphi_s)\mathrm{cn}\xi_n\mathrm{dn}\xi_n}{1-m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right)\\
&+ b_1\left(\frac{\mathrm{cn}\xi_n\mathrm{cn}(\varphi_s)-\mathrm{sn}\xi_n\mathrm{sn}(\varphi_s)\mathrm{dn}\xi_n\mathrm{dn}(\varphi_s)}{1-m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right)+c_1\left(\frac{\mathrm{dn}\xi_n\mathrm{dn}(\varphi_s)-m^2\mathrm{sn}\xi_n\mathrm{sn}(\varphi_s)\mathrm{cn}\xi_n\mathrm{dn}\xi_n}{1-m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right),\\
v_{n-1} =& a_0 + a_1\left(\frac{\mathrm{sn}\xi_n\mathrm{cn}(\varphi_s)\mathrm{dn}(\varphi_s)-\mathrm{sn}(\varphi_s)\mathrm{cn}\xi_n\mathrm{dn}\xi_n}{1-m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right)\\
&+ b_1\left(\frac{\mathrm{cn}\xi_n\mathrm{cn}(\varphi_s)+\mathrm{sn}\xi_n\mathrm{sn}(\varphi_s)\mathrm{dn}\xi_n\mathrm{dn}(\varphi_s)}{1-m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right)+c_1\left(\frac{\mathrm{dn}\xi_n\mathrm{dn}(\varphi_s)+m^2\mathrm{sn}\xi_n\mathrm{sn}(\varphi_s)\mathrm{cn}\xi_n\mathrm{dn}\xi_n}{1-m^2\mathrm{sn}^2\xi_n\mathrm{sn}^2(\varphi_s)}\right). \quad (13)
\end{aligned}$$

Substituting Eq.(13) into Eq.(12), clearing the denominator and setting the coefficients of power terms like $\mathrm{cn}\xi_n\mathrm{dn}\xi_n\mathrm{sn}\xi_n^i (i=0,1,2)$ to zero yield a system of algebraic equations as follows:

$$\begin{aligned}
&4\mathrm{sn}(d_1)a_1b_1c_1m^2\mathrm{cn}(d_1)+4\mathrm{sn}(d_1)a_1c_1b_1\mathrm{dn}(d_1)m^2+4\mathrm{sn}(d_1)b_1c_1a_1m^2=0,\\
&2\mathrm{sn}(d_1)c_1^3m^4\mathrm{cn}(d_1)+kc_1m^4\mathrm{sn}(d_1)^2+4\mathrm{sn}(d_1)b_1^2c_1\mathrm{dn}(d_1)m^2\\
&+2\mathrm{sn}(d_1)b_1^2c_1m^2\mathrm{cn}(d_1)-4\mathrm{sn}(d_1)a_1^2c_1m^2-2\mathrm{sn}(d_1)a_1^2c_1m^2\mathrm{cn}(d_1)=0,\\
&4\mathrm{sn}(d_1)b_1c_1^2m^2\mathrm{cn}(d_1)-2\mathrm{sn}(d_1)a_1^2b_1\mathrm{dn}(d_1)-4\mathrm{sn}(d_1)a_1^2b_1+kb_1m^2\mathrm{sn}(d_1)^2\\
&+2\mathrm{sn}(d_1)b_1^3\mathrm{dn}(d_1)+2\mathrm{sn}(d_1)c_1^2b_1\mathrm{dn}(d_1)m^2=0,\\
&4\mathrm{sn}(d_1)a_0b_1c_1m^2\mathrm{cn}(d_1)+4\mathrm{sn}(d_1)a_0c_1b_1\mathrm{dn}(d_1)m^2=0,\\
&2\mathrm{sn}(d_1)a_1^3-2\mathrm{sn}(d_1)c_1^2a_1m^2-ka_1m^2\mathrm{sn}(d_1)^2-4\mathrm{sn}(d_1)a_1b_1^2\mathrm{dn}(d_1)\\
&-2\mathrm{sn}(d_1)b_1^2a_1-4\mathrm{sn}(d_1)a_1c_1^2m^2\mathrm{cn}(d_1)=0,\\
&-4\mathrm{sn}(d_1)a_0a_1c_1m^2\mathrm{cn}(d_1)-4\mathrm{sn}(d_1)a_0c_1a_1m^2=0,\\
&-4\mathrm{sn}(d_1)a_0a_1b_1\mathrm{dn}(d_1)-4\mathrm{sn}(d_1)a_0b_1a_1=0,\\
&-4\mathrm{sn}(d_1)b_1c_1a_1-4\mathrm{sn}(d_1)a_1c_1b_1\mathrm{dn}(d_1)-4\mathrm{sn}(d_1)a_1b_1c_1m^2\mathrm{cn}(d_1)-4\mathrm{sn}(d_1)b_1c_1a_1m^2=0,\\
&-4\mathrm{sn}(d_1)a_0b_1^2\mathrm{dn}(d_1)+4\mathrm{sn}(d_1)a_0a_1^2-4\mathrm{sn}(d_1)a_0c_1^2m^2\mathrm{cn}(d_1)=0,\\
&2\mathrm{sn}(d_1)\alpha c_1m^2\mathrm{cn}(d_1)-2\mathrm{sn}(d_1)b_1^2c_1m^2\mathrm{cn}(d_1)-kc_1m^2-2\mathrm{sn}(d_1)a_0^2c_1m^2\mathrm{cn}(d_1)\\
&+4\mathrm{sn}(d_1)a_1^2c_1-4\mathrm{sn}(d_1)b_1^2c_1\mathrm{dn}(d_1)-2\mathrm{sn}(d_1)c_1^3m^2\mathrm{cn}(d_1)=0,\\
&-4\mathrm{sn}(d_1)b_1c_1^2m^2\mathrm{cn}(d_1)+2\mathrm{sn}(d_1)\alpha b_1\mathrm{dn}(d_1)-2\mathrm{sn}(d_1)b_1^3\mathrm{dn}(d_1)-kb_1\\
&+4\mathrm{sn}(d_1)a_1^2b_1-2\mathrm{sn}(d_1)c_1^2b_1\mathrm{dn}(d_1)-2\mathrm{sn}(d_1)a_0^2b_1\mathrm{dn}(d_1)=0,\\
&-4\mathrm{sn}(d_1)a_0c_1b_1\mathrm{dn}(d_1)-4\mathrm{sn}(d_1)a_0b_1c_1m^2\mathrm{cn}(d_1)=0,\\
&ka_1+2\mathrm{sn}(d_1)a_0^2a_1+2\mathrm{sn}(d_1)c_1^2a_1+2\mathrm{sn}(d_1)b_1^2a_1-2\mathrm{sn}(d_1)\alpha a_1=0,\\
&\mathrm{sn}(d_1)a_0c_1a_1=\mathrm{sn}(d_1)b_1c_1a_1=\mathrm{sn}(d_1)a_0b_1a_1=0. \quad (14)
\end{aligned}$$

From the above, we obtain three types of solutions

$$\begin{aligned}
&a_0=0, a_1=\pm m\sqrt{\alpha}\mathrm{sn}(d_1), b_1=c_1=0, k=2\alpha\mathrm{sn}(d_1);\\
&a_0=0, a_1=c_1=0, b_1=\pm\sqrt{\frac{\alpha}{(m^2\mathrm{sn}^2(d_1)-1)}}\mathrm{sn}(d_1)m, k=\frac{-2\alpha\mathrm{sn}(d_1)\mathrm{dn}(d_1)}{(m^2\mathrm{sn}^2(d_1)-1)};\\
&a_0=0, a_1=b_1=0, c_1=\pm\sqrt{\frac{\alpha}{(\mathrm{sn}^2(d_1)-1)}}\mathrm{sn}(d_1), k=\frac{-2\alpha\mathrm{sn}(d_1)\mathrm{cn}(d_1)}{(\mathrm{sn}^2(d_1)-1)},
\end{aligned}$$

where d_1 is an arbitrary constant. Then, the closed form solutions of Eq.(11) are

$$u(n,t)_1 = \pm m\sqrt{\alpha}\mathrm{sn}(d_1)\mathrm{sn}(nd_1+2\alpha\mathrm{sn}(d_1)t+\delta); \quad (15)$$

$$u(n,t)_2 = \pm\sqrt{\frac{\alpha}{(m^2\mathrm{sn}^2(d_1)-1)}}\mathrm{sn}(d_1)m\mathrm{cn}\left(nd_1-\frac{2\alpha\mathrm{sn}(d_1)\mathrm{dn}(d_1)}{(m^2\mathrm{sn}^2(d_1)-1)}t+\delta\right); \quad (16)$$

$$u(n,t)_3 = \pm\sqrt{\frac{\alpha}{(\mathrm{sn}^2(d_1)-1)}}\mathrm{sn}(d_1)\mathrm{dn}\left(nd_1-\frac{2\alpha\mathrm{sn}(d_1)\mathrm{cn}(d_1)}{(\mathrm{sn}^2(d_1)-1)}t+\delta\right). \quad (17)$$

When $m \to 1$, $u(n,t)_1$ degenerate as

$$\tilde{u}(n,t)_1 = \pm\sqrt{\alpha}\tanh(d_1)\tanh(nd_1 + 2\alpha\tanh(d_1)t + \delta), \tag{18}$$

which are kink-type excitations of the discrete mKdV lattice equation in Ref.[17]. When $m \to 1$, $u(n,t)_2$ degenerate as

$$\tilde{u}(n,t)_2 = \pm\sqrt{\frac{\alpha}{(\tanh^2(d_1)-1)}}\tanh(d_1)\mathrm{sech}\left(nd_1 - \frac{2\alpha\tanh(d_1)\mathrm{sech}(d_1)}{(\tanh^2(d_1)-1)}t + \delta\right), \tag{19}$$

which are bell-type excitations of the discrete mKdV lattice equation.

4. Summary

In this paper, we have successfully utilized the modified Jacobian elliptic function approach to construct solutions for nonlinear discrete mKdV lattice, and obtained Jacobian elliptic function doubly periodic solutions, kink-type solitary solutions and bell-type solitary solutions. We have obtained not only many kinds of solutions, but also a procedure which is readily computable. Owing to the importance of DDEs, the above proposed method is worthy of further study, and it may help us find new and interesting solutions to a given nonlinear discrete system.

References

[1] Cardener C S, Kruskal J M and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095

[2] Wahquist H D and Estabrook F B 1971 *Phys. Lett.* A **31** 1386

[3] Fan E G 2001 *Phys. Lett.* A **282** 18

[4] Malfliet W 1992 *Am. J. Phys.* **60** 659

[5] Hereman W 1996 *Comput. Phys. Commun.* **65** 143

[6] Wang M L 1995 *Phys. Lett.* A **199** 169100

[7] Fan E G 2000 *Acta Phys. Sin.* **49** 1049 (in Chinese)

[8] Zhang J F, Wang W H and Zheng C L 2002 *Acta Phys. Sin.* **51** 2676 (in Chinese)

[9] Tang X Y, Lou S Y and Zhang Y 2002 *Phys. Rev.* **E66** 046601

[10] Zheng C L, Zhu J M, Zhang J F and Chen L Q 2003 *Commun. Theor. Phys.* **39** 216

[11] Zhang J F, Zheng C L, Meng J P and Fang J P 2003 *Chin. Phys. Lett.* **20** 448

[12] Fu Z T, Liu S K, Liu S D and Zhao Q 2001 *Phys. Lett.* A **290** 72

[13] Liu S D, Fu Z T, Liu S K and Zhao Q 2002 *Acta Phys. Sin.* **51** 718 (in Chinese)

[14] Fan E G and Hon Y C 2002 *Phys. Lett.* A **292** 335

[15] Zhu J M, Ma Z Y, Fang J P, Zheng C L and Zhang J F 2004 *Chin. Phys.* **13** 798

[16] Fermi E, Pasta J and Ulam S 1965 *Collected Papers of Enrico Fermi* (Chicago: University of Chicago Press) p978

[17] Dai C Q, Yang Q and Zhang J F 2004 *Zeitschrift für Naturforschung* A **59a** 635

[18] Ablowitz M J and Ladik J F 1977 *Stud. Appl. Math.* **57** 1