Vol 14 No 1, January 2005
1009-1963/2005/14(01)/0012-05

Chinese Physics

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Form invariance of Raitzin's canonical equations of a nonholonomic mechanical system*

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(Received 8 June 2004; revised manuscript received 25 August 2004)

A form invariance of Raitzin's canonical equations of a nonholonomic mechanical system is studied. The Raitzin canonical equations of the system are established. The definition and criterion of the form invariance in the system under infinitesimal transformations of groups are given. The relation between the form invariance and the conserved quantity of the system is obtained and an example is also given to illustrate the application of the result.

Keywords: nonholonomic system, Raitzin's canonical equation, form invariance, conserved quantity
PACC: 0320, 0200

1. Introduction

The conservation laws for a dynamical system play an important role in the fields of mathematics, mechanics, and physics. Furthermore, their study has become one of most important directions in modern analytical mechanics. There are four main methods used to seek the conserved quantities of the dynamical system: the Noether theory,[1] the Lie symmetry,[2] the theory of integrating factors,[3] and the Hojman method.[4] Significant achievements have been obtained in the study of these methods in recent years.[5–13]

The symmetry method is a modern method in seeking conservation laws. Mei's form invariance[14] (or Mei symmetry) means that the form of an equation of motion remains invariant under the infinitesimal transformations of groups. It is different from the Noether symmetry and can also lead to a conserved quantity under certain conditions.

In Refs.[15–24], the form invariances of the Lagrangian equations, the Appell equations, the Gibbs–Appell equations, the Nielsen equations, the Chaplygin equations and the Hamilton canonical equations have been studied.

In 1961, a Spanish scholar Raitzin proposed a new form of canonical variables r and s and established new differential equations of motion, which are called Raitzin's canonical equations.[25] In practice, compared with the Hamilton canonical equations, some dynamical problems were solved just simply by applying the equations. Therefore, in analytical mechanics, Raitzin's canonical equations are significant theoretically and practically. At present, some results have been obtained.[26,27] In this paper, a form invariance of Raitzin's canonical equations of a nonholonomic mechanical system is studied.

2. Raitzin's canonical equations of the corresponding holonomic system

We consider a mechanical system composed of N particles, and its configuration is determined by n generalized coordinates $q_1, q_2, \cdots, q_n$. The motion of the system is subject to the following g ideal bilateral nonholonomic constraints of Chetaev type

$$f_\beta(t, q, \dot{q}) = 0, \quad \frac{\partial f_\beta}{\partial \dot{q}_j}\delta q_j = 0, \qquad (1)$$

$$(\beta = 1, 2, \cdots g, j = 1, 2, \cdots, n).$$

The equations of motion of the system can be written in Routh's form[28]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_j}, \quad (j = 1, 2, \cdots, n), \qquad (2)$$

*Project supported by the Heilongjiang Natural Science Foundation, China (Grant No 9507).

where L is the Lagrangian of the system, Q_j are the nonpotential generalized forces and λ_β are the constrained multipliers. Before we integrate the differential equations of motion, we obtain λ_β form Eqs.(1) and (2), and express it as a function of t, q and $\dot{q}$. Thus, Eqs.(2) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j + \Lambda_j, \quad (j = 1, 2, \cdots, n), \tag{3}$$

where $\Lambda_j = \lambda_\beta(t, q, q_j)\dfrac{\partial f_\beta}{\partial \dot{q}_j}$ are the generalized nonholonomic constraint forces, and have been written as functions of the variables t, q and $\dot{q}$.

Equations (3) are called differential equations of motion of the holonomic system corresponding to the nonholonomic systems (1) and (2). The integrals of the nonholonomic systems (1) and (2) can be found according to the corresponding holonomic system (3), only if we apply the restriction of the nonholonomic constraints (1) to the initial conditions.

Now, we introduce Raitzin's canonical variables and function[25] as follows:

$$r_j = \frac{\partial L}{\partial q_j}, \qquad s_j = \dot{q}_j, \tag{4}$$

$$R(t, r, s) = L - r_j q_j. \tag{5}$$

From Eqs.(3), (4) and (5), we have

$$s_j = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j}, \tag{6}$$

$$r_j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial s_j} - \tilde{Q}_j - \tilde{\Lambda}_j, \quad (j = 1, 2, \cdots, n). \tag{7}$$

Here, the symbol "$\sim$" is the expression obtained by substituting t, r, s for q, $\dot{q}$.

Equations (6) and (7) are called Raitzin's canonical equations of motion of holonomic system corresponding to the nonholonomic systems (1) and (2).

3. Form invariance of Raitzin's canonical equations of the corresponding holonomic system

Introduce the infinitesimal transformations for t, q and r,

$$t^* = t + \Delta t, \quad q_j^*(t^*) = q_j(t) + \Delta q_j,$$

$$r_j^*(t^*) = r_j(t) + \Delta r_j, \qquad (j = 1, 2, \cdots, n). \tag{8}$$

Their expanding forms are

$$t^* = t + \varepsilon\xi_0(t, r, s), \quad q_j^*(t^*) = q_j(t) + \varepsilon\xi_j(t, r, s),$$

$$r_j^*(t^*) = r_j(t) + \varepsilon\eta_j(t, r, s), \tag{9}$$

where ε is an infinitesimal parameter, and ξ_0, ξ_j and η_j are infinitesimal generators. Under transformations (9), $R, \tilde{Q}_j$ and $\tilde{\Lambda}_j$ respectively become

$$R^* = R(t^*, r^*, s^*), \quad \tilde{Q}_j^* = \tilde{Q}_j(t^*, r^*, s^*),$$

$$\tilde{\Lambda}_j^* = \tilde{\Lambda}_j(t^*, r^*, s^*). \tag{10}$$

3.1 Definition

If the form of Raitzin's canonical equations (6) and (7) remains invariant under infinitesimal transformations (9), i.e.

$$s_j^* = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial r_j}, \tag{11}$$

$$r_j^* = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial r_j} - \tilde{Q}_j^* - \tilde{\Lambda}_j^*, \quad (j = 1, 2, \cdots, n), \tag{12}$$

then the invariance is called the form invariance of Raitzin's canonical equations (6) and (7) of the corresponding holonomic system. We introduce an infinitesimal transformation vector of generator

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_j\frac{\partial}{\partial s_j} + \eta_j\frac{\partial}{\partial r_j}. \tag{13}$$

Expanding R^*, $\tilde{Q}_j^*$ and $\tilde{\Lambda}_j^*$ under infinitesimal transformations (9), we have

$$R^* = R + \varepsilon[X^{(0)}(R)] + O(\varepsilon^2), \tag{14}$$

$$\tilde{Q}_j^* = \tilde{Q}_j + \varepsilon[X^{(0)}(\tilde{Q}_j)] + O(\varepsilon^2), \tag{15}$$

$$\tilde{\Lambda}_j^* = \tilde{\Lambda}_j + \varepsilon[X^{(0)}(\tilde{\Lambda}_j)] + O(\varepsilon^2), \tag{16}$$

then the following criterion can be obtained.

3.2. Criterion

If R, $\tilde{Q}_j$, $\tilde{\Lambda}_j$ and the infinitesimal generators ξ_0, ξ_j, η_j satisfy the following relations:

$$\dot{\xi}_j = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial r_j}[X^{(0)}(R)], \tag{17}$$

$$\eta_j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial s_j}[X^{(0)}(R)] - X^{(0)}[\tilde{Q}_j + \tilde{\Lambda}_j], \tag{18}$$

then the Raitzin canonical equations (6) and (7) of the corresponding holonomic system are form invariant under infinitesimal transformations (9).

Proof

From transformations (9), differentiating the $q_j^*(t^*)$ with respect to time t, we obtain

$$\dot{q}_j^*(t^*) = \dot{q}_j(t) + \varepsilon\dot{\xi}_j(t, r, s). \tag{19}$$

Then, Eq.(19) can be written as

$$s_j^*(t^*) = s_j(t) + \varepsilon\dot{\xi}_j(t,r,s). \tag{20}$$

Substituting Eqs.(14) and (20) into Eq.(11), using Eq.(6), and ignoring ε^2 and the higher-order infinitesimal terms, we have

$$\begin{aligned} &s_j^* + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial r_j} = s_j + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial r_j} \\ &+ \varepsilon\left\{\dot{\xi}_j + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial r_j}[X^{(0)}(R)]\right\} = 0. \end{aligned} \tag{21}$$

Using the same theory, we have

$$\begin{aligned} &r_j^* - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial s_j} + \tilde{Q}_j^* + \tilde{\Lambda}_j^* = r_j - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} + \tilde{Q}_j + \tilde{\Lambda}_j \\ &+ \varepsilon\left\{\eta_j - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial s_j}[X^{(0)}(R)] + X^{(0)}[\tilde{Q}_j + \tilde{\Lambda}_j]\right\} = 0. \end{aligned} \tag{22}$$

In general, the form invariance of a dynamical system does not always imply the conserved quantity, it can lead to a conserved quantity only under certain conditions.

3.3. Proposition

If the corresponding holonomic system given by Eqs.(6) and (7) is form invariant under infinitesimal transformations (9), and there exists a gauge function $G_{\mathrm{F}} = G_{\mathrm{F}}(t,r,s)$ satisfying the Noether identity[29]

$$\begin{aligned} &\left(\frac{\partial R}{\partial r_j} + q_j\right)\bar{\eta}_j + \left(r_j - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} + \tilde{Q}_j + \tilde{\Lambda}_j\right)\bar{\xi}_j \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial s_j}\bar{\xi}_j + L\xi_0\right) = -\dot{G}_{\mathrm{F}}, \\ &\qquad (j = 1,2,\cdots,n), \end{aligned} \tag{23}$$

where

$$\bar{\eta}_j = n_j - \dot{r}_j\xi_0, \quad \bar{\xi}_j = \xi_j - s_j\xi_0, \tag{24}$$

then the symmetry is called Noether symmetry, which leads to the following conserved quantity:

$$I = \frac{\partial R}{\partial s_j}\xi_j + R\xi_0 + \left(r_j q_j - \frac{\partial R}{\partial s_j}s_j\right)\xi_0 + G_{\mathrm{F}} = \mathrm{const.} \tag{25}$$

4. Form invariance of Raitzin's canonical equations of the nonholonomic system

According to the Raitzin variables (4), the nonholonomic constraints (1) and constrained multipliers λ_β can be written as

$$\tilde{f}_\beta(t,r,s) = f_\beta(t,q(t,r,s),s) = 0, \tag{26}$$

$$\tilde{\lambda}_\beta(t,r,s) = \lambda_\beta(t,q(t,r,s),s), \tag{27}$$

and then from nonholonomic systems (2), (26) and (27) we obtain Raitzin's canonical equations of a nonholonomic system as follows:

$$s_j = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j}, \tag{28}$$

$$\begin{aligned} &r_j = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} - \tilde{Q}_j - \tilde{\lambda}_\beta\frac{\partial \tilde{f}_\beta}{\partial s_j}, \\ &(j = 1,2,\cdots,n;\beta = 1,2,\cdots,g). \end{aligned} \tag{29}$$

Under the transformations (9), R becomes

$$R^* = R(t^*,r^*,s^*). \tag{30}$$

We can also obtain the expressions for $\tilde{Q}_j$, $\tilde{\Lambda}_j$, $\tilde{f}_\beta$ and $\tilde{\lambda}_\beta$ which are similar to Eq.(30) in form.

4.1. Definition

If the forms of the constraint equations (26) and Raitzin's canonical equations (6) and (7) of the corresponding holonomic system remain invariant under infinitesimal transformations (9), i.e.

$$\tilde{f}_\beta^* = \tilde{f}_\beta(t^*,r^*,s^*) = 0, \quad (\beta = 1,2,\cdots,g), \tag{31}$$

$$s_j^* = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial r_j}, \tag{32}$$

$$r_j^* = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial s_j} - \tilde{Q}_j^* - \tilde{\Lambda}_j^*, \quad (j = 1,2,\cdots,n), \tag{33}$$

then the invariance is called the form invariance of Raitzin's canonical equations (28) and (29) of the nonholonomic system.

Expanding $\tilde{f}_\beta^*$ under infinitesimal transformations (9), we have

$$\tilde{f}_\beta^* = \tilde{f}_\beta + \varepsilon[X^{(0)}(\tilde{f}_\beta)] + O(\varepsilon^2). \tag{34}$$

Substituting Eq.(34) into Eq.(31), using Eq.(26), and ignoring ε^2 and the higher-order infinitesimal terms, we have

$$X^{(0)}[\tilde{f}_\beta(t,r,s)] = 0. \tag{35}$$

4.2 Criterion

For the nonholonomic system given by Eqs.(26), (28) and (29), if the infinitesimal generators ξ_0, ξ_j and η_j satisfy Eqs.(17), (18) and (35) then the nonholonomic system is form invariant under the infinitesimal transformation (9).

4.3. Proposition

If the nonholonomic system given by Eqs.(26), (28) and (29) is form invariant under infinitesimal transformation (9), and there exists a gauge function $G_{\rm F} = G_{\rm F}(t, r, s)$ satisfying the Noether identity (23), then the Noether symmetry leads to the same conserved quantity as Eq.(25).

Through the above, the form invariance of the corresponding holonomic systems (6) and (7) should satisfy the conditions (17) and (18).

The form invariance of the nonholonomic systems (26), (28) and (29), needs to satisfy not only conditions (17), (18) but also condition (35). It seems that the form invariance of the nonholonomic system must be the form invariance of the corresponding holonomic system. On the contrary, it is uncertain.

5. Example

Suppose that the Lagrangian of a mechanical system is

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3, \tag{36}$$

and the nonholonomic constraint is

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0. \tag{37}$$

We try to find the conserved quantity of the system.

The first step is to establish the Raitzin's canonical equations of the system.

Equations (2) give

$$\begin{aligned} &m\ddot{q}_1 = 2\lambda\dot{q}_1, \quad m\ddot{q}_2 = 2\lambda\dot{q}_2, \\ &m\ddot{q}_3 + mg = -2\lambda\dot{q}_3. \end{aligned} \tag{38}$$

From Eqs.(37) and (38), we obtain

$$\lambda = -\frac{mg}{4\dot{q}_3}. \tag{39}$$

Therefore, we have

$$\begin{aligned} Q_1 + \Lambda_1 &= -\frac{mg\dot{q}_1}{2\dot{q}_3}, \\ Q_2 + \Lambda_2 &= -\frac{mg\dot{q}_2}{2\dot{q}_3}, \\ Q_3 + \Lambda_3 &= \frac{1}{2}mg. \end{aligned} \tag{40}$$

Introducing the Raitzin canonical variables and function as

$$\begin{aligned} &r_1 = \frac{\partial L}{\partial q_1} = 0, \quad r_2 = \frac{\partial L}{\partial q_2} = 0, \quad r_3 = \frac{\partial L}{\partial q_3} = -mg, \\ &s_1 = \dot{q}_1, \quad s_2 = \dot{q}_2, \quad s_3 = \dot{q}_3, \end{aligned} \tag{41}$$

$$R = L - r_j q_j = \frac{1}{2}m(s_1^2 + s_2^2 + s_2^3), \tag{42}$$

we have

$$\begin{aligned} r_1 &= \frac{\rm d}{{\rm d}t}\frac{\partial R}{\partial s_1} - (\tilde{Q}_1 + \tilde{\Lambda}_1) = m\dot{s}_1 + \frac{1}{2}mg\frac{s_1}{s_3}, \\ r_2 &= \frac{\rm d}{{\rm d}t}\frac{\partial R}{\partial s_2} - (\tilde{Q}_2 + \tilde{\Lambda}_2) = m\dot{s}_2 + \frac{1}{2}mg\frac{s_2}{s_3}, \\ r_3 &= \frac{\rm d}{{\rm d}t}\frac{\partial R}{\partial s_3} - (\tilde{Q}_3 + \tilde{\Lambda}_3) = m\dot{s}_3 - \frac{1}{2}mg, \end{aligned} \tag{43}$$

$$\begin{aligned} s_1 &= \dot{q}_1 = -\frac{\rm d}{{\rm d}t}\frac{\partial R}{\partial r_1}, \\ s_2 &= \dot{q}_2 = -\frac{\rm d}{{\rm d}t}\frac{\partial R}{\partial r_2}, \\ s_3 &= \dot{q}_3 = -\frac{\rm d}{{\rm d}t}\frac{\partial R}{\partial r_3}, \end{aligned} \tag{44}$$

$$\begin{aligned} \tilde{Q}_1 + \tilde{\Lambda}_1 &= \frac{-1}{2}mg\frac{s_1}{s_3}, \\ \tilde{Q}_2 + \tilde{\Lambda}_2 &= \frac{-1}{2}mg\frac{s_2}{s_3}, \end{aligned}$$

$$\tilde{Q}_3 + \tilde{\Lambda}_3 = \frac{1}{2}mg, \tag{45}$$

$$\tilde{f} = s_1^2 + s_2^2 - s_2^3 = 0. \tag{46}$$

The second step is to study the form invariance of the system.

Taking

$$\xi_0 = 1, \quad \xi_1 = \xi_2 = \xi_3 = 0, \quad \eta_1 = \eta_2 = \eta_3 = 0, \tag{47}$$

we have

$$\begin{aligned} X^{(0)}[R] =& \xi_0\frac{\partial R}{\partial t} + \xi_1\frac{\partial R}{\partial s_1} + \xi_2\frac{\partial R}{\partial s_2} + \xi_3\frac{\partial R}{\partial s_3} \\ &+ \eta_1\frac{\partial R}{\partial r_1} + \eta_2\frac{\partial R}{\partial r_2} + \eta_3\frac{\partial R}{\partial r_3} = \frac{\partial R}{\partial t} = 0, \end{aligned} \tag{48}$$

$$\frac{\rm d}{{\rm d}t}\frac{\partial}{\partial s_j}[X^{(0)}(R)] = 0, \frac{\rm d}{{\rm d}t}\frac{\partial}{\partial r_j}[X^{(0)}(R)] = 0, \quad (j = 1, 2, 3), \tag{49}$$

$$\begin{aligned} X^{(0)}[\tilde{Q}_1 + \tilde{\Lambda}_1] &= \frac{\partial\left(-\frac{1}{2}mg\frac{s_1}{s_3}\right)}{\partial t} = 0, \\ X^{(0)}[\tilde{Q}_2 + \tilde{\Lambda}_2] &= \frac{\partial\left(-\frac{1}{2}mg\frac{s_2}{s_3}\right)}{\partial t} = 0, \\ X^{(0)}[\tilde{Q}_3 + \tilde{\Lambda}_3] &= \frac{\partial\left(\frac{1}{2}mg\right)}{\partial t} = 0, \end{aligned}$$

i.e.

$$X^{(0)}[\tilde{Q}_j + \tilde{\Lambda}_j] = 0, \quad (j = 1, 2, 3), \tag{50}$$

$$X^{(0)}[\tilde{f}] = \frac{\partial\tilde{f}}{\partial t} = 0. \tag{51}$$

The generators (47) satisfy the conditions (17), (18) and (35), therefore, the nonholonomic system is form invariant.

The third step is to seek the conserved quantity of the system.

Formula (23) gives

$$G = 0. \tag{52}$$

From formula (25), we obtain the conserved quantity as follows:

$$I = -\frac{1}{2}m(s_1^2 + s_2^2 + s_3^2) - mgq_3 = \text{const.} \tag{53}$$

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