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Localized Lie symmetries and conserved quantities for the finite-degree-of-freedom systems*

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This paper focuses on the study of localized Lie symmetries under the infinitesimal transformation of an infinite continuous group for the finite-degree-of-freedom systems. Based on an invariance of differential equation under an infinitesimal transformation, we present the localized Lie symmetries including direct and inverse problems for the finite degree-of-freedom mechanical systems. We also give the definitions, determining equations, structural equation and conserved laws of localized Lie symmetries, and further, the Lie symmetries under the infinitesimal transformation of a finite continuous group derived from localized Lie symmetry. Finally, an example is discussed to illustrate these results.

Keywords: localized Lie symmetry, infinite continuous group, conservation law, finite degree of freedom system

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1. Introduction

The symmetrical principle plays a central role in the study of mathematics and physics. The research on conserved quantities of mechanical systems not only has important theoretic significance, but also can reveal the physical essence in them.[1,2] The approach of Lie symmetries was proposed in the 19th century,[3] but its application in mechanics did not start until 1979.[4] In recent years, many results have been obtained in the studies of Lie symmetries.[4−20] Lately, Mei[21] studied Noether symmetries as well as Lie symmetries of many constrained mechanical systems, and obtained relationships between Noether symmetries and Lie symmetries, and between the symmetries and conserved quantities. The methods of searching for conservation laws via studying Lie symmetries have become an important tool in mechanics.

However, the previous researches on Lie symmetries of dynamical systems were all based on invariance of differential equations under the infinitesimal transformations of finite continuous groups with r arbitrary small parameters ε, thus the symmetries are called global Lie symmetries. The symmetries are given by the Lie determining equations, and are applied to seek conserved quantities of the finite-degree-of-freedom dynamical systems.

The localized Noether symmetry was introduced by Noether in 1918.[22] Noether's second theorem is concerned with the invariance of the action integral under the infinitesimal transformation of an infinite continuous group parametrized with r arbitrary small functions $\varepsilon^{\alpha}(t)$ $(\alpha = 1, \cdots, r)$ and the derivatives of $\varepsilon^{\alpha}(t)$. If an action integral is invariant under the action of such a group, there will exist r differential identities (Noether identities). Noether's second theorem has been discussed by Hilbert, Bergmann and Anderson in connection with electrodynamics and general relativity, [23−25] by Drobot and Rybarski with respect to hydromechanics,[26] by Sundermeyer for gauge field theory,[27] and by Li for the non-invariant action integral, the Yang–Mills field and quantum systems.[28−31]

The approach of localized Noether symmetries has been widely used in modern physics,[32] whereas

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the approach of localized Lie symmetry in various continuous fields is only just beginning. We call the invariance of differential equations under the infinitesimal transformation of an infinite continuous group parametrized with r arbitrary small functions $\varepsilon^{\alpha}(t)$ and the derivatives of $\varepsilon^{\alpha}(t)$, the localized Lie symmetry, which will be used to seek conserved quantities of continuous dynamical systems as well as physical systems. This theory may be applied to various fields of theoretical physics such as electrodynamics, general relativity, hydromechanics, and gauge field theory. In this paper we will discuss only the localized Lie symmetries for the finite-degree-of-freedom systems, and then derive the determining equations, structure equation and conservation law of localized Lie symmetries. Then we give the relations between localized Noether and Lie symmetries. We also study the inverse problem of the symmetries, and present an example to illustrate the symmetries.

2. Equation of motion of a system

An n degree-of-freedom mechanical system with Lagrangian $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ $\left(\boldsymbol{q} = q^1, q^2, \cdots, q^n\right)$, subjected to a non-potential generalized force $Q'_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, has equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q'_i \ (i = 1, \cdots, n). \tag{1}$$

Suppose that Eq.(1) is non-singular, i.e.

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k}\right) \neq 0. \tag{2}$$

Equation (1) can be put in the form

$$\ddot{q}^i = h_i(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{3}$$

3. Localized Noether symmetry of a system

We assume that the Hamiltonian action of a finite-degree-of-freedom system is

$$I = \int_{t_1}^{t_2} L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\mathrm{d}t, \tag{4}$$

and introduce the infinitesimal transformation which is a single parameter transformation of an infinite continuous group with respect to time and generalized coordinates

$$\begin{cases} t^{\bullet} = t + \Delta t = t + R\varepsilon(t), \\ q^{i\bullet}(t) = q^i(t) + \Delta q^i(t) = q^i(t) + S^i\varepsilon(t), \end{cases} \tag{5}$$

where $\varepsilon(t)$ is an arbitrary small function, and R and S^i are linear differential operators, which can be written as

$$\begin{cases} R = a_k D^k = a_0 + a_1 D + a_2 D^2 + \cdots + a_K D^K \\ S^i = b^i_l D^l = b^i_0 + b^i_1 D + b^i_2 D^2 + \cdots + b^i_L D^L \end{cases}$$

$$\left(D = \frac{\mathrm{d}}{\mathrm{d}t}\right). \tag{6}$$

Here, $a_k = a_k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ $(k = 0, 1, \cdots, K)$, $b^i_l = b^i_l(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ $(l = 0, 1, \cdots, L)$ are infinitesimal generators. Equation (5) gives the infinitesimal transformation for finite continuous group of the system when the arbitrary small function $\varepsilon(t)$ degenerates to a small parameter, and then $R = a_0, S^i = b^i_0$.

Based on the invariance of the Hamiltonian action and the variance of Lagrangian, there is a total derivative term $\dfrac{\mathrm{d}\Omega^{\bullet}}{\mathrm{d}t}$, where $\Omega^{\bullet} = \Omega\varepsilon(t)$, and Ω is a linear differential operator

$$\Omega = e^i_j D^j = e^i_0 + e^i_1 D + e^i_2 D^2 + \cdots + e^i_J D^J. \tag{7}$$

From the transformation formula (5), one can obtain[32]

$$\tilde{S}^i\left(\frac{\delta I}{\delta q^i}\right) - \tilde{R}\left(\dot{q}^i\frac{\delta I}{\delta q^i}\right) = 0 \quad (i = 1, \cdots, n), \tag{8}$$

where $\tilde{R}$ and $\tilde{S}^i$ are the adjoint operators with respect to R and S^i respectively, defined by

$$\int_{t_1}^{t_2} f(t) Rg(t)\mathrm{d}t = \int_{t_1}^{t_2} g(t)\tilde{R}f(t)\mathrm{d}t + [\cdots]_{t_1}^{t_2}, \tag{9}$$

where $f(t)$ and $g(t)$ are slick functions in the domain [t_1,t_2], and $[\cdots]_{t_1}^{t_2}$ expresses the end point functions. $\dfrac{\delta I}{\delta q^i}$ is an Euler–Lagrange equation in Eq.(8), i.e.

$$\frac{\delta I}{\delta q^i} = \frac{\partial L}{\partial q^i} - D\left(\frac{\partial L}{\partial \dot{q}^i}\right). \tag{10}$$

Expression (8) is called the generalized Noether identical equations of the system under the infinitesimal transformation (5).

Then, the secondary Noether theorem of the finite-degree-of-freedom systems is as follows.

Theorem 1 There exists the differential identical equation (8) associated with the functional derivative $\dfrac{\delta I}{\delta q^i}$, if the Hamiltonian action of a finite degree-of-freedom system remains invariant under the infinitesimal transformation (5).

For the invariable systems mentioned above, the Noether identical equations indicate that the derivatives of the action for a system with respect to q^i (or the Euler–Lagrange expression) are not independent, and are under restriction of Noether identical equations.

4. Direct problems of localized Lie symmetries

We give the definition of localized Lie symmetry for a finite-degree-of-freedom system.

Definition Transformation (5) is called localized Lie symmetrical transformation corresponding to a finite-degree-of-freedom system when differential equations (3) remain invariant under the infinitesimal transformation (5).

We now show that the transformation of the infinite continuous group includes an arbitrary function $\varepsilon(t)$ and kth-order derivatives of $\varepsilon(t)$.

Firstly, we study the direct problem of localized Lie symmetries for the system. Let us introduce the vector of the differential operator for the infinitesimal transformation (5)

$$X^{(0)\bullet} = R\varepsilon\frac{\partial}{\partial t} + S_k\varepsilon\frac{\partial}{\partial q^k} \quad (k = 1, \cdots, n), \tag{11}$$

its first-order expanding formula is

$$X^{(1)\bullet} = X^{(0)\bullet} + \left(D(S_k\varepsilon) - \dot{q}^k D(R\varepsilon)\right)\frac{\partial}{\partial \dot{q}^k}, \tag{12}$$

and the second-order expanding form is

$$\begin{aligned} X^{(2)\bullet} =& X^{(1)\bullet} + [D(D(S_k\varepsilon) - \dot{q}^k D(R\varepsilon)) \\ & - \ddot{q}^k D(R\varepsilon)]\frac{\partial}{\partial \ddot{q}^k}. \end{aligned} \tag{13}$$

The vectors of the differential operators $X^{(0)\bullet}$, $X^{(1)\bullet}$ and $X^{(2)\bullet}$ include arbitrary function $\varepsilon(t)$ and the derivative of $\varepsilon(t)$.

Based on the invariance of differential equations (3) under the infinitesimal transformation (5), it follows that

$$\begin{aligned} & D^2(S_k\varepsilon) - \dot{q}^k D^2(R\varepsilon) - 2h_k D(R\varepsilon) \\ =& X^{(1)\bullet}(h_k) \quad (k = 1, \cdots, n), \end{aligned} \tag{14}$$

if and only if

$$X^{(2)\bullet}\left[\ddot{q}^k - h_k(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\right]_{\ddot{q}^k = h_k\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right)} = 0. \tag{15}$$

Equation (14) is called the determining equation of localized Lie symmetries for a finite-degree-of-freedom system when the differential operators R and $S^i\,(i = 1, \cdots, L)$ satisfy the infinitesimal transformation (5).

Criterion Transformation (5) is called localized Lie symmetric transformation corresponding to a finite degree-of-freedom system, if the linear differential operators R and S_k satisfy determining equation (14).

Equation (14) is the determining equation of Lie symmetries under the infinitesimal transformation of the finite continuous group, if the arbitrary function $\varepsilon(t)$ degenerates to a small parameter in transformation (5).

Theorem 2 For the linear differential operators R and S_k of the infinitesimal transformation (5) of a finite-degree-of-freedom system, if they satisfy the determining equation (14), and there is a gauge function $G^\bullet = \varepsilon G$ which is also a linear differential operator satisfying the structural equation

$$\begin{aligned} & X^{(1)\bullet}(L) + D(R\varepsilon)L + \left(S_k\varepsilon - \dot{q}^k R\varepsilon\right)Q'_k + DG^\bullet = 0 \\ & \qquad (k = 1, \cdots, n), \end{aligned} \tag{16}$$

this system then possesses Noether-type conserved connection (or conserved quantity) corresponding to localized Lie symmetries for the finite-degree-of-freedom system

$$I^\bullet = R\varepsilon L + \left(S_k\varepsilon - \dot{q}^k R\varepsilon\right)\frac{\partial L}{\partial \dot{q}^k} + G^\bullet = \text{const}, \tag{17}$$

and the conserved connection (17) is different from the usual conserved quantity of Lie symmetries. We now show that the relationship between conserved quantities of motion and physical parameter of the system includes an arbitrary function $\varepsilon(t)$ and the derivative of $\varepsilon(t)$ in the conserved quantity.

Proof

$$\begin{aligned} \frac{\mathrm{d}I^\bullet}{\mathrm{d}t} =& D(R\varepsilon)L + R\varepsilon\left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q^k}\dot{q}^k + \frac{\partial L}{\partial \dot{q}^k}\ddot{q}^k\right) \\ & + \left(S_k\varepsilon - \dot{q}^k R\varepsilon\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^k}\right) \\ & + \left[D(S_k\varepsilon) - \dot{q}^k D(R\varepsilon) - \ddot{q}^k R\varepsilon\right]\frac{\partial L}{\partial \dot{q}^k} \\ & - X^{(1)}(L) - D(R\varepsilon)L - \left(S_k\varepsilon - \dot{q}^k R\varepsilon\right)Q'_k \\ =& \left(S_k\varepsilon - \dot{q}^k R\varepsilon\right)\left(D\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} - Q'_k\right) = 0. \end{aligned}$$

This work can give usual Lie symmetries under the transformation of finite continuous group, if the arbitrary small function $\varepsilon(t)$ degenerates to a small parameter ε in transformation (5) of infinite continuous group, and then Eqs.(14), (16) and (17) lead

respectively to the determining equations, structural equation and conserved quantities of Lie symmetries under the infinitesimal transformation of finite continuous groups.

However, we find that the linear differential operators S_k, R and G are quite difficult to obtain from Eqs.(14) and (16), so we use the Killing equations of localized Lie symmetries of the system.

We can obtain the simple equations (Killing equations), which include the linear differential operators S_k, R and G, other than Eq.(16). One expands structural equation (16), and chooses terms of $\ddot{q}^k$ or not, then the Killing equations of localized Lie symmetry of the system are obtained by

$$\begin{aligned}
&L\left(\frac{\partial(R\varepsilon)}{\partial t}+\frac{\partial(R\varepsilon)}{\partial q^l}\dot{q}^l\right)+R\varepsilon\frac{\partial L}{\partial t}+S_k\varepsilon\frac{\partial L}{\partial q^k}\\
&+\left(S_k\varepsilon-\dot{q}^kR\varepsilon\right)Q'_k\\
&+\frac{\partial L}{\partial\dot{q}^k}\left(\frac{\partial(S_k\varepsilon)}{\partial t}+\frac{\partial(S_k\varepsilon)}{\partial q^l}\dot{q}^l\right.\\
&\left.-\dot{q}^k\frac{\partial(R\varepsilon)}{\partial t}-\dot{q}^k\frac{\partial(R\varepsilon)}{\partial q^l}\dot{q}^l\right)\\
=&-\frac{\partial(G\varepsilon)}{\partial t}-\frac{\partial(G\varepsilon)}{\partial q^k}\dot{q}^k,\\
&L\frac{\partial(R\varepsilon)}{\partial\dot{q}^k}+\frac{\partial L}{\partial\dot{q}^k}\left(\frac{\partial(S_k\varepsilon)}{\partial\dot{q}^l}-\dot{q}^k\frac{\partial(R\varepsilon)}{\partial\dot{q}^l}\right)\\
=&-\frac{\partial(G\varepsilon)}{\partial\dot{q}^k}.
\end{aligned}\tag{18}$$

Generally speaking, Lie symmetries do not always possess a Noether-type conserved quantity, however the localized Lie symmetries do possess a Noether-type conserved quantity, and one can find a gauge function in which $\dot{G}$ equals zero, or the total differential of a function in the structure equation.

5. Inverse problems of localized Lie symmetries

Secondly, we study the inverse problem of localized Lie symmetries. To find the localized Lie symmetries from the known conserved quantity is the inverse problem of localized Lie symmetries.

Suppose that the conserved quantity of a finite-degree-of-freedom system is

$$I^\bullet=I^\bullet(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\text{const},\tag{19}$$

then we have

$$\frac{\mathrm{d}I^\bullet}{\mathrm{d}t}=\frac{\partial I^\bullet}{\partial t}+\frac{\partial I^\bullet}{\partial q^k}\dot{q}^k+\frac{\partial I^\bullet}{\partial\dot{q}^k}\ddot{q}^k=0.\tag{20}$$

Multiplying Eq.(1) by

$$\overline{S_k}=S_k\varepsilon-\dot{q}^kR\varepsilon,\tag{21}$$

and then summing for k leads to

$$\overline{S_k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^k}-\frac{\partial L}{\partial q^k}-Q''_k\right)=0.\tag{22}$$

Combining Eqs.(20) and (22), expanding the resulting equation, and letting the coefficients of $\ddot{q}_k$ go to zero, one has

$$\overline{S_k}\frac{\partial^2L}{\partial\dot{q}^k\partial\dot{q}^l}-\frac{\partial I^\bullet}{\partial\dot{q}^l}=0\quad(l=1,\cdots,n),\tag{23}$$

and

$$\overline{S_k}=\overline{h}_{kl}\frac{\partial I^\bullet}{\partial\dot{q}^l},\tag{24}$$

where $h_{kl}=\dfrac{\partial^2L}{\partial\dot{q}^k\partial\dot{q}^l}$ is called Hess matrix and satisfies

$$\overline{h}_{kl}h_{lj}=\delta_{kj}.\tag{25}$$

Letting Eq.(17) be equal to the conserved connection (19), i.e.

$$LR\varepsilon+\frac{\partial L}{\partial\dot{q}^k}\left(S_k\varepsilon-\dot{q}^kR\varepsilon\right)+G^\bullet=I^\bullet,\tag{26}$$

then one can obtain the linear differential operators R and S_k from Eqs.(24) and (26).

The localized Lie symmetry of the finite-degree-of-freedom system is determined by substituting the linear differential operators R and S_k into determining equation (14).

Theorem 3 The symmetries are called localized Lie symmetries corresponding to the finite-degree-of-freedom systems, if the linear differential operators R and S_k of the infinitesimal transformation (5) are determined by Eqs.(24), (26) and determining equation (14).

We would show that the finite-degree-of-freedom systems may not be localized Lie symmetries when the conserved quantities of the systems are given, and then the inverse problem of the localized Lie symmetries of the system may have solution or may not.

We can derive the inverse problem of Lie symmetries under the infinitesimal transformation of finite continuous group from the localized Lie symmetry of the system when we make the arbitrary small function $\varepsilon(t)$ equal to a small parameter ε.

6. The relationships between localized Noether and Lie symmetries

Now we give the relationships between localized Noether and Lie symmetries

Theorem 4 In a finite-degree-of-freedom system which possesses localized Lie symmetry, if its linear differential operators R and S_k are adjoint operators, and satisfy the generalized Noether identical equation (8), then the system also possesses localized Noether symmetry.

Theorem 5 In a finite-degree-of-freedom system which possesses localized Noether symmetry, if its linear differential operators R and S_k satisfy the determining equation (14), then the system also possesses localized Lie symmetry.

The proofs of Theorems 4 and 5 are obvious.

7. An example

Suppose that the generalized coordinates q^1 and q^2 can specify the configuration of a system, its Lagrange function being

$$L = \frac{1}{2}\left(\dot{q}^1\right)^2 + \dot{q}^1 q^2 + \frac{1}{2}(q^1 - q^2)^2, \tag{27}$$

then the equations of motion are

$$\begin{aligned} \ddot{q}^1 &= q^1 - q^2 - \dot{q}^2 = h_1, \\ \dot{q}^1 &= q^1 - q^2 = h_2. \end{aligned} \tag{28}$$

Substituting Eq.(28) into the determining equation (14) yields

$$\begin{cases} D^2(S_1\varepsilon) - \dot{q}^1 D^2(R\varepsilon) - 2\left(q^1 - q^2 - \dot{q}^2\right) DR\varepsilon \\ = S_1\varepsilon - S_2\varepsilon - (D(S_2\varepsilon) - \dot{q}^2 DR\varepsilon), \\ DS_1\varepsilon - \dot{q}^1 DR\varepsilon = S_1\varepsilon - S_2\varepsilon, \end{cases} \tag{29}$$

and Eq.(29) has solution of

$$R = 0,\ S_1 = 1,\ S_2 = 1 - D. \tag{30}$$

Substituting Eqs.(30) and (27) into Eq.(16) yields

$$\begin{aligned} &\varepsilon\left(q^1 - q^2\right) + (1 - D)\varepsilon\left(\dot{q}^1 - \left(q^1 - q^2\right)\right) \\ &+ \dot{\varepsilon}\left(\dot{q}^1 + q^2\right) + DG^{\bullet} = 0, \end{aligned} \tag{31}$$

and then one obtains

$$G^{\bullet} = -\varepsilon q^1. \tag{32}$$

Substituting Eqs.(30) and (32) into Eq.(17), one has

$$I^{\bullet} = \varepsilon\left(\dot{q}^1 + q^2\right) + \varepsilon q^1 = \varepsilon\left(\dot{q}^1 - q^1 + q^2\right) = a\varepsilon, \tag{33}$$

i.e, the system possesses a Noether-type conserved quantity

$$I = \dot{q}^1 - q^1 + q^2 = \text{const.} \tag{34}$$

Then the conservation law of mechanical energy for the system is given by Eq.(34)

Substituting Eqs.(27) and (30) into Eq.(8) yields

$$\begin{aligned} &\frac{\delta I}{\delta q^1} + \frac{\delta I}{\delta q^2} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta I}{\delta q^2}\right) \\ =&q^1 - q^2 - \ddot{q}^1 - \dot{q}^2 + \dot{q}^1 + q^2 - q^1 + \ddot{q}^1 + \dot{q}^2 - \dot{q}^1 = 0, \end{aligned} \tag{35}$$

i.e. Eq.(30) satisfies generalized Noether identical equations of the system, and then the infinitesimal transformation (30) is called a localized Lie symmetric transformation as well as the localized Noether symmetric transformation for the finite-degree-of-freedom system, and the system possesses a Noether-type conserved quantity (34).

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