Vol 13 No 9, September 2004
1009-1963/2004/13(09)/1382-04

Chinese Physics

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Fractal localized structures related to *Jacobian* elliptic functions in the higher-order Broer–Kaup system*

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(Received 22 December 2003; revised manuscript received 6 April 2004)

This work reveals a novel phenomenon—that the localized coherent structures of a (2+1)-dimensional physical model possesses fractal behaviours. To clarify the interesting phenomenon, we take the (2+1)-dimensional higher-order Broer–Kaup system as a concrete example. Starting from a Bäcklund transformation, we obtain a linear equation, and then a general solution of the system is derived. From this some special localized excitations with fractal behaviours are obtained by introducing some types of lower-dimensional fractal patterns that related to *Jacobian* elliptic functions.

Keywords: higher-order Broer–Kaup system, Bäcklund transformation, variable separation approach, *Jacobian* elliptic function, fractal

PACC: 0340K, 0365G

1. Introduction

Fractals and solitons are two important parts of nonlinearity.[1−4] They are widely applied in many natural sciences, particularly in fluid dynamics, plasma physics, field theory, optics and condensed matter physics, etc.[5−7] Conventionally, the two aspects are treated independently, since one often considers that solitons are the basic excitations of an integrable model while fractals are the elementary behaviours of a nonintegrable system. In other words, one does not analyse the possibility of existence of fractals in a soliton system. However, the above consideration may not be complete, especially in some higher dimensions. In the recent study of soliton systems, scientists[8−18] have found that some characteristic lower-dimensional arbitrary functions exist in exact excitation of some two-dimensional integrable models. This means that any lower-dimensional fractal solutions can be used to construct the exact solution of a higher-dimensional integrable model, which also implies that any exotic behaviour may propagate along this characteristic.

To verify the above viewpoints, we take the (2+1)-dimensional higher-order Broer–Kaup system (HBKs) as a concrete example,

$$u_{yt}+4(u_{xx}+u^3-3uu_x+3uw)_{xy}+12(uv)_{xx}=0, \quad (1)$$

$$v_t+4(v_{xx}+3u^2v+3uv_x+3vw)_x=0, \quad (2)$$

$$w_y-v_x=0, \quad (3)$$

which is first obtained from the inner parameter dependent symmetry constraints of the KP equation.[19] When we take $y=x$, the system (1)–(3) is reduced to the usual (1+1)-dimensional HBK system. In Refs.[20–22], some exact soliton-like solutions and abundant localized coherent solutions of the HBKs were obtained through the homogeneous balance method and the general variable-separated approach respectively.[23−32]

In order to investigate some localized excitations with fractal behaviours of Eqs.(1)–(3), a linear equation of the HBKs is derived firstly via an extended homogeneous balance approach. According to the homogeneous balance principle (which can also be obtained

*Project supported by the Foundation of '151 Talent Engineering' of Zhengjiang Province, China and by the National Natural Science Foundation of China (Grant No 10172056).

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through the standard truncated Painlevé expansion), we begin with the Bäcklund transformation

$$\begin{aligned} u &= (\ln f)_x + u_0(x,t), \\ v &= (\ln f)_{xy}, \\ w &= (\ln f)_{xx} + w_0(x,t), \end{aligned} \tag{4}$$

where $f \equiv f(x,y,t)$ is a function of variables x, y and t to be determined, $u_0 \equiv u_0(x,t), w_0 \equiv w_0(x,t)$ are two arbitrary functions of x and t. Substituting Eq.(4) into Eqs.(1)–(3), we can obtain a single equation

$$\begin{aligned} &(12u_0^2 f_x + 12w_0 f_x + 12u_0 f_{xx} + f_t + 4f_{xxx})_{xy} f^2 \\ &- ((12u_0^2 f_x + 12w_0 f_x \\ &+ 12u_0 f_{xx} + f_t + 4f_{xxx})_y f_x \\ &+ ((12u_0^2 f_x + 12w_0 f_x \\ &+ 12u_0 f_{xx} + f_t + 4f_{xxx}) f_y)_x) f \\ &+ 2(12u_0^2 f_x + 12w_0 f_x \\ &+ 12u_0 f_{xx} + f_t + 4f_{xxx}) f_x f_y = 0. \end{aligned} \tag{5}$$

It can be easily seen that if f satisfies

$$12u_0^2 f_x + 12w_0 f_x + 12u_0 f_{xx} + f_t + 4f_{xxx} = 0, \tag{6}$$

Eq.(5) is satisfied automatically.

We take f as follows:

$$f(x,y,t) = a_0 + a_1 p(x,t) + a_2 q(y,t) + a_3 p(x,t) q(y,t), \tag{7}$$

where the variable separation functions $p \equiv p(x,t)$ and $q \equiv q(y,t)$ are functions of the indicated variables, while a_0, a_1, a_2, a_3 are arbitrary constants.

Substituting the ansatz (7) into Eq.(6), we obtain two variable-separated equations

$$\begin{aligned} p_t = &- 12u_0^2 p_x - 12w_0 p_x - 12u_0 p_{xx} \\ &- 4p_{xxx} - c_0(a_2 + a_3 p), \end{aligned} \tag{8}$$

$$q_t = c_0(a_1 + a_3 q). \tag{9}$$

where $c_0 \equiv c_0(t)$ is an arbitrary function of time t. Equation (9) has the solution

$$q = \frac{Q(y)}{a_3}(\mathrm{e}^{\int a_3 c_0 \mathrm{d}t} - a_1), \tag{10}$$

where $\mathrm{e}^{\int a_3 c_0 \mathrm{d}t} - a_1 \neq 0$ and $Q(y)$ is an arbitrary function of y. Although it is not easy to obtain a general solution of Eq.(8) for any fixed u_0 and w_0, we can treat the problem in an alternative way. Since u_0 and w_0 are arbitrary seed solutions, we can regard p as an arbitrary function of (x,t), then the seed solution w_0 can be fixed by Eq.(8), which reads

$$\begin{aligned} w_0 = &- \frac{1}{12p_x}(12u_0^2 p_x + 12u_0 p_{xx} + p_t \\ &+ 4p_{xxx} + c_0(a_2 + a_3 p)). \end{aligned} \tag{11}$$

Finally, we obtain the variable-separated solution of HBKs

$$u = \frac{(a_1 + a_3 q)p_x}{a_0 + a_1 p + a_2 q + a_3 pq} + u_0, \tag{12}$$

$$v = \frac{(a_0 a_3 - a_1 a_2)p_x q_y}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \tag{13}$$

$$\begin{aligned} w = &\frac{(a_1 + a_3 q)p_{xx}}{a_0 + a_1 p + a_2 q + a_3 pq} \\ &- \frac{(a_1 + a_3 q)^2 p_x^2}{(a_0 + a_1 p + a_2 q + a_3 pq)^2} + w_0, \end{aligned} \tag{14}$$

where p, u_0 are arbitrary functions of x and t, q is determined by Eq.(10) and w_0 is expressed by Eq.(11).

Here, one of the most important things may be that the physical quantity v is given by Eq.(13). Because of the arbitrariness of $p(x,t)$ and $q(y,t)$ (which contains an arbitrary function $Q(y)$) in Eq.(13), the physical quantity v obviously possesses quite rich structures. Now an important problem is whether we can find some new types of soliton structures which possess fractal localized excitations for the HBK system, i.e. the fractal localized excitations of the HBK system. The answer is apparently positive, since $p(x,t)$, $Q(y)$ are arbitrary functions. Here, we cite some interesting possible fractal patterns which are just related to several *Jacobian* elliptic functions. For simplicity, we set $a_0 = a_2 = a_3 = c_0 = 1, a_1 = 0$ in Eq.(13).

(1) Regular fractal dromions

In a (2+1)-dimensional system, one of the most important basic excitations is the so-called dromion, which is exponentially localized in all directions. Recently, it has been found that many lower-dimensional piecewise smooth functions with fractal structure can be used to construct exact localized solutions of the higher-dimensional soliton systems which also possess fractal structures. This situation also occurs in the HBKs. If we appropriately select the arbitrary functions p and q, we can find that some special types of fractal dromions for the physical quantity v can be revealed. For example, if we take

$$p(x,t) = 1 + \mathrm{e}^{[(x+at)\mathrm{sn}(\ln(x^2),m)]}, \tag{15}$$

$$q(y,t) = \{1 + \mathrm{e}^{[y\mathrm{sn}(\ln(y^2),m)}\}\mathrm{e}^t, \tag{16}$$

(where a is an arbitrary constant in Eq.(15), then a simple fractal dromion can be obtained. Figure 1(a) shows a plot of this special type of fractal dromion structure for the physical quantity v given by Eq.(13) under conditions (15) and (16). Figure 1(b) shows the density of the fractal structure of the dromion in the region $\{x = [-0.015, 0.015],\ y = [-0.015, 0.015]\}$. To observe the self-similar structure of the fractal dromion clearly, we enlarge a small region near the centre of Fig.1(b). We reduce the region of Fig.1(b) to $\{x = [-0.005, 0.005],\ y = [-0.005, 0.005]\}$, $\{x = [-0.001, 0.001],\ y = [-0.001, 0.001]\}$ and so on. A totally similar structure to that presented in Fig.1(b) can be found.

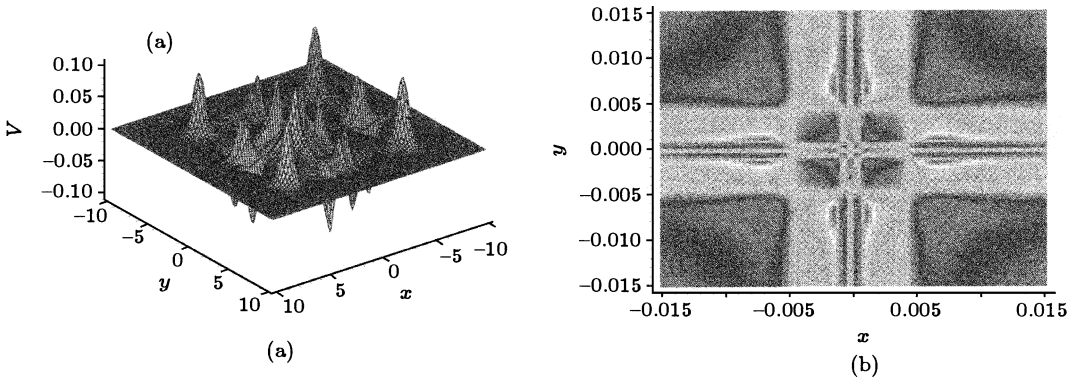


Fig.1. (a)Fractal dromion structure for the physical quantity v under conditions (15) and (16) and $t = 0$, $m = 0.5$. (b)Density of the fractal structures of dromion in the region $\{x = [-0.015, 0.015],\ y = [-0.015, 0.015]\}$.

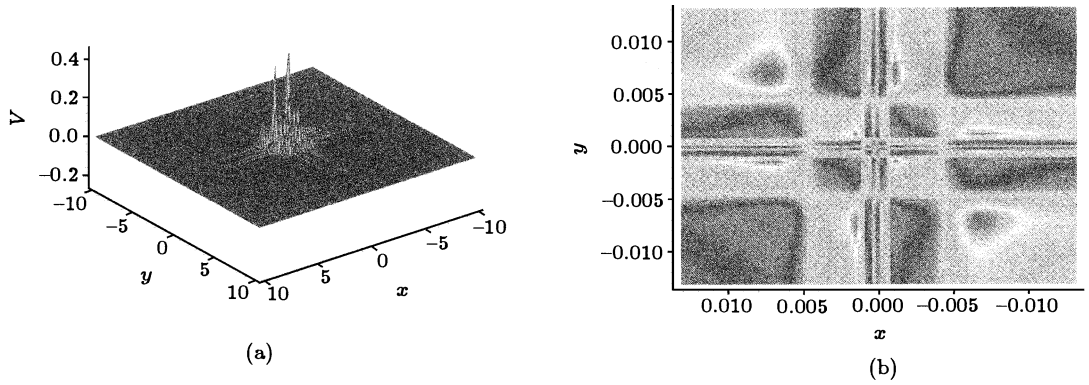


Fig.2. (a)Fractal lump structure for the physical quantity v under conditions (17) and (18) and $t = 0$, $m = 0.5$. (b)Density of the fractal structures of lump in the region $\{x = [-0.013, 0.013],\ y = [-0.013, 0.013]\}$.

(2) Regular fractal lumps

It is well known that lump solution has been formed by rational functions. This localized coherent soliton structure is another type of significantly localized excitation. If we select functions p and q of the physical quantity v in Eq.(13) appropriately, we can find some types of lump solutions with fractal behaviours. Figure 2(a) shows a fractal lump structure for the physical quantity v given by Eq.(13), where p and q in solution (13) are selected as

$$p(x,t) = 1 + \frac{|x+bt|}{1+x^4}[x - \mathrm{sn}(\ln(x^2), m)], \qquad (17)$$

$$q(y,t) = \left\{1 + \frac{|y|}{1+y^4}[y - \mathrm{sn}(\ln(y^2), m)]\right\}\mathrm{e}^t, \qquad (18)$$

(where b is an arbitrary constant in Eq.(17). From Fig.2(a), we can see that the solution is localized in all directions. Near the centre there are infinitely many peaks which are distributed in a fractal manner. Figure 2(b) presents a density plot of the structure of the fractal lump in the region $\{x = [-0.013, 0.013],\ y = [-0.013, 0.013]\}$. More detailed studies can show the self-similar structure of the lump. We reduce the region of Fig.2(b) to $\{x = [-0.0045, 0.0045],\ y = [-0.0045, 0.0045]\}$, $\{x = [-0.001, 0.001],\ y = [-0.001, 0.001]\}$ and so on. A totally similar structure

to that plotted in Fig.2(b) can be found.

In conclusion, through introducing a Bäcklund transformation of the HBK system, we have obtained a rather general variable-separated solution. From this, by selecting several appropriate functions, we have found some new localized excitations which possess fractal behaviours. Because of the importance of *Jacobian* functions in nonlinear science, we are sure that these new fractal behaviours would be significant. As is known, fractals not only belong to the realms of mathematics or computer graphics, but also exist nearly everywhere in nature, such as in fluid turbulence, crystal growth patterns, human veins, fern shapes, galaxy clustering, cloud structures and in numerous other examples. Traditionally, fractals are the opposite circumstances to solitons in nonlinear science since solitons are the representatives of integrable systems while fractals represent the behaviours of nonintegrable systems. However, in this paper, we have found some fractal structures for localized solutions of the (2+1)-dimensional integrable HBKs. Why do these localized excitations possess such kinds of fractal behaviours? If one considers the boundary and/or initial conditions of the fractal solutions obtained here, one can find that the initial and/or boundary conditions possess the fractal property. This means that these fractal behaviours of the localized excitations of the integrable models come from the nonintegrable boundary and/or initial conditions. Due to the wide applications of the soliton and fractal theory in many physics fields such as fluid dynamics, plasma physics, field theory, optics, etc., it is a significant work to learn more about the localized fractal excitations of the 'integrable' physical model with 'nonintegrable' boundary and/or initial conditions such as fractal boundary and/or initial conditions.

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