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New families of non-travelling wave solutions to the (2+1)-dimensional modified dispersive water-wave system*

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In this paper, we introduce a further generalized projective Riccati equation method and apply it to solve the (2+1)-dimensional modified dispersive water-wave system. Many new types of non-travelling wave solutions are obtained for this system.

Keywords: projective Riccati equation method, (2+1)-dimensional modified dispersive water-wave system, non-travelling wave solution

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1. Introduction

It is well known that many dynamic problems in physics and other fields are usually characterized by nonlinear partial differential equations (NLPDEs) of evolution.[1] To understand the physical mechanism of these problems one has to study the solutions to the associated equations. Looking for analytical solutions to nonlinear physical models has long been a major concern for both mathematicians and physicists since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Much work has been done over the last years on the subject of obtaining special solutions of a nonlinear partial differential equation(PDE). Some of the most important methods are the inverse scattering transformation (IST),[2] bilinear method,[3] Bäcklund and Darboux transformations,[4−6] tanh method,[7] sine-cosine method,[8] extended tanh-function method[9−10] and so on.

In 1992, Conte and Musette[11] presented a new method to seek more new solitary wave solutions of NLPDEs that can be expressed as a polynomial of two elementary functions which satisfy a projective Riccati equation.[12] The method has been applied to find many solitary wave solutions of many equations.[11,13] Recently, Yan[14] improved this method and obtained more types of solutions which include not only the results obtained by using the above-mentioned method but also other types of solutions such as singular solitary wave solutions, periodic wave solutions and rational solutions. More recently, Fu *et al*[15] gave a more detailed discussion on this method, and obtained a large number of new kinds of solutions.

In this paper, we further generalize this method, and apply it to (2+1)-dimensional nonlinear evolution equations. By using the generalized method, we derive some new travelling wave as well as non-travelling wave solutions to the equations.

This paper is organized as follows: In Section 2, we describe briefly the further generalized projective Riccati equation method. In Section 3, we apply the method to the (2+1)-dimensional modified dispersive water-wave system and bring out many new travelling wave as well as non-travelling wave solutions to the equations. In the last section, we summarize our results and give some comments and discussions.

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2. The further generalized projective Riccati equation method

Here we briefly describe the further generalized projective Riccati equation method.

For a given nonlinear evolution equation in three variables x, y, and t,

$$E(u, u_t, u_x, u_y, u_{xx}, u_{xt}, u_{xy}, \ldots,) = 0, \tag{1}$$

we seek the following formal solutions of the given equation by the ansätz,

$$u(x,y,t) = A_0 + \sum_{i=1}^{n} F^{i-1}(\xi)[A_i F(\xi) + B_i G(\xi)],$$
$$A_n^2 + B_n^2 \neq 0, \tag{2}$$

where n is an integer to be determined by balancing the highest order derivative terms with the nonlinear terms in Eq.(1), while $A_0 = A_0(x,y,t)$, $A_i = A_i(x,y,t), B_i = B_i(x,y,t)$, $(i = 1,2,\ldots,n)$, and $\xi = \xi(x,y,t)$ are all differentiable functions, and F, G satisfy

$$F'(\xi) = pF(\xi)G(\xi), \tag{3}$$

$$G'(\xi) = q + pG^2(\xi) - rF(\xi), \tag{4}$$

where $p \neq 0$ is a real constant, q and r are two real constants. When $p = -1$ and $q = 1$, Eqs.(3) and (4) reduce to the coupled equations given in Refs.[11,13], and when $p = \pm 1$ and $q \geq 0$, Eqs.(3) and (4) reduce to the coupled equations given in Ref[14].

For Eqs.(3) and (4), there is a relation between F and G,

$$G^2 = -\frac{1}{p}[q - 2rF + \frac{r^2 + \delta}{q}F^2], \tag{5}$$

where $\delta = \pm 1$.

Eqs.(3) and (4) have the following general solutions:

Case 1. $q \neq 0$, $pq < 0$

$$F_1(\xi) = \frac{q}{r + \sinh(\sqrt{-pq}\xi)},$$
$$G_1(\xi) = -\frac{1}{p} \times \frac{\sqrt{-pq}\cosh(\sqrt{-pq}\xi)}{r + \sinh(\sqrt{-pq}\xi)}, \ (\delta = 1), \tag{6}$$

$$F_2(\xi) = \frac{q}{r + \cosh(\sqrt{-pq}\xi)},$$
$$G_2(\xi) = -\frac{1}{p} \times \frac{\sqrt{-pq}\sinh(\sqrt{-pq}\xi)}{r + \cosh(\sqrt{-pq}\xi)}, \ (\delta = -1). \tag{7}$$

Case 2. $q \neq 0$, $pq > 0$

$$F_3(\xi) = \frac{q}{r + \sin(\sqrt{pq}\xi)},$$
$$G_3 = -\frac{1}{p} \times \frac{\sqrt{pq}\cos(\sqrt{pq}\xi)}{r + \sin(\sqrt{pq}\xi)}, \ (\delta = -1), \tag{8}$$

$$F_4(\xi) = \frac{q}{r + \cos(\sqrt{pq}\xi)},$$
$$G_4 = \frac{1}{p} \times \frac{\sqrt{pq}\sin(\sqrt{pq}\xi)}{r + \cos(\sqrt{pq}\xi)}, \ (\delta = -1). \tag{9}$$

Case 3. $q = 0$

$$F_5(\xi) = \frac{1}{\frac{pr}{2}\xi^2 + a_1\xi + a_0},$$
$$G_5(\xi) = -\frac{1}{p} \times \frac{pr\xi + a_1}{\frac{pr}{2}\xi^2 + a_1\xi + a_0}. \tag{10}$$

It is easy to see that the ansätz (2) is more general. By using this further generalized projective Riccati equation method, one can derive the travelling wave as well as non-travelling wave solutions to the equations.

3. New solutions to the (2+1)-dimensional modified dispersive water-wave system

Let us consider the (2+1)-dimensional modified dispersive water-wave system[16]

$$u_{yt} + u_{xxy} - 2v_{xx} - (u^2)_{xy} = 0,$$
$$v_t - v_{xx} - 2(uv)_x = 0, \tag{11}$$

which was used to model the nonlinear and dispersive, long gravity waves travelling in two horizontal directions on shallow water of uniform depth, and can also be derived from the celebrated Kadomtsev–Petviashvili (KP) equation by the symmetry constraint.[17] It is worth mentioning that the system has been widely applied in many branches of physics, such as plasma physics, fluid dynamics, nonlinear optics, etc. So a good understanding of more solutions of system (11) is very helpful, especially for coastal and civil engineers to apply the nonlinear water model to harbour and coastal design. Meanwhile, finding more types of solutions of system (11) is of fundamental interest in fluid dynamics. In Ref.[18], abundant structures of the localized solution of this system was obtained by using the variable separation approach.

To solve the (2+1)-dimensional modified dispersive water-wave system, we take the following

Bäcklund transformations of Eqs.(11):

$$u = \frac{f_x}{f} + u_0(x,t), v = \frac{f_{xy}}{f} - \frac{f_x f_y}{f^2}, \tag{12}$$

which can be obtained from the standard Painlevé truncation expansion with $u_0(x,t)$, an arbitrary function of x,t.

It is easy to deduce from (12) that

$$v = u_y, \tag{13}$$

which leads to a simple transformation from u to v. Substituting (13) into system (11), we can change system (11) into a single differential equation:

$$u_{yt} - (u^2)_{xy} - u_{xxy} = 0, \tag{14}$$

which is equivalent to

$$u_t - u_{xx} - 2uu_x = R(x,t), \tag{15}$$

where $R(x,t)$ is an arbitrary function of the indicated variables. In what follows, we only consider the case $R(x,t) = R(t)$.

It is easy to find that solutions of Eq.(15) are also solutions of Eq.(14), so we can find solutions of Eqs.(11) by solving Eq.(15).

To seek for the solutions of Eqs.(15), according to the method described in Section 2, we assume that

$$\begin{aligned} u(x,y,t) =& f(y,t) + A(y)F[\xi(x,y,t)] \\ &+ B(y)G[\xi(x,y,t)], \end{aligned} \tag{16}$$

where $A_0(x,y,t) = f(y,t)$, $A_1(x,y,t) = A(y)$, $B_1(x,y,t) = B(y)$, $\xi(x,y,t) = H(y)x + K(y,t)$. Substituting (16) into (15) along with (3)–(5), we have

$$\begin{aligned} & u_t - u_{xx} - 2uu_x - R(t) \\ =& \{(4AHB + 2pAH^2)(\delta + r^2)\}\frac{N^3}{q} \\ &+ \{(2HB^2 + 2pH^2B)(\delta + r^2) - 2pqHA^2\}\frac{MN^2}{q} \\ &- \{qrABH + (BK_t - 2fBH)(\delta + r^2) \\ &+ 3pqrAH^2\}\frac{N^2}{q} \\ &- \{2pfAH - pAK_t + 2rB^2H + prBH^2\}MN \\ &+ \{pqAH^2 + rBK_t + 2qBHA - 2rfBH\}N \\ &+ f_t - R(t) = 0, \end{aligned} \tag{17}$$

which leads to a set of first order ordinary differential equations:

$$\begin{aligned} & (4AHB + 2pAH^2)(\delta + r^2) = 0, \\ & (2HB^2 + 2pH^2B)(\delta + r^2) - 2pqHA^2 = 0, \\ & qrABH + (BK_t - 2fBH)(\delta + r^2) + 3pqrAH^2 = 0, \\ & 2pfAH - pAK_t + 2rB^2H + prBH^2 = 0, \\ & pqAH^2 + rBK_t + 2qBHA - 2rfBH = 0, \\ & f_t - R(t) = 0. \end{aligned} \tag{18}$$

Solving this set of equations, we choose $B(y)$, $F_1(y)$, $F_2(y)$ as arbitrary functions and find that:

For $\delta = 1$,

$$\begin{aligned} K(y,t) =& \int -4\frac{B(y)\int R(t)\mathrm{d}t}{p}\mathrm{d}t \\ & - 4\frac{B(y)F_1(y)t}{p} + F_2(y), B(y) = B(y), \\ A(y) =& \pm \frac{\sqrt{-qp(r^2+1)}B(y)}{pq}, \\ H(y) =& -2\frac{B(y)}{p}, \\ f(y,t) =& \int R(t)\mathrm{d}t + F_1(y). \end{aligned} \tag{19}$$

For $\delta = -1$,

$$\begin{aligned} K(y,t) =& \int -4\frac{B(y)\int R(t)\mathrm{d}t}{p}\mathrm{d}t \\ & - 4\frac{B(y)F_1(y)t}{p} + F_2(y), B(y) = B(y), \\ A(y) =& \pm \frac{\sqrt{-qp(r^2-1)}B(y)}{pq}, \\ H(y) =& -2\frac{B(y)}{p}, \\ f(y,t) =& \int R(t)\mathrm{d}t + F_1(y). \end{aligned} \tag{20}$$

Substituting (19) and (20) into (16) and using (13), (6)–(9), we derive the following solutions of system (11):

Case 1. $pq < 0$, $\delta = 1$

$$\begin{aligned} u_{11}(x,y,t) =& f(y,t) + \frac{qA(y)}{r + \sinh(\sqrt{-pq}\xi)} \\ & - \frac{B(y)}{p} \times \frac{\sqrt{-pq}\cosh(\sqrt{-pq}\xi)}{r + \sinh(\sqrt{-pq}\xi)}, \\ v_{11}(x,y,t) =& u_{11y}(x,y,t), \end{aligned} \tag{21}$$

where $\xi(x,y,t) = H(y)x + K(y,t)$, $f(y,t)$, $A(y)$, $B(y)$, $H(y)$ and $K(y,t)$ are determined by (19).

Case 2. $pq<0$, $\delta=-1$, $|r|>1$

$$\begin{aligned}u_{21}(x,y,t)=&f(y,t)+\frac{qA(y)}{r+\cosh(\sqrt{-pq}\xi)}\\&-\frac{B(y)}{p}\times\frac{\sqrt{-pq}\sinh(\sqrt{-pq}\xi)}{r+\cosh(\sqrt{-pq}\xi)},\\v_{21}(x,y,t)=&u_{21y}(x,y,t).\end{aligned}\tag{22}$$

$pq>0$, $\delta=-1$, $0<|r|<1$

$$\begin{aligned}u_{22}(x,y,t)=&f(y,t)+\frac{qA(y)}{r+\sin(\sqrt{pq}\xi)}\\&-\frac{B(y)}{p}\times\frac{\sqrt{pq}\cos(\sqrt{pq}\xi)}{r+\sin(\sqrt{pq}\xi)},\\v_{22}(x,y,t)=&u_{22y}(x,y,t),\end{aligned}\tag{23}$$

$$\begin{aligned}u_{23}(x,y,t)=&f(y,t)+\frac{qA(y)}{r+\cos(\sqrt{pq}\xi)}\\&+\frac{B(y)}{p}\times\frac{\sqrt{pq}\sin(\sqrt{pq}\xi)}{r+\cos(\sqrt{pq}\xi)},\\v_{23}(x,y,t)=&u_{23y}(x,y,t),\end{aligned}\tag{24}$$

where $\xi(x,y,t)=H(y)x+K(y,t)$, $f(y,t)$, $A(y)$, $B(y)$, $H(y)$ and $K(y,t)$ are determined by (20).

In addition, when we take $r=0$ in (17), we have

$$\begin{aligned}&u_t-u_{xx}-2uu_x-R(t)\\=&\{2\delta(4AHB+2pAH^2)\}\frac{N^3}{q}\\&+\{\delta(2HB^2+2pH^2B)-2pqHA^2\}\frac{MN^2}{q}\\&-\{\delta(BK_t-2fBH)\}\frac{N^2}{q}\\&-\{2pfAH-pAK_t\}MN\\&+\{pqAH^2+2qBHA\}N\\&+f_t-R(t)=0,\end{aligned}\tag{25}$$

and a set of first order ordinary differential equations:

$$\begin{aligned}&2\delta(4AHB+2pAH^2)=0,\\&\delta(2HB^2+2pH^2B)-2pqHA^2=0,\\&\delta(BK_t-2fBH)=0,\\&2pfAH-pAK_t=0,\\&pqAH^2+2qBHA=0,\\&f_t-R(t)=0.\end{aligned}\tag{26}$$

Solving this set of equations, we choose $H(y)$, $F_1(y)$, $F_2(y)$ as arbitrary functions and find that:

For $r=0$, $\delta=1$,

$$\begin{aligned}K(y,t)=&2H(y)\int\int R(t)\mathrm{d}t\mathrm{d}t\\&+2H(y)F_1(y)t+F_2(y),\\A(y)=&\pm\frac{\sqrt{-qp}H(y)}{2q},\\B(y)=&-\frac{1}{2}pH(y),\\f(y,t)=&\int R(t)\mathrm{d}t+F_1(y),\\H(y)=&H(y);\end{aligned}\tag{27}$$

and

$$\begin{aligned}K(y,t)=&2H(y)\int\int R(t)\mathrm{d}t\mathrm{d}t\\&+2H(y)F_1(y)t+F_2(y),\\A(y)=&0,\\B(y)=&-pH(y),\\f(y,t)=&\int R(t)\mathrm{d}t+F_1(y),\\H(y)=&H(y).\end{aligned}\tag{28}$$

For $r=0$, $\delta=-1$,

$$\begin{aligned}K(y,t)=&2H(y)\int\int R(t)\mathrm{d}t\mathrm{d}t\\&+2H(y)F_1(y)t+F_2(y),\\A(y)=&\pm\frac{\sqrt{qp}H(y)}{2q},\\B(y)=&-\frac{1}{2}pH(y),\\f(y,t)=&\int R(t)\mathrm{d}t+F_1(y),\\H(y)=&H(y);\end{aligned}\tag{29}$$

and

$$\begin{aligned}K(y,t)=&2H(y)\int\int R(t)\mathrm{d}t\mathrm{d}t\\&+2H(y)F_1(y)t+F_2(y),\\A(y)=&0,\\B(y)=&-pH(y),\\f(y,t)=&\int R(t)\mathrm{d}t+F_1(y),\\H(y)=&H(y).\end{aligned}\tag{30}$$

Substituting (27)–(30) into (16) and using (13), (6)–(9), we derive the following solutions of system (11):

Case 3. $pq<0$, $r=0$, $\delta=1$

$$\begin{aligned}u_{31}(x,y,t)=&f(y,t)+qA(y)\mathrm{sech}(\sqrt{-pq}\xi)\\&-\frac{\sqrt{-pq}B(y)}{p}\tanh(\sqrt{-pq}\xi),\\v_{31}(x,y,t)=&u_{31y}(x,y,t),\end{aligned}\tag{31}$$

where $\xi(x,y,t) = H(y)x + K(y,t)$, $f(y,t)$, $A(y)$, $B(y)$, $H(y)$ and $K(y,t)$ are determined by (27);

$$u_{32}(x,y,t) = f(y,t) - \frac{\sqrt{-pq}B(y)}{p}\tanh(\sqrt{-pq}\xi),$$
$$v_{32}(x,y,t) = u_{32y}(x,y,t), \quad (32)$$

where $\xi(x,y,t) = H(y)x + K(y,t)$, $f(y,t)$, $B(y)$, $H(y)$ and $K(y,t)$ are determined by (28).

Case 4. $pq < 0$, $r = 0$, $\delta = -1$

$$u_{41}(x,y,t) = f(y,t) - \frac{\sqrt{-pq}B(y)}{p}\coth(\sqrt{-pq}\xi),$$
$$v_{41}(x,y,t) = u_{41y}, \quad (33)$$

where $\xi(x,y,t) = H(y)x + K(y,t)$, $f(y,t)$, $B(y)$, $H(y)$ and $K(y,t)$ are determined by (30).

$pq > 0$, $r = 0$, $\delta = -1$

$$u_{42}(x,y,t) = f(y,t) + qA(y)\csc(\sqrt{pq}\xi) - \frac{\sqrt{pq}B(y)}{p}\cot(\sqrt{pq}\xi),$$
$$v_{42}(x,y,t) = u_{42y}, \quad (34)$$

$$u_{43}(x,y,t) = f(y,t) + qA(y)\sec(\sqrt{pq}\xi) + \frac{\sqrt{pq}B(y)}{p}\tan(\sqrt{pq}\xi),$$
$$v_{43}(x,y,t) = u_{43y}, \quad (35)$$

where $\xi(x,y,t) = H(y)x + K(y,t)$, $f(y,t)$, $A(y)$, $B(y)$, $H(y)$ and $K(y,t)$ are determined by(29).

$$u_{44}(x,y,t) = f(y,t) - \frac{\sqrt{pq}B(y)}{p}\cot(\sqrt{pq}\xi),$$
$$v_{44}(x,y,t) = u_{44y}, \quad (36)$$

$$u_{45}(x,y,t) = f(y,t + \frac{\sqrt{pq}B(y)}{p}\tan(\sqrt{pq}\xi),$$
$$v_{45}(x,y,t) = u_{45y}, \quad (37)$$

where $\xi(x,y,t) = H(y)x + K(y,t)$, $f(y,t)$, $B(y)$, $H(y)$ and $K(y,t)$ are determined by(30).

Obviously, the solutions (32)–(37) can be obtained by the method given in Ref.[19]. But the other solutions presented in this paper cannot be obtained by the methods in Refs.[2–15] and [19], these solutions are new types of non-travelling wave solutions. In addition, when we take the coefficient functions in the ansätz (16) to be constant, we derive the solutions that can be obtained in Refs.[11, 13, 14, 15]. By the generalized method, the travelling wave as well as non-travelling wave solutions to the equations can be derived.

4. Conclusion and discussion

In summary, in this paper, we introduce a further generalized projective Riccati equation method and apply it to solve the problem of (2+1)-dimensional modified dispersive water-wave system. Many new types of travelling and non-travelling wave solutions are obtained for this system. It is obvious that our method can be also applied to other (2+1)-dimensional or (3+1)-dimensional nonlinear wave equations.

References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[2] Gardner C S, Green J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095
[3] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[4] Lamb G L 1971 *Rev. Mod. Phys.* **43** 99
[5] Wadati M, Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 419
[6] Gu C H *et al* 1999 *Darboux Transformation in Soliton Theory and its Geometric Applications* (Shanghai: Shanghai Scientific and Technical Publishers)
[7] Malfliet W 1992 *Amer. J. Phys.* **60** 650
[8] Yan C T 1996 *Phys. Lett.* A **224** 77
[9] Fan E 2000 *Phys. Lett.* A **277** 212
[10] Fan E 2001 *Z. Naturforsch.* **56**A 312
[11] Conte R and Musette M 1992 *J. Phys.* A **25** 5609
[12] Bountis T C *et al* 1986 *J. Math. Phys.* **27** 1215
[13] Zhang G X, Li Z B and Duan Y S 2002 *Sci. Chin.* A **30** 1103
[14] Yan Z Y 2003 *Chaos, Solitons Fractals* **16** 759
[15] Fu Z *et al* 2004 *Chaos, Solitons Fractals* **20** 301
[16] Durovsky V G and Konopelchenko E G 1994 *J. Phys.* A **27** 4619
[17] Lou S Y and Wu X B 1997 *J. Math. Phys.* **38** 6401
[18] Zheng C L 2003 *Commun. Theor. Phys.* **40** 25
[19] Li D S and Zhang H Q 2004 *Chin. Phys.* **13** 984