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A new class of quantum states generated by m-fold application of Bogoliubov's transformation*

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By applying higher powers of the Bogoliubov transformation operator $b^\dagger = \nu^* a + \mu^* a^\dagger$ to the two-photon coherent states (or minimum uncertainty squeezed states) we construct a new type of quantum state which we call the generalized excited two-photon coherent states. Analytic expressions for the quantum statistical properties are derived, and through numerical computation the phase space quasi-probability distributions are found. These states can exhibit highly nonclassical behaviour depending on the degree of excitation m and other parameters. For particular values of two parameters λ and ρ, these generalized states reduce to other classes of coherent states formerly reported. Our theory thus presents a much broader approach to these types of quantum states.

Keywords: generalized excited two-photon coherent states, pseudo-creation operator, nonclassical statistical properties

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1. Introduction

Many different types of nonclassical states in quantum optics have been created and discussed in the past years, and they continue to be of great interest due to their scientific significance as well as potential application values.[1−4] Some of these states were obtained by finding the eigenstates of certain operators,[5−10] and some by exciting other quantum states.[11−14] In recent years superpositions of quantum states[15] and multimode quantum states[16] have been investigated as important types of nonclassical fields. Squeezed states,[17] Fock states,[18] and various coherent states derived from them[19] have already been realized experimentally, while schemes to generate other quantum states have been proposed.[20−23] These states are of special interest since they exhibit quite significant and unusual physical properties. In particular, the generation of large quantum states is a new and important part of quantum optics since they manifest highly nonclassical behaviour and yet begin to approach the realm of mesoscopic physics. Through a more general theoretical treatment we define a new class of quantum states that is broader than the excited two-photon coherent states.[13] They are obtained by repeated application of the Bogoliubov transformation operator $b^\dagger$, also known as the quasi-creation operator, on its eigenstate (these minimum uncertainty squeezed states were called two-photon coherent states by Yuen[6]), to obtain what we shall call the generalized excited two-photon coherent (GETC) states. Two-photon coherent states can be generated by two-photon parametric down conversion processes in media with second-order optical nonlinearity.[24] In Sec.2, we first give the necessary mathematical formulae for their derivation. We then make use of the photon number matrix elements of the correlated two-photon coherent states, and obtain expressions for their quantum properties. In Sec.3 we discuss the characteristic properties such as squeezing and sub-Poissonian distribution. We also examine the nonclassical behaviour of several quasiprobability distributions, and introduce a set of combination parameters μ' and ν' to distinguish between these GETC states and other nonclassical states in Sec.4. Our results show that this new class of quantum states ex-

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hibits remarkable nonclassical characteristics different from those of previously known states.[5,6,11,13]

2. Generalized excited two-photon coherent states

For a single mode radiation field, the minimum uncertainty two-photon coherent states $|\beta,\mu,\nu\rangle$ are defined as the eigenstates of the operator $b(\mu,\nu)$ with the eigenvalue β, that satisfy the eigenvalue equation[6]

$$b(\mu,\nu)|\beta,\mu,\nu\rangle=\beta|\beta,\mu,\nu\rangle. \tag{1}$$

The Bogoliubov transformation operator b is a linear transformation of the annihilation and creation operators, defined by

$$b=b(\mu,\nu)=\mu a+\nu a^{\dagger}, \tag{2}$$

where $a(a^{\dagger})$ is the boson annihilation (creation) operator and the complex parameters μ and ν obey $|\mu|^2-|\nu|^2=1$, $\mu,\nu\in C$. The basic operators b and $b^{\dagger}$ satisfy the same canonical commutation relation as $a,a^{\dagger}$, i.e.

$$[b(\mu,\nu),b^{\dagger}(\mu^*,\nu^*)]=1. \tag{3}$$

The Bogoliubov transformation operator b was first used by Bogoliubov in solid state physics[25] and later by Yuen to derive his two-photon coherent states. We now reconstruct a modified Bogoliubov transformation operator b by replacing μ with μ' and ν with ν', so that

$$b(\mu',\nu')=\mu' a+\nu' a^{\dagger}, \tag{4}$$

with

$$|\mu'|^2-|\nu'|^2=1,\mu',\nu'\in C. \tag{5}$$

The Hermitian adjoint operator $b^{\dagger}(\mu',\nu')$ is the quasi-creation operator

$$b^{\dagger}(\mu'^*,\nu'^*)=\nu'^* a+\mu'^* a^{\dagger}, \tag{6}$$

and the canonical commutation relation is also satisfied:

$$[b(\mu',\nu'),b^{\dagger}(\mu'^*,\nu'^*)]=1. \tag{7}$$

Defining the parameters λ and ρ such that

$$\lambda=\mu'\mu^*-\nu'\nu^*,\rho=\mu\nu'-\mu'\nu, \tag{8}$$

and

$$|\lambda|^2-|\rho|^2=1, \tag{9}$$

from Eqs.(4)–(7) we rewrite Eq.(4) as

$$b(\mu',\nu')=\lambda b(\mu,\nu)+\rho b^{\dagger}(\mu^*,\nu^*), \tag{10}$$

while the commutator in (7) is still invariant.

We now apply the Bogoliubov transformation operator $b^{\dagger}(\mu'^*,\nu'^*)$ m-times to the two-photon coherent states based on the above representation and obtain a new type of quantum state, the generalized excited two-photon coherent states $|\beta,m,\mu',\nu'\rangle_{\rm GETC}$. This may be written as

$$\begin{aligned}&|\beta,m,\mu',\nu'\rangle_{\rm GETC}\\ =&\frac{b^{\dagger m}(\mu'^*,\nu'^*)|\beta,\mu,\nu\rangle}{\sqrt{\langle\beta,\mu,\nu|b^m(\mu',\nu')b^{\dagger m}(\mu'^*,\nu'^*)|\beta,\mu,\nu\rangle}},\end{aligned} \tag{11}$$

where m is a non-negative integer. It is easy to see that the GETC states for any complex numbers μ' and ν' encompass quite a broad class of radiation states. When $\mu'=\mu$ and $\nu'=\nu$, corresponding to $\lambda=1$ and $\rho=0$, the state reduces to

$$\begin{aligned}&|\beta,m,\mu'=\mu,\nu'=\nu\rangle_{\rm GETC}\\ =&\frac{b^{\dagger m}(\mu^*,\nu^*)|\beta,\mu,\nu\rangle}{\sqrt{\langle\beta,\mu,\nu|b^m(\mu,\nu)b^{\dagger m}(\mu^*,\nu^*)|\beta,\mu,\nu\rangle}},\end{aligned} \tag{12}$$

which is just the excited coherent state (see Ref.[13]). When $\mu'=1,\nu'=0$, the state $|\beta,m,\mu',\nu'\rangle_{\rm GETC}$ is reduced to the excited two-photon coherent state[13]

$$\begin{aligned}&|\beta,m,\mu'=1,\nu'=0\rangle_{\rm GETC}\\ =&\frac{a^{\dagger m}|\beta,\mu,\nu\rangle}{\sqrt{\langle\beta,\mu,\nu|a^m a^{\dagger m}|\beta,\mu,\nu\rangle}}.\end{aligned} \tag{13}$$

When $\mu=1,\nu=0$, corresponding to $\lambda=\mu',\rho=\nu'$, and relabelling μ' and ν' as μ and ν, the state $|\beta,m,\mu',\nu'\rangle_{\rm GETC}$ reduces to the generalized excited coherent state.[13]

Considering Eq.(11) for the GETC states, we first need to derive some necessary mathematical formulae and numerical quantities for later discussion. From Eqs.(8), (9) and (10), and the method used in Refs.[11,13,26], we obtain

$$\begin{aligned}b^{\dagger^m}(\mu'^*,\nu'^*)=&[\rho^* b(\mu,\nu)+\lambda^* b^{\dagger}(\mu^*,\nu^*)]^m\\ =&\sum_{k=0}^{[\frac{m}{2}]}\sum_{M=0}^{m-2k}d^m_{M,m-2k-M}(\rho^*,\lambda^*)\\ &\times[b^{\dagger}(\mu^*,\nu^*)]^M[b(\mu,\nu)]^{m-2k-M},\end{aligned} \tag{14}$$

where $d^m_{M,N}(\rho^*,\lambda^*)$ is a c-number factor defined by

$$\begin{aligned}d^m_{M,N}(\rho^*,\lambda^*)=&\frac{m!}{M!N!(m-M-N)!!}\\ &\times(\rho^*)^{\frac{1}{2}(m-M+N)}(\lambda^*)^{\frac{1}{2}(m+M-N)}.\end{aligned} \tag{15}$$

We utilize the matrix elements of the two-photon coherent states

$$A^{n,m}_{\xi,\eta}(\alpha^*,\beta) = \langle\alpha,\mu,\nu|b^n(\mu',\nu')b^{\xi}(\mu,\nu) \times b^{\dagger\eta}(\mu^{*},\nu^{*})b^{\dagger m}(\mu'^{*},\nu'^{*})|\beta,\mu,\nu\rangle, \quad (16)$$

where ξ and η are non-negative integers, and

$$A^{m,n}_{\eta,\xi}(\beta^*,\alpha) = (A^{n,m}_{\xi,\eta}(\alpha^*,\beta))^*. \quad (17)$$

For convenience we hereafter denote $A^{m,m}_{0,0}(\beta^*,\beta)$ simply by $A^m(\beta)$. From Eqs.(14)–(16) and (1) we obtain

$$A^{n,m}_{\xi,\eta}(\alpha^*,\beta) = \langle\alpha,\mu,\nu|\beta,\mu,\nu\rangle \times \sum_{s=0}^{m}\sum_{t=0}^{n} F^{m+\eta-s}_{n+\xi-t}(\beta,\alpha^*,-\alpha^*\beta) \times R^m_s(\lambda^*,\rho^*,\beta)S^n_t(\lambda,\rho,\alpha^*), \quad (18)$$

where

$$F^p_q(\beta,\alpha^*,-\alpha^*\beta) = \begin{cases} p!\,\beta^{(q-p)}L^{(q-p)}_p(-\alpha^*\beta), q \geq p, \\ q!(\alpha^*)^{p-q}L^{(p-q)}_q(-\alpha^*\beta), q \leq p. \end{cases} \quad (19)$$

$$R^m_s(\lambda^*,\rho^*,\beta) = (\lambda^*)^m \binom{m}{s}\left(\mathrm{i}\sqrt{\frac{\rho^*}{\lambda^*}}\right)^s H_s\left(\frac{1}{\mathrm{i}}\sqrt{\frac{\rho^*}{\lambda^*}}\beta\right), \quad (20)$$

$$S^n_t(\lambda,\rho,\alpha^*) = \lambda^n \binom{n}{t}\left(\mathrm{i}\sqrt{\frac{\rho}{\lambda}}\right)^t H_t\left(\frac{1}{\mathrm{i}}\sqrt{\frac{\rho}{\lambda}}\alpha^*\right). \quad (21)$$

In deriving Eqs.(19)–(21), we have employed the associated Laguerre polynomial $L^{(k)}_m(x)$ and Hermite polynomial $H_m(x)$ of order m, and the normally ordered expression of the operator

$$b^n(\mu,\nu)b^{\dagger m}(\mu^*,\nu^*) = \begin{cases} n!m!\sum\limits_{l=0}^{m}\dfrac{b^{\dagger l}(\mu^*,\nu^*)b^{l+n-m}(\mu,\nu)}{(m-l)!l!(l+n-m)!}, n \geq m \\ n!m!\sum\limits_{l=0}^{n}\dfrac{b^{\dagger(l+m-n)}(\mu^*,\nu^*)b^{l}(\mu,\nu)}{(n-l)!l!(l+m-n)!}, n \leq m. \end{cases} \quad (22)$$

The scalar product of two GETC states is given by

$${}_{\mathrm{GETC}}\langle\alpha,n|\beta,m\rangle_{\mathrm{GETC}} = \frac{A^{n,m}_{0,0}(\alpha^*,\beta)}{\sqrt{A^n(\alpha)A^m(\beta)}}. \quad (23)$$

Below we shall discuss the statistical properties of these states.

3. Non-classical statistical properties

To discuss the quantum properties of the GETC states we consider the phase quadratures x and y of an arbitrary state $|\Psi\rangle$ of the quantum field

$$x = \frac{1}{2}(a\mathrm{e}^{\mathrm{i}\phi} + a^\dagger\mathrm{e}^{-\mathrm{i}\phi}) \quad y = \frac{1}{2\mathrm{i}}(a\mathrm{e}^{\mathrm{i}\phi} - a^\dagger\mathrm{e}^{-\mathrm{i}\phi}), \quad (24)$$

and their variances $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$ and $(\Delta y)^2 = \langle y^2\rangle - \langle y\rangle^2$. We also note that the operators a and b ($a^\dagger$ and $b^\dagger$) are mutually convertible through a unitary transformation. The phase quadratures satisfy the commutation relation $[x,y] = \mathrm{i}/2$, and the uncertainty relation of quantum mechanics $(\Delta x)^2(\Delta y)^2 \geq 1/16$. We say that the state is squeezed in the quadrature x (or y) if $(\Delta x)^2 \leq 1/4$ (or $(\Delta y)^2 \leq 1/4$)). For the GETC states the mean-square fluctuation $(\Delta x)^2$ of x (or $(\Delta y)^2$ of y) in the state $|\beta,\mu,\nu\rangle$ as a function of $|\beta|$ can be expressed analytically as

$$(\Delta x)^2_{\mathrm{GETC}} = \frac{1}{4(A^m(\beta))^2} \times \{\mathrm{e}^{2\mathrm{i}\phi}[B^{m,m}_{2,0}(\beta^*,\beta)A^m(\beta) - (B^{m,m}_{1,0}(\beta^*,\beta))^2] + \mathrm{e}^{-2\mathrm{i}\phi}[B^{m,m}_{0,2}(\beta^*,\beta)A^m(\beta) - (B^{m,m}_{0,1}(\beta^*,\beta))^2] + 2[B^{m,m}_{1,1}(\beta^*,\beta)A^m(\beta) - |B^{m,m}_{1,0}(\beta^*,\beta)|^2] - (A^m(\beta))^2\}, \quad (25)$$

where the functions $B(\beta)$ are a set of the sums of the matrix elements $A^m(\beta)$:

$$B^{m,m}_{1,0}(\beta^*,\beta) = \mu^* A^{m,m}_{1,0}(\beta^*,\beta) - \nu A^{m,m}_{0,1}(\beta^*,\beta), \quad (26)$$

$$B^{m,m}_{0,1}(\beta^*,\beta) = (B^{m,m}_{1,0}(\beta^*,\beta))^*, \quad (27)$$

$$B^{m,m}_{2,0}(\beta^*,\beta) = (\mu^*)^2 A^{m,m}_{2,0}(\beta^*,\beta) + \nu^2 A^{m,m}_{0,2}(\beta^*,\beta) - 2\mu^*\nu A^{m,m}_{1,1}(\beta^*,\beta) + \mu^*\nu A^m(\beta), \quad (28)$$

$$B^{m,m}_{0,2}(\beta^*,\beta) = (B^{m,m}_{2,0}(\beta^*,\beta))^*, \quad (29)$$

$$B^{m,m}_{1,1}(\beta^*,\beta) = -(\mu\nu)^* A^{m,m}_{2,0}(\beta^*,\beta) - (\mu\nu)A^{m,m}_{0,2}(\beta^*,\beta) + (2|\nu|^2+1)A^{m,m}_{1,1}(\beta^*,\beta) - |\nu|^2 A^m(\beta). \quad (30)$$

Figure 1 shows the results of numerical computation of $(\Delta x)^2_{\rm GETC}$ as a function of $|\beta|$ for different values of the parameters m, μ, ν, μ' and ν' for various phase values such that $\phi + \arg\beta = \pi$, where $\phi = \frac{1}{2}\arg\frac{\mu}{\nu}$, and $\arg\mu' = -\arg\nu'$. We see that, putting $m = 0$, $\nu = \nu' = 0$ we return to a coherent state[5]). We also see that the values of $(\Delta x)^2_{\rm GETC}$ for $m = 5$ in the cases of $\nu = \nu' = 0$ become the same as those of the photon-added coherent states of Agarwal and Tara.[11] For $\nu' = 0$, $\nu = -0.4$, and $m = 0$, Yuen's two-photon coherent states are recovered. When we take $m = 5$, $\nu' = 0$ and $\nu = -0.4$, we obtain the same results as Xin's excited two-photon coherent states.[13] Plots are also shown for the cases when $\nu' \neq 0$ and $\nu \neq 0$ which correspond to the new GETC states. For $m = 5$ and $\nu = \nu' = -1.0$, and another general situation with $m = 5, \nu = -0.4$ and $\nu' = -1.0$, we see that the corresponding values of $(\Delta x)^2_{\rm GETC}$ vary over a wider range of $|\beta|$; for $0.5 < |\beta| < 2$ the fluctuations of the field quadrature x are more strongly squeezed than for the two-photon coherent states,[6] the photon-added coherent states,[11] the excited two-photon coherent states or the generalized excited coherent states.[13] We also note that for given m and ν, with proper choice of the parameters ν' and μ', the position of the dip for the GETC states can be moved towards the variance axis. That is, the squeezing degree increases as $|\nu'|$ increases, and the solid curves associated with the GETC states not only represent a very broad class of nonclassical states but also exhibit remarkable squeezing effects.

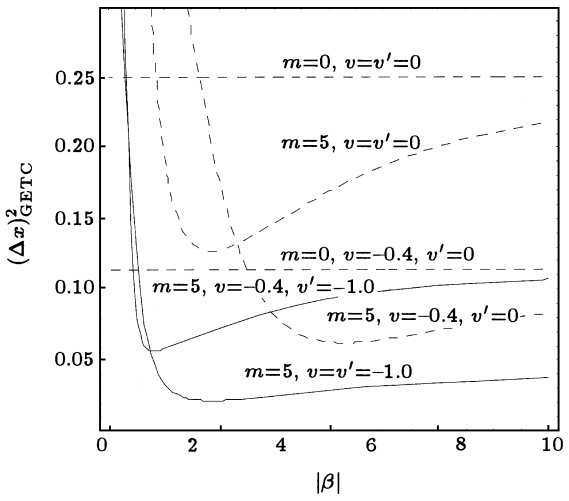


Fig.1. Variance $(\Delta x)^2_{\rm GETC}$ of the GETC states as a function of $|\beta|$ for m=0, 5. Solid curves: $\nu = -0.4$, $\nu' = -1.0$ and $\nu = \nu' = -1.0$.

To examine the photon-number distribution of these states we calculate their Q parameter according to the definition below (equivalent to the Fano factor used in some earlier papers):

$$Q = \frac{\langle\psi|(a^\dagger a)^2|\psi\rangle - (\langle\psi|a^\dagger a|\psi\rangle)^2}{\langle\psi|a^\dagger a|\psi\rangle}. \tag{31}$$

For the GETC states we thus obtain

$$\begin{aligned} Q_{\rm GETC} =& \frac{1}{A^m(\beta)[B^{m,m}_{1,1}(\beta^*,\beta) - A^m(\beta)]} \\ & \times \{A^m(\beta)B^{m,m}_{2,2}(\beta^*,\beta) \\ & - B^{m,m}_{1,1}(\beta^*,\beta)[A^m(\beta) + B^{m,m}_{1,1}(\beta^*,\beta)]\}. \end{aligned} \tag{32}$$

We note also the relationship between $B^{m,m}_{2,2}(\beta^*,\beta)$ and $A^{m,m}_{i,j}(\beta^*,\beta)$:

$$\begin{aligned} B^{m,m}_{2,2}(\beta^*,\beta) =& ((\mu\nu)^*)^2 A^{m,m}_{4,0}(\beta^*,\beta) \\ & + (\mu\nu)^2 A^{m,m}_{0,4}(\beta^*,\beta) \\ & + (6|\nu|^4 + 6|\nu|^2 + 1)A^{m,m}_{2,2}(\beta^*,\beta) \\ & - 2(2|\nu|^2 + 1)[(\mu\nu)^* A^{m,m}_{3,1}(\beta^*,\beta) \\ & - (\mu\nu)A^{m,m}_{1,3}(\beta^*,\beta)] \\ & + (6|\nu|^2 + 1)[(\mu\nu)^* A^{m,m}_{2,0}(\beta^*,\beta) \\ & + (\mu\nu)A^{m,m}_{0,2}(\beta^*,\beta)] \\ & - 2|\nu|^2(5|\nu|^2 + 3)A^{m,m}_{1,1}(\beta^*,\beta) \\ & + |\nu|^2(2|\nu|^2 + 1)A^m(\beta). \end{aligned} \tag{33}$$

Figure 2 shows the values of $Q_{\rm GETC}$ for various values of ν and ν' with $m = 0$ and $m = 5$. The phases of μ, ν and μ', ν' are chosen to be the same as those mentioned above when $m \neq 0, \nu \neq 0$ or $\nu' \neq 0$. It is evident that rich sub-Poissonian statistics exist for the GETC states, and the value of $Q_{\rm GETC}$ is always less than 1. This is a significant quantum effect.

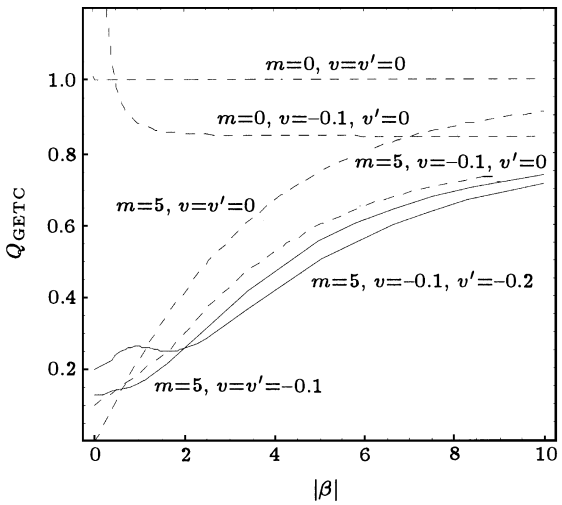


Fig.2. $Q_{\rm GETC}$ parameter for $m = 0, 5$ in the GETC states as a function of $|\beta|$. Solid curves: $\nu = -0.1$, $\nu' = -0.2$ and $\nu = \nu' = -0.1$.

4. Quasiprobability distribution functions

The quasiprobability distribution functions are extremely useful in the mathematical description of quantum mechanical systems and in displaying the quantum nature of the nonclassical field. They include the P function, the Q function and the Wigner function which are closely related to the mathematical description of physical systems. We shall first consider the quasiprobability distribution function $P_{\rm GETC}$ associated with the GETC state representation. The quasiprobability distribution function is obtained through the density matrix of the state $\rho = \frac{1}{\pi}\int P(z)|z,\mu,\nu\rangle\langle z,\mu,\nu|{\rm d}^2z$, where $|z,\mu,\nu\rangle$ is Yuen's two-photon coherent state. Using the double-Fourier transform,[27] we have

$$P_{\rm GETC}(z) = \frac{\exp(|z|^2-|\beta|^2)}{A^m(\beta)} \times \sum_{s=0}^{m}\sum_{t=0}^{m} R_s^m(\lambda^*,\rho^*,\beta)S_t^n(\lambda,\rho,\beta^*) \times (-1)^{s+t}\frac{\partial^{2m-s-t}}{\partial z^{m-s}\partial z^{*m-t}}\delta^2(z-\beta). \quad (34)$$

From the double δ function we see that $P_{\rm GETC}(z)$ becomes extremely singular, which is a typical characteristic of a nonclassical quantum state. Obviously, for $\nu'=0$ and $\mu'=1$, the distribution function $P_{\rm GETC}(z)$ is reduced to $P_{\rm eg}(z)$ which is the P function of the excited two-photon coherent states.[13] The quasiprobability Q function, which is given by the diagonal element of the density matrix ρ associated with the GETC states in the Yuen's state basis, is

$$Q_{\rm GETC}(z) = \frac{|z|^{2m}\exp(-|z-\beta|^2)}{A^m(\beta)} \times \sum_{s=0}^{m}\sum_{t=0}^{m}\frac{R_s^m(\lambda^*,\rho^*,\beta)S_t^n(\lambda,\rho,\beta^*)}{z^{*s}z^t}. \quad (35)$$

It may be seen that the form of $Q_{\rm GETC}(z)$ is no longer centred at $z=\beta$, which is a signature of the nonclassical radiation states.

We note that the squeezing properties of the GETC states can be seen even more clearly by projecting the three-dimensional plot of $Q_{\rm GETC}(z)$ onto the two-dimensional plane.[28] Figures 3(a)–3(c) show

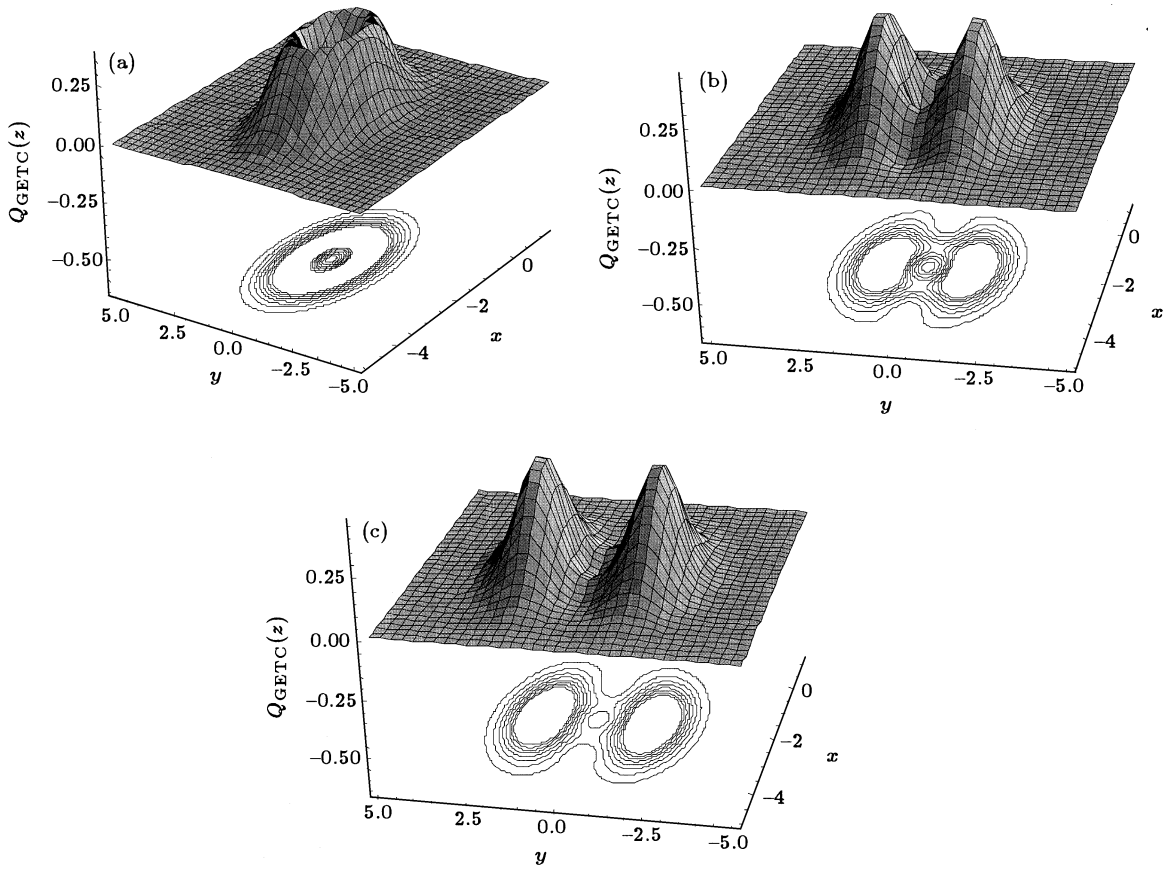


Fig.3. The $Q_{\rm GETC}(z)$ function of the GETC states at $\beta_1=-2$, β_2=0, $\nu'=3$, ν=5 and $\arg\nu=\arg\mu, \arg\nu'=-\arg\mu'$ for (a) m=1, (b) m=2 and (c) m=5.

the distribution of the $Q_{\rm GETC}(z)$ contours for different m. Since $Q_{\rm GETC}(z)$ is a function of a complex number $z = x + {\rm i}y$, we see that as m increases, the contours are squeezed in the y direction when $\nu' = 3, \nu = 5$ and $\beta = -2$ with $\arg\mu' = -\arg\nu', \arg\mu = \arg\nu$. Specifically, we observe that at $m = 1$ the contours are deformed to an oval shape (over the range of $-4.6 \le x \le 0.8, -2.5 \le y \le 2.5$), and at $m = 2$ have begun to split into two peaks. As m increases further, the two peaks are quite distinct and separated, as can be seen in Fig.3(c) for $m = 5$, with squeezing in the y direction in both peaks.

The Wigner function in the Bogoliubov state basis is also a useful measure for studying the nonclassical features of radiation fields. It is expressed through the density operator ρ associated with the representation[6,29]

$$W(z) = \frac{2}{\pi^2}{\rm e}^{2|z|^2}\int {\rm d}^2\beta\langle -\beta,\mu,\nu|\rho|\beta,\mu,\nu\rangle \times \exp[2(\beta^* z - \beta z^*)]. \quad (36)$$

Using the Fourier transformation in the complex number region and after the integration, we derive the expression for the Wigner function to be

$$W_{\rm GETC}(z) = \frac{2\exp(-2|z-\beta|^2)}{\pi A^m(\beta)} \times \sum_{s=0}^{m}\sum_{t=0}^{m}\{G_{m-s,m-t}(2z-\beta) \times R_s^m(\lambda^*,\rho^*,\beta)S_t^n(\lambda,\rho,\beta^*)\}, \quad (37)$$

where $G_{p,q}(z)$ is given by

$$G_{p,q}(z) = \begin{cases} (-1)^p p! L_p^{(q-p)}(|z|^2), q \ge p, \\ (-1)^q q! L_q^{(p-q)}(|z|^2), q \le p. \end{cases} \quad (38)$$

Exact analytical expressions for the GETC states when μ' and ν' are varied can be obtained as special cases from expression (37). Let us look at the special case of $\arg\mu' = -\arg\nu'$, $\arg\mu = \arg\nu$ and $\arg\beta = \arg(\beta_1 + {\rm i}\beta_2) = \pi$, corresponding to $\beta_1 = -2$ and $\beta_2 = 0$. In Figs.4 and 5 we give three-dimensional plots of $W_{\rm GETC}(z)$ for selected values of m, μ', ν', μ, ν and $|\beta|$. We see in Fig.4(a) that $W_{\rm GETC}(z)$ has the shape of a volcanic peak with a deep dip in the centre, corresponding to the rise and collapse of the wave packet, for $m = 1$, $\nu' = 3$ and $\nu = 5$. However, we observe in Figs.4(a) and 4(b) that for a given ν', ν and $|\beta|$, as the degree of excitation increases ($m > 1$), the peak of the $W_{\rm GETC}(z)$ function begins to split into two wave peaks which then move in opposite directions, and many oscillations occur between the two split peaks, with some dips becoming negative even. The greater the value of m, the farther the distance between the two peaks. These features are very pronounced and narrow. Furthermore, from the plot of $W_{\rm GETC}(z)$ for a fixed m but decreasing ν', the structure is seen to collapse and split into two peaks. The results of numerical computation of $W_{\rm GETC}(z)$ for various values of $m, \beta_1, \beta_2, \nu'$ and ν are demonstrated in Figs.4 and 5, where we see that $W_{\rm GETC}(z)$ of the GETC states is a function of a complex number $z = x + {\rm i}y$, and in a certain region of (x, y), the function is always negative. Obviously, these results are all completely different from a Gaussian distribution.

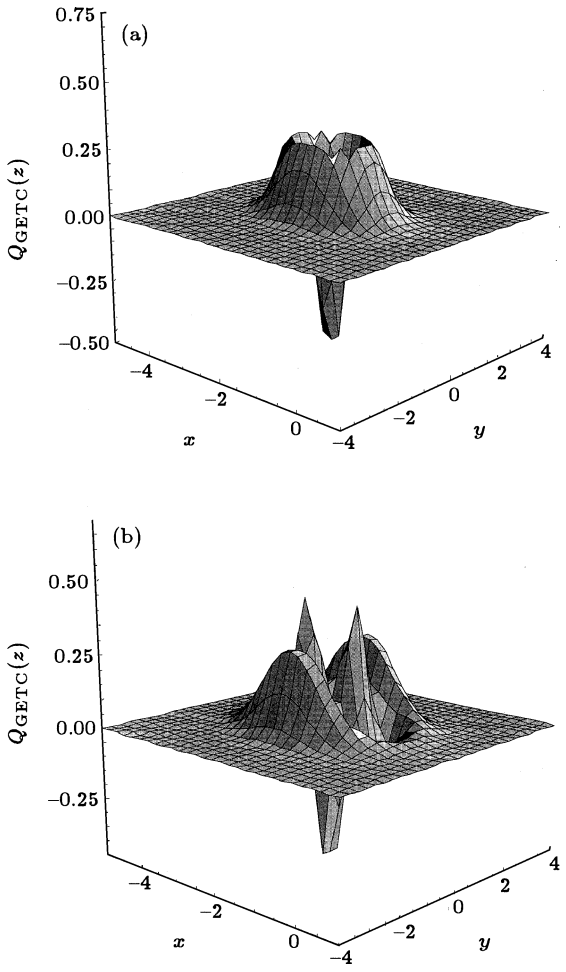


Fig.4. The Wigner function $W_{\rm GETC}$ in the case of $\beta_1 = -2$, β_2=0 and ν=5, ν'=3 for (a) m=1 and (b) m=3.

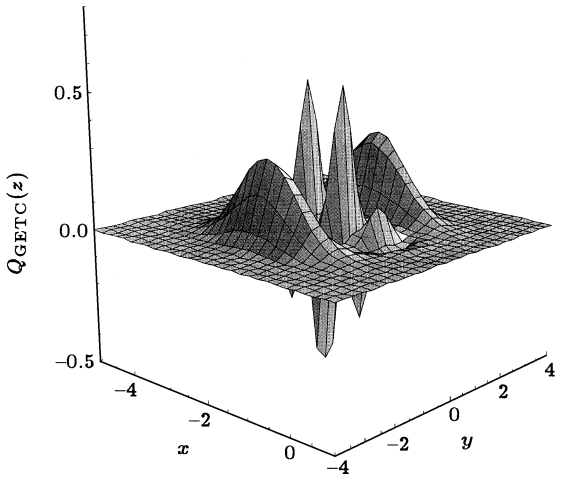


Fig.5. The Wigner function W_{GETC} in the case of $\beta_1 = -2$, $\beta_2 = 0$ and ν=5, ν'=3 for m=5.

5. Discussion and conclusion

We have constructed mathematically a new type of quantum state, the generalized excited two-photon coherent states, generated by allowing Bogoliubov's pseudo-creation operator $b^\dagger = \nu'^* a + \mu'^* a^\dagger$ to act m times on the two-photon coherent states, where μ' and ν' are variable parameters. From numerical computation we see that, with proper choice of m, ν, ν' and their phases, in a certain region these states can exhibit a very high degree of squeezing, together with significant sub-Poissonian distributions and other nonclassical properties.

For limiting values of the parameters the GETC states reduce to the two-photon coherent states, excited two-photon coherent states, excited coherent states, and coherent states. The quasiprobability distribution functions P, Q and $W_{\mathrm{GETC}}(z)$ also reveal the nonclassical properties of the GETC states. In particular, the dependence of the Wigner function on the m, ν and ν' parameters is clearly demonstrated; the splitting and oscillation of the peaks are the results of quantum interference which indicate the nonclassical nature of the quantum state.

To realize the GETC states experimentally we first need to prepare the two-photon coherent states, then replace a and $a^\dagger$ by the Bogoliubov transformation operators $b = \mu' a + \nu' a^\dagger$ and $b^\dagger = \nu'^* a + \mu'^* a^\dagger$, respectively, in the interaction Hamiltonian operator of the atom-field system.[10,30] With the rapid development of sophisticated ion and atom trapping technology it is conceivable that the GETC states may one day be realized experimentally through excitation of multilevel ions or atoms in a high field optical cavity.

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