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Global synchronization of Chua's chaotic delay network by using linear matrix inequality*

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Global synchronization of Chua's chaotic dynamical networks with coupling delays is investigated in this paper. Unlike other approaches, where only local results were obtained, the network is found to be not linearized in this paper. Instead, the global synchronization is obtained by using the linear matrix inequality theory. Moreover, some quite simple linear-state-error feedback controllers for global synchronization are derived, which can be easily constructed based on the minimum eigenvalue of the coupling matrix. A simulation of Chua's chaotic network with global coupling delays in nodes is finally given, which is used to verify the theoretical results of the proposed global synchronization scheme.

Keywords: global synchronization, Chua's chaotic network, coupling delays, linear matrix inequality
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1. Introduction

Synchronization of complex networks of dynamical systems has received a great deal of attention from the nonlinear dynamics community in the past decade. Special attention has been focused on the synchronization of chaotic dynamical systems, particularly the large-scale and complex networks of chaotic oscillators.[1−4] Recently, synchronization in the different small-word and scale-free dynamical network models has also been carefully studied.[5−11] In these studies, a common approach is to linearize the nonlinear dynamical nodes around the network synchronized state, thereby obtaining some local synchronization and stability results. This approach will fail if the dynamical system at the node is unknown or uncertain, since linearization cannot be done in such cases.

In this paper, a direct approach without using linearization is proposed and investigated for global synchronization of Chua's chaotic delay network. Moreover, by using the linear matrix inequality theory, some quite simple linear-state-error feedback controllers for global synchronization are derived, which can be easily constructed based on the minimum eigenvalue of the coupling matrix. A simulation is finally given to verify the theoretical result of the proposed global synchronization scheme.

2. The network model

Chua's chaotic circuit is described by

$$\begin{aligned}\dot{x}_1 &= p\left(-x_1 + x_2 - f\left(x_1\right)\right),\\ \dot{x}_2 &= x_1 - x_2 + x_3,\\ \dot{x}_3 &= -qx_2,\end{aligned} \tag{1}$$

where

$$f\left(x_1\right) = m_0 x_1 + \frac{1}{2}\left(m_1 - m_0\right)\left(\left|x_1 + 1\right| - \left|x_1 - 1\right|\right)$$

with $m_0 < 0$ and $m_1 < 0$.

We use Chua's chaotic circuit as a dynamical node in network, and obtain a dynamical network consisting of N identical nodes with coupling delays, in which each node is a three-dimensional dynamical system. The network is of the form

$$\begin{pmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{pmatrix} = \begin{pmatrix}-p & p & 0\\ 1 & -1 & 1\\ 0 & -q & 0\end{pmatrix}\begin{pmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{pmatrix} + \begin{pmatrix}-pf\left(x_{i1}\right)\\ 0\\ 0\end{pmatrix} + \sum_{j=1}^{N} c_{ij}x_j\left(t-\tau\right) + v_i, i = 1,\ldots,N, \tag{2}$$

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where $x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}} \in R^3$ represents the state vector of the ith node, and $v_i \in R^3, i = 1, 2, \ldots, N$, are the controllers to be designed. In this model, it is required that the coupling coefficients satisfy

$$\sum_{j=1}^{N} c_{ij} = 0, c_{ij} \geq 0 (i \neq j), i, j = 1, 2, \ldots, N. \quad (3)$$

This ensures that the synchronized state $x_1 = x_2 = \cdots = x_N = s(t)$, as $t \to \infty$. Here, $s(t) = \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix}^{\mathrm{T}}$ satisfies Chua's circuit (1).

The objective is to find the controllers such that the solutions of network (1) globally asymptotically synchronize with each other, in the sense that

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for all } i, j = 1, 2, \ldots, N. \quad (4)$$

To proceed, define

$$\begin{aligned} e_i(t) =& x_i(t) - s(t), \\ e_i(t-\tau) =& x_i(t-\tau) - s(t-\tau), \quad i = 1, 2, \ldots, N. \end{aligned}$$

We have the dynamical error equations

$$\begin{aligned} \dot{e}_i =& A e_i + B \tilde{f}(x_{i1}, s_1) \\ &+ \sum_{j=1}^{N} c_{ij} e_j(t-\tau) + v_i, i = 1, \ldots, N, \end{aligned} \quad (5)$$

where

$$A = \begin{pmatrix} -p & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -p \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{f}(x_{i1}, s) = f(x_{i1}(t)) - f(s_1(t)), i = 1, 2, \ldots, N.$$

3. The main result

The main result of this paper on global synchronization of network (1) is derived in this section.

Let

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2N} \\ \vdots & & & \\ c_{N1} & c_{N2} & \cdots & c_{NN} \end{pmatrix}. \quad (6)$$

If there is a connection between node i and node $j\,(j \neq i)$, then $\mathrm{sign}(c_{ij}) = \mathrm{sign}(c_{ji}) = 1$, otherwise, $\mathrm{sign}(c_{ij}) = \mathrm{sign}(c_{ji}) = 0$. If the degree d_i is defined as the number of connection of node i, then

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \mathrm{sign}(c_{ij}) = \sum_{\substack{i=1 \\ j \neq i}}^{N} \mathrm{sign}(c_{ji}) = d_i, \quad i, j = 1, \ldots, N. \quad (7)$$

It follows from Eq.(5) that

$$\dot{e}(t) = A e(t) + B F(e(t)) + e(t-\tau) C^{\mathrm{T}} + V, \quad (8)$$

where

$$\begin{aligned} e(t) &= (e_1(t), e_2(t), \ldots, e_N(t)), \\ e(t-\tau) &= (e_1(t-\tau), e_2(t-\tau), \ldots, e_N(t-\tau)), \end{aligned}$$

$$\begin{aligned} F(e(t)) &= \left(\tilde{f}(x_{11}, s_1), \ldots, \tilde{f}(x_{N1}, s_1)\right), \\ V &= (v_1, v_2, \ldots, v_N). \end{aligned}$$

Since $C = C^{\mathrm{T}}$ is a real symmetric matrix with Eq.(3), there exists a unitary matrix $U = (u_1, u_2, \ldots, u_N)$ such that

$$C = U \Lambda U^{\mathrm{T}},$$

where $U^{\mathrm{T}} U = U U^{\mathrm{T}} = I$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, $0 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, which will be determined later.

Now, introducing a transformation $Y(t) = e(t) U$, $Y(t-\tau) = e(t-\tau) U$ along with Eq.(8), we have

$$\dot{Y}(t) = A Y(t) + B F(e(t)) U + Y(t-\tau) \Lambda + V U, \quad (9)$$

where $Y(t) = (y_1, \ldots, y_N)$, $y_k \in R^3$, $k = 1, \ldots, N$. That is,

$$\dot{y}_k(t) = A y_k(t) + B F(e(t)) u_k + \lambda_k y_k(t-\tau) + V u_k,$$

$$k = 1, 2, \ldots, N. \quad (10)$$

Theorem 1 If controllers

$$V u_k = \frac{1}{2} \left(-\mu_{\max}\left(A^{\mathrm{T}} + A\right) + \alpha \lambda_N\right) y_k(t),$$

$$k = 1, 2, \ldots, N \text{ or} \quad (11)$$

$$v_i = \frac{1}{2} \left(-\mu_{\max}\left(A^{\mathrm{T}} + A\right) + \alpha \lambda_N\right) e_i(t),$$

$$i = 1, 2, \ldots, N \quad (12)$$

satisfy the following condition:

$$\alpha > N + 1, \quad \text{and} \quad \lambda_N < \frac{2\gamma p N}{N + 1 - \alpha}, \quad (13)$$

then the network (1) synchronizes in the global sense of formula (4), where $\gamma = |m_0| + |m_1 - m_0|$, $\mu_{\max}\left(A^{\mathrm{T}} + A\right)$ is the largest eigenvalue of matrix $A^{\mathrm{T}} + A$.

Proof Firstly, note that the eigenvalues $0 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ of coupling matrix C. For system (10), choose a Lyapunov function of the form

$$V = \sum_{k=1}^{N} y_k^{\mathrm{T}}(t)\, y_k(t) - \lambda_N \sum_{k=1}^{N} \int_0^{\tau} y_k^{\mathrm{T}}(t-s)\, y_k(t-s)\, \mathrm{d}s. \tag{14}$$

The time derivative of Eq.(14) along the trajectory of systems (10) is

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} =& \sum_{k=1}^{N} (y_k^{\mathrm{T}}(A^{\mathrm{T}} + A) y_k - \lambda_N y_k^{\mathrm{T}}(t) y_k(t) \\
&+ 2 y_k^{\mathrm{T}}(t) B F(e(t)) u_k \\
&+ 2\lambda_k y_k^{\mathrm{T}}(t) y_k(t-\tau)) \\
&+ \sum_{k=1}^{N} (2 y_k^{\mathrm{T}}(t) V u_k + \lambda_N y_k^{\mathrm{T}}(t-\tau) y(t-\tau)) \\
\leq& \sum_{k=1}^{N} ((\mu_{\max}(A^{\mathrm{T}} + A) - \lambda_N) \|y_k(t)\|^2 \\
&- 2\lambda_N \|y_k(t)\| \|y_k(t-\tau)\| \\
&+ 2\|y_k(t)\| \|B F(e(t)) u_k\|) \\
&+ \sum_{k=1}^{N} (2 y_k^{\mathrm{T}} V u_k + \lambda_N \|y(t-\tau)\|^2).
\end{aligned} \tag{15}$$

Since there exists a nonnegative constant γ such that the function $f(x)$ in network (1) satisfies

$$\begin{aligned}
&|f(x_{i1}) - f(s_1)| \leq \gamma |x_{i1} - s_1|, \quad \text{for all} \\
&x_{i1}(t), s_1(t) \in R, i = 1, 2, \ldots, N,
\end{aligned} \tag{16}$$

where $\gamma = |m_0| + |m_1 - m_0|$, then we have

$$\|F(e(t)) u_k\| \leq \sum_{i=1}^{N} \gamma |e_{i1}| \leq \gamma \sum_{i=1}^{N} \|e_i\| \leq \gamma \sum_{i=1}^{N} \|y_i\|. \tag{17}$$

Substituting Eqs.(11), (16) and (11) into Eq.(15) we have

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} \leq& \sum_{k=1}^{N} ((\alpha\lambda_N - \lambda_N) \|y_k(t)\|^2 \\
&- 2\lambda_N \|y_k(t)\| \|y_k(t-\tau)\| + \lambda_N \|y_k(t-\tau)\|^2) \\
&+ \sum_{k=1}^{N} 2\gamma p \|y_k\| \sum_{j=1}^{N} \|y_j\|.
\end{aligned} \tag{18}$$

The right-hand side of Eq.(8) is quadratic in terms of

$$\begin{aligned}
&\|y_1(t)\|, \|y_2(t)\|, \ldots, \|y_N(t)\|, \\
&\|y_1(t-\tau)\|, \|y_2(t-\tau)\|, \ldots, \|y_N(t-\tau)\|,
\end{aligned}$$

which can be rewritten as

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq -\frac{1}{2} \phi^{\mathrm{T}} M \phi, \tag{19}$$

where

$$\begin{aligned}
\phi =& (\|y_1(t)\|, \ldots, \|y_N(t)\|, \\
&\|y_1(t-\tau)\|, \ldots, \|y_N(t-\tau)\|)^{\mathrm{T}}
\end{aligned}$$

and

$$M = \begin{bmatrix} S^{\mathrm{T}} + S & D \\ D^{\mathrm{T}} & -2\lambda_N I \end{bmatrix},$$

which is an $2N \times 2N$ matrix, with elements in matrix $S^{\mathrm{T}} + S$ given by

$$s_{ij} = \begin{cases} (1-\alpha)\lambda_N - 2\gamma p, & i = j \leq N, \\ -2\gamma p, & i \neq j, i, j = 1, 2, \ldots, N, \end{cases} \tag{20}$$

and D is an $N \times N$ matrix:

$$D = \begin{bmatrix} 2\lambda_N & 2\lambda_N & \cdots & 2\lambda_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{21}$$

Choosing

$$\lambda_N < \frac{2\gamma p N}{1-\alpha}, \alpha > 1, \tag{22}$$

we obtain the linear matrix inequality about λ_N,

$$S + S^{\mathrm{T}} > 0.$$

Since the linear matrix inequality about λ_N

$$M = \begin{bmatrix} S^{\mathrm{T}} + S & D \\ D^{\mathrm{T}} & -2\lambda_N I \end{bmatrix} > 0$$

is equivalent to the following two linear matrix inequalities:

$$S + S^{\mathrm{T}} > 0, \quad \text{and} \quad S + S^{\mathrm{T}} + \frac{1}{2\lambda_N} D D^{\mathrm{T}} > 0, \tag{23}$$

the linear matrix inequalities (23) hold if $\alpha > N + 1$, and $\lambda_N < \dfrac{2\gamma p N}{N + 1 - \alpha}$. It then follows that $\dfrac{\mathrm{d}V}{\mathrm{d}t} < 0$. Hence, the solutions of the system are globally uniformly asymptotically stable about $y_k = 0, k = 1, \ldots, N$.

Consequently,

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad i, j = 1, 2, \ldots, N.$$

The proof is thus completed.

Remark: (a) Since

$$\begin{aligned}
VU =& V(u_1, u_2, \ldots, u_N) \\
=& \frac{1}{2}(-\mu_{\max}(A^{\mathrm{T}} + A) + \alpha\lambda_N) Y
\end{aligned}$$

and

$$UU^{\mathrm{T}} = I,$$

we obtain

$$\begin{aligned} V =& \frac{1}{2}(-\mu_{\max}(A^{\mathrm{T}} + A) + \alpha\lambda_N)YU^{\mathrm{T}} \\ =& \frac{1}{2}(-\mu_{\max}(A^{\mathrm{T}} + A) + \alpha\lambda_N)eUU^{\mathrm{T}}. \end{aligned}$$

It follows that

$$v_i = \frac{1}{2}(-\mu_{\max}(A^{\mathrm{T}} + A) + \alpha\lambda_N)e_i(t), i = 1, 2, \ldots, N.$$

(b) The controller (12) is constructed in terms of the minimum eigenvalue of the coupling matrix which depends upon the inequality (13). Therefore, when there exists an error in the controller, Theorem 1 still holds.

4. Simulation

Now, we verify the scheme proposed in Sec.3. In simulation, we assume that Chua's chaotic dynamical network, with coupling delays, is globally connected, namely all elements of the coupling matrix C are nonzero: $c_{ij} \neq 0 (i \neq j), i, j = 1, 2, \ldots, N$; then the matrix C becomes

$$C = \begin{pmatrix} -\frac{N-1}{2}c & \frac{c}{2} & \cdots & \frac{c}{2} \\ \frac{c}{2} & & & \\ \vdots & & & \\ \frac{c}{2} & \frac{c}{2} & \cdots & -\frac{N-1}{2}c \end{pmatrix}.$$

It can be easily verified that the eigenvalues of this matrix C are

$$0 = \rho_1 > \rho_2 = \cdots = \rho_N = -\frac{c}{2}N, \quad \lambda_N(C) = -\frac{c}{2}N.$$

According to Theorem 1, we have $\lambda_N(C) < \frac{2\gamma pN}{N+1-\alpha}$, if we choose $c > \frac{4\gamma p}{\alpha - N - 1}$ and $\alpha > N + 1$. One can construct the needed controllers as follows:

$$v_i = \frac{1}{2}(-\mu_{\max}(A^{\mathrm{T}} + A) + \alpha\lambda_N(C))e_i(t),$$

$$i = 1, 2, \ldots, N.$$

According to the theoretical results, the network (2) globally synchronizes in the sense of formula (4).

RK4 method is used for the simulation with time step equal to 0.0002. In simulation, we take the number of nodes N=30 and delay time $\tau = 0.004$. To ensure the chaotic behaviour of Chua's circuit (1), we select the parameters as p=10, q=14.7, $m_0 = -0.68$, $m_1 = -1.27$. Simulation has confirmed the theoretical results, as shown in Figs.1–3.

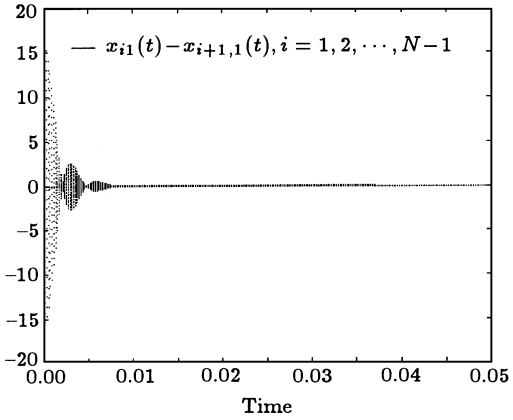


Fig.1. The error states of $x_{i1}(t) - x_{i+1,1}(t), i = 1, 2, \ldots, N-1$ with coupling delays.

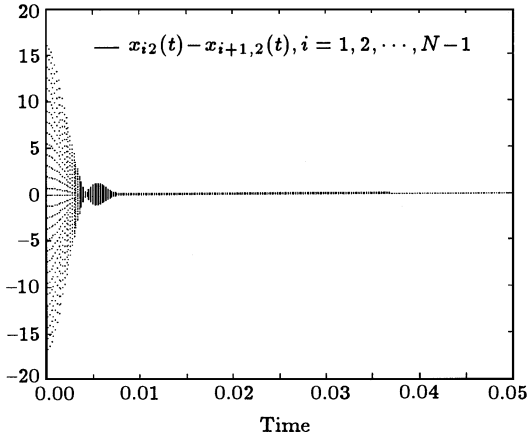


Fig.2. The error states of $x_{i2}(t) - x_{i+1,2}(t), i = 1, 2, \ldots, N-1$ with coupling delays.

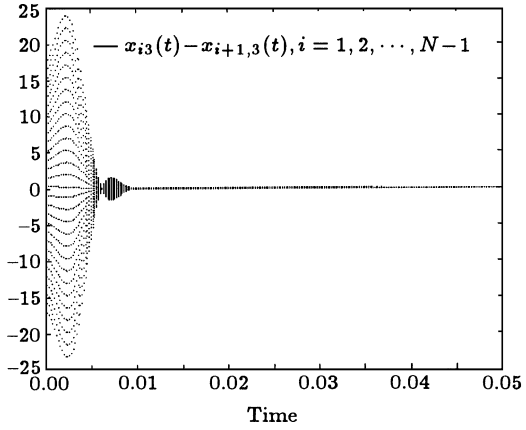


Fig.3. The error states of $x_{i3}(t) - x_{i+1,3}(t), i = 1, 2, \ldots, N-1$ with coupling delays.

5. Conclusions

Differing from the common approach, where linearization is a necessary step and only local results were obtained, a nonlinear and large-scale Chua's delayed network is studied in this paper without linearization. Global synchronization and asymptotic stability of Chua's network with coupling delays are investigated under a simple sufficient condition, which has been derived based on the linear matrix inequality. Moreover, some fairly simple linear state-error feedback controllers for global synchronization and stability have been designed for Chua's network, which can be easily constructed based on the minimum eigenvalue of the coupling matrix. When Chua's network with global coupling delays in nodes are known, the theoretical results of the proposed global synchronization scheme can be verified.

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