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Chaotifying a linear time-invariant system by the state feedback controller and sawtooth function

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Another algorithm for chaotification of any given linear time-invariant discrete-time systems is presented. The new chaotification algorithm uses the decentralized control and the continuous sawtooth function, which can generate discrete chaos with an arbitrarily desired amplitude bound. Based on the Marotto theorem, we mathematically prove that the controlled system is chaotic in the sense of Li and Yorke. Finally, a simple example is used to illustrate the effectiveness of the proposed theory and method.

Keywords: anticontrol of chaos, chaos control, chaotification
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1. Introduction

There have been increasing interest and rapid development in the research area of controlling and synchronizing chaotic dynamical systems in the last decade.[1,2] In particular, chaotification (also called 'anticontrol of chaos') by means of making an originally non-chaotic dynamical system chaotic, or enhancing and retaining existing chaos, has attracted some special attention lately due to the great potential of chaos in some novel time- and/or energy-critical applications.[1,2] Chaos, under appropriate monitoring, can provide a system designer with a variety of special properties, richness of flexibility, and a great many opportunities.[1,2] Dynamical systems (maps) with chaotic and bifurcate behaviours have been reported to be very useful in some practical applications, such as secure communication.[3−7] This provides a strong motivation for the current research on chaotification of the discrete-time dynamical systems[2,8−10] and the continuous-time dynamical systems.[11,12] In the algorithm of Chen and Lai,[8,9] mod-operation is used in order to keep the motion confined to a compact region of state space. In practical implementations, reproducing the modulus function in a chaotic map circuit may not be easy because the modulus function is discontinuous. In this paper, the discontinuous modulus function is replaced by a continuous sawtooth function. Based on the Marotto theorem,[13] we mathematically prove that the modified Chen–Lai feedback anticontrol method yields chaos with an arbitrarily desired amplitude bound in the sense of Li and Yorke.[14] Finally, a simple example is used to illustrate the effectiveness of the proposed method.

2. Description of the problem

Consider a time-invariant linear state transfer:

$$x_k \to x_{k+1} = Ax_k. \tag{1}$$

Clearly, $x^* = 0$ is a fixed point of the state transfer (1). The chaotification problem under investigation is to design the state-feedback control input sequence $\{u_k\}_{k=0}^{\infty}$ that will be added to the right-hand side of formula (1) with the form

$$u_k = B_k x_k, \tag{2}$$

where $\{B_k\}$ are $n \times n$ constant matrices to be determined, and the state transfer $F : R^n \to R^n$ such that the composite state transfer

$$x_k \to x_{k+1} = F(Ax_k + u_k) \tag{3}$$

behaves chaotically. Apparently, if the state transfer $F : R^n \to R^n$ is linear, the composite state transfer (1) is impossible to produce chaotic state series.

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To obtain chaotic state series, Chen and Lai[8,9] suggested to first design the control variable $u_k = B_k x_k$ such that all the Lyapunov exponents of the state transfer (3) without F are strictly positive, and then to employ the mod transformation, i.e. $F(y) = y(\text{mod } 1)$. The Chen–Lai anticontrol algorithm[8,9] can be described as follows:

Step 1 Design a positive feedback controller

$$u_k = B_k x_k = \sigma_k x_k, \tag{4}$$

with positive number $\sigma_k \equiv N + e^c$, $c > 0$ for all $k = 0, 1, 2, \ldots$, where $\|A\|_\infty \leq N$.

Step 2 Then, apply the mod-operation and obtain the composite state transfer:

$$x_{k+1} = (Ax_k + B_k x_k)(\text{mod } 1). \tag{5}$$

Wang and Chen[10] have proved that the algorithm of Chen and Lai[8,9] can generate chaos in the sense of Li and Yorke.[14] In this paper, instead of the discontinuous modulus function, we use a continuous sawtooth function to realize F. Therefore, Step 2 is modified as follows:

$$x_{k+1} = C\Phi(Ax_k + u_k), \tag{6}$$

where $\Phi(y) = [\varphi(y_1), \ldots, \varphi(y_n)]^{\mathrm{T}}$, for any $y = [y_1, \ldots, y_n]^{\mathrm{T}} \in R^n$. $\varphi : R \to R$ is a continuous sawtooth function defined as

$$\varphi(y) = (-1)^m (y - 2m\varepsilon)$$

$$\text{for} \quad (2m-1)\varepsilon \leq y \leq (2m+1)\varepsilon,$$

$$m = 0, \ \pm 1, \ \pm 2 \cdots.$$

$C = \text{diag}(c_1, c_2, \ldots, c_n) \in R^{n \times n}$, $c_i \neq 0$ is a constant diagonal matrix designed such that in each dimension the state variable of the system (6) has the desired amplitude bound. Later, we will apply the Marotto theorem[13] to prove that the controlled system (6) with

$$u_k = \sigma x_k, \tag{7}$$

where $\sigma > 0$ is a constant to be designed, is chaotic in the sense of Li and Yorke.[14]

3. Verification of chaos in the sense of Li-Yorke

In this section, we apply the Marotto theorem[13] to prove that the controlled system (6) and (7) is chaotic in the sense of Li and Yorke.

3.1. Chaos in the sense of Li and Yorke: the Marotto theorem

To introduce the Marotto theorem, consider a general n-dimensional discrete-time autonomous system of the form

$$x_{k+1} = g(x_k). \tag{8}$$

Let g^t denote the t times of compositions of g with itself. A point, x^*, is said to be a p-periodic point of g if $g^p(x^*) = x^*$ but $g^t(x^*) \neq x^*$ for $1 \leq t < p$. If $p = 1$, that is, $g(x^*) = x^*$, x^* is called a fixed point. Let $Dg(x)$ and $\det(Dg(x))$ be the Jacobian of g at point x and its determinant, respectively, and let $B_r(x)$ denote a closed ball in R^n of radius r centred at point x.

Definition[13] A fixed point x^* of Eq.(8) is said to be a snap-back repeller if

(a) there exist real numbers $r > 0$ and $s > 1$ such that $g(x)$ is differentiable and expanded in the neighbourhood of the fixed point x^*, in the sense that

$$\|g(x) - g(y)\|_\infty > s \|x - y\|_\infty, \quad \forall x, y \in B_r(x^*); \tag{9}$$

(b) there exists a point $x^0 \in B_r(x^*)$, with $x^0 \neq x^*$, such that $g^M(x^0) = x^*$ and $\det(Dg^M(x^0)) \neq 0$ for some positive integer M.

The following theorem of Marotto is of central importance in this paper.

Theorem (Marotto)[8] If system (8) has a snap-back repeller, then the system is chaotic in the sense of Li–Yorke:

(a) there exists a positive integer N such that for every integer $p \geq N$, system (8) has p-periodic points;

(b) there exists a 'scrambled set' (an uncountable invariant set S containing no periodic points) such that

(b1) $g(S) \subset S$,

(b2) for every $x, y \in S$ with $x \neq y$,

$$\lim_{k \to \infty} \sup \left\| g^k(x) - g^k(y) \right\| > 0,$$

(b3) for every $x \in S$ and any periodic point y_{per} of system (8)

$$\lim_{k \to \infty} \sup \left\| g^k(x) - g^k(y_{\text{per}}) \right\| > 0;$$

(c) there exists an uncountable subset S_0 of S such that for any $x, y \in S_0$,

$$\lim_{k \to \infty} \inf \left\| g^k(x) - g^k(y) \right\| = 0.$$

3.2. Verification of chaos in the designed controlled system

Theorem 1 The controlled system (6) and (7) is chaotic in the sense of Li and Yorke, if

$$\sigma > \|A\|_\infty + \left\|C^{-1}\right\|_\infty + \sqrt{4\left\|C^{-1}\right\| \cdot \|A\|_\infty + \left\|C^{-1}\right\|_\infty^2}. \tag{10}$$

Remark It is surprising to note that σ is independent of ε.

Proof In the controlled system (6) and (7) one has

$$x_{k+1} = C\,\Phi(Ax_k + \sigma\, x_k) \equiv g(x_k); \tag{11}$$

apparently, $x^* = 0$ is a fixed point of the controlled system. Condition (10) implies that $A + \sigma\, I$ are invertible matrices. Let $b = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^{\mathrm{T}}$ and let

$$x^0 = 2\varepsilon(\sigma\, I + A)^{-1}(C^{-1}(\sigma\, I + A)^{-1}b). \tag{12}$$

In what follows, we prove that

(a) There exist constants r, $0 < r < \dfrac{\varepsilon}{\|A\|_\infty + \sigma}$, and $s > 1$ such that $g(x)$ is differentiable and expanded in the neighbourhood of the fixed point $x^* = 0$, in the sense that

$$\|g(x) - g(y)\|_\infty > s\,\|x - y\|_\infty, \quad \forall x, y \in B_r(0).$$

(b) $g^2(x^0) = 0 = x^*$ and $\det(Dg^2(x^0)) \neq 0$.

Proof of (a) Note that

$$x^0 = 2\varepsilon\sigma^{-1}(I + \sigma^{-1}A)^{-1}(\sigma^{-1}C^{-1}(I + \sigma^{-1}A)^{-1}b). \tag{13}$$

Since $\left\|\sigma^{-1}A\right\|_\infty < 1$, it follows from condition (10) that

$$\begin{aligned}\left\|x^0\right\|_\infty =& \|2\varepsilon\sigma^{-1}(I + \sigma^{-1}A)^{-1} \\ & \times (\sigma^{-1}C^{-1}(I + \sigma^{-1}A)^{-1}b)\|_\infty \\ \leq& 2\varepsilon\frac{\sigma^{-1}}{1 - \sigma^{-1}\|A\|_\infty}\frac{\sigma^{-1}\left\|C^{-1}\right\|_\infty}{1 - \sigma^{-1}\|A\|_\infty} \\ =& \frac{2\varepsilon\left\|C^{-1}\right\|_\infty}{(\sigma - \|A\|_\infty)^2} < \frac{\varepsilon}{\|A\|_\infty + \sigma}.\end{aligned} \tag{14}$$

Let r be chosen such that

$$\left\|x^0\right\|_\infty \leq r < \frac{\varepsilon}{\|A\|_\infty + \sigma}. \tag{15}$$

Then $x^0 \in B_r(0)$ and for any $x \in B_r(0)$

$$\begin{aligned}&\|Ax + \sigma Ix\|_\infty \leq \|A + \sigma I\|_\infty \|x\|_\infty \\ &< (\|A\|_\infty + \sigma)\frac{\varepsilon}{\|A\|_\infty + \sigma} = \varepsilon.\end{aligned} \tag{16}$$

Therefore, it follows from the definition of g that, for any $x \in B_r(0)$

$$g(x) = C(Ax + \sigma x) = C(A + \sigma I)x \tag{17}$$

and

$$Dg(x) = C(A + \sigma I). \tag{18}$$

For any $x, y \in B_r(0)$, we have

$$\begin{aligned}&\|g(x) - g(y)\|_\infty \\ =& \|C(A + \sigma I)(x - y)\|_\infty \\ \geq& \left\|(C(A + \sigma I))^{-1}\right\|_\infty^{-1} \times \|x - y\|_\infty.\end{aligned} \tag{19}$$

Let s be chosen such that

$$1 < s \leq \frac{\sigma - \|A\|_\infty}{\left\|C^{-1}\right\|_\infty}. \tag{20}$$

Then, $g(x)$ is expanded in the neighbourhood of the fixed point $x^* = 0$, in the sense that $\|g(x) - g(y)\|_\infty > s\,\|x - y\|_\infty$, $\forall x, y \in B_r(0)$.

Proof of (b) Observe that

$$x^1 = g(x^0) = C\,\Phi((A + \sigma I)x^0) = 2\varepsilon(A + \sigma I)^{-1}b. \tag{21}$$

We only need to prove $g(x^1) = 0$. Note that

$$(A + \sigma)x^1 = 2(A + \sigma)(\sigma I + A)^{-1}b = 2b. \tag{22}$$

Therefore, according to the definition of g, we can obtain

$$g(x^1) = C\,\Phi(Ax^1 + \sigma x^1) = C\,\Phi(2b) = 0. \tag{23}$$

Moreover, g^2 is differentiable at x^0 with

$$\begin{aligned}\det(Dg^2(x^0)) =& \det(Dg(x^0)) \times \det(Dg(x^1)) \\ =& -(\det(A + \sigma I))^2 \times (\det C)^2 \neq 0.\end{aligned} \tag{24}$$

Thus, based on the Marotto theorem, (a) and (b) together imply that the controlled system (6) and (7) is chaotic in the sense of Li and Yorke.

4. A simulation example

Consider a three-dimensional discrete-time system described by

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \\ x_{k+1}^3 \end{bmatrix} = Ax_k = \begin{bmatrix} 0.1 & 2 & 0.3 \\ -0.2 & 1 & 0.1 \\ 0.4 & 2 & 0.3 \end{bmatrix}\begin{bmatrix} x_k^1 \\ x_k^2 \\ x_k^3 \end{bmatrix}. \tag{25}$$

Since $\|A\|_\infty = 0.7645 < 1$, the uncontrolled system (25) is stable. Based on the system (25), we use the

proposed method to construct a chaotic state transfer described by

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \\ x_{k+1}^3 \end{bmatrix} = \begin{bmatrix} c_1\varphi(0.1x_k^1 + 2x_k^2 + 0.3x_k^3 + \sigma x_k^1) \\ c_2\varphi(-0.2x_k^1 + x_k^2 + 0.1x_k^3 + \sigma x_k^2) \\ c_3\varphi(0.4x_k^1 + 2x_k^2 + 0.3x_k^3 + \sigma x_k^3) \end{bmatrix}. \tag{26}$$

We expect that the amplitude bound of x_k^1, x_k^2, x_k^3 are 0.1, 0.4, 0.8, respectively. When $\varepsilon = 1$ in function $\varphi(\cdot)$, we obtain $c = \mathrm{diag}(0.1,\ 0.4,\ 0.8)$ and know from Theorem 1 that the composite state transfer (26) is chaotic if $\sigma > 7.1351$. In the simulation, we choose $\sigma = 10$. Figure 1 shows the simulation results. In Fig.1, (a), (b) and (c) represent the evolution of the states x_k^1, x_k^2 and x_k^3, respectively; (d) shows the maximum magnitude of the control input, and (e) shows the strange attractor of the controlled system. We can see that the controlled system displays complex chaotic behaviours.

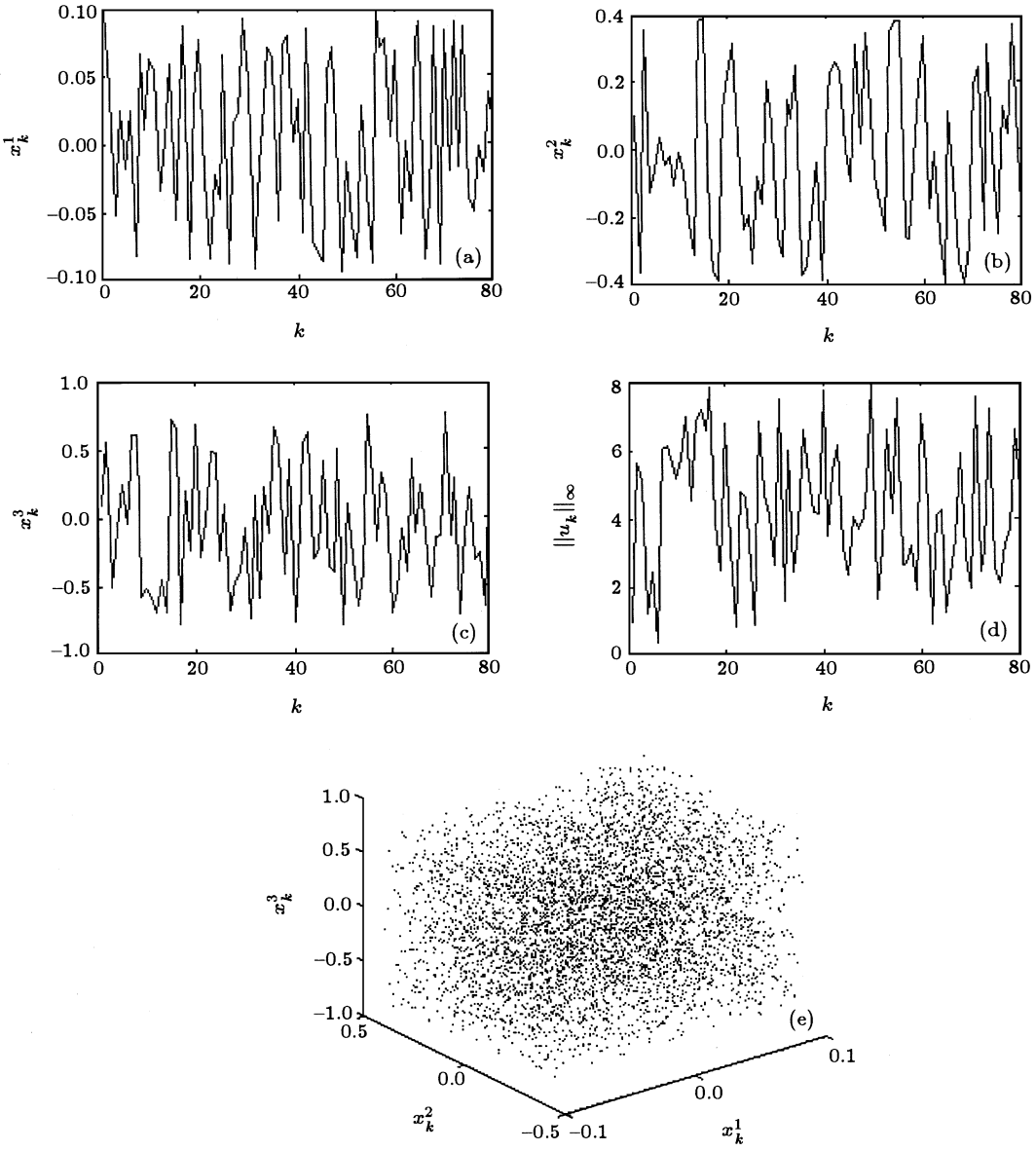


Fig.1. Simulation results.

5. Conclusions

The theory and experiment have demonstrated that the proposed method can make a linear time-invariant system chaotic in the sense of Li and Yorke and have an arbitrarily desired amplitude bound.

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