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Lie symmetry and Mei symmetry of a rotational relativistic system in phase space

Li Hong(李 红) and Fang Jian-Hui(方建会)

College of Physics Science and Technology, University of Petroleum, Dongying 257061, *China*

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The Lie symmetry and the Mei symmetry of a rotational relativistic system in phase space are studied. The definition, criterion and conserved quantity of the Lie symmetry and the Mei symmetry of a rotational relativistic system in phase space are given. The relation between the Lie symmetry and the Mei symmetry is found. An example is given to illustrate the application of the result.

Keywords: rotational relativistic system, Lie symmetry, Mei symmetry, conserved quantity, phase space

PACC: 0330, 0320

1. Introduction

There is a close relation between the symmetry and the conserved quantity in a mechanical system. The study of the theory and application of the symmetry and conserved quantity of a mechanical system is an important topic in mathematics, mechanics and physics. There are mainly the Noether symmetry,[1] the Lie symmetry[2] and the Mei symmetry (or the form invariance) proposed by professor Mei recently.[3] A series of important results have been obtained on the study of the Noether symmetry, the Lie symmetry and the Mei symmetry for classical mechanical systems.[4−16] Recently, the Lie symmetry of the rotational relativistic Hamilton system has been studied in Ref.[17]. The Mei symmetry and Lie symmetry of the rotational relativistic variable mass system have been studied in Ref.[18]. The form invariance and Lie symmetry of the relativistic mechanical system has been studied in Ref.[19]. In this paper, we study the Lie symmetry and the Mei symmetry of a rotational relativistic system in phase space. The definition, criterion and conserved quantity of the Lie symmetry and the Mei symmetry of the rotational relativistic system in phase space are given. The relation between the Lie symmetry and the Mei symmetry is found. An example is given to illustrate the application of the result.

2. The dynamical equation of the system in phase space

We consider a mechanical system consisting of N particles. We suppose that, at time t, the particle i rotates about the fixed axis oz with angular velocity ω_i; its limiting angular velocity is Γ_i, classical moment of inertia is I_{0i} and relativistic moment of inertia is

$$I_i = \frac{I_{0i}}{\sqrt{1-\omega_i^2/\Gamma_i^2}} \quad (i = 1, \ldots, N). \tag{1}$$

Suppose that the configuration of the mechanical system is determined by n generalized coordinates $q_s (s = 1, \ldots, n)$. With the ideal holonomic constraints or without any constraints, the Lagrange equations of the rotational relativistic system are as follows:[20]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r}{\partial \dot{q}_s} - \frac{\partial L_r}{\partial q_s} = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \ldots, n), \tag{2}$$

where Q_s is the non-potential generalized force, and L_r is the generalized Lagrangian of the rotational relativistic system

$$L_r = \sum_{i=1}^{N} I_{0i}\Gamma_i^2\left(1 - \sqrt{1-\omega_i^2/\Gamma_i^2}\right) - V. \tag{3}$$

Introducing generalized momentum p_s and Hamiltonian H, we have

$$p_s = \frac{\partial L_r}{\partial \dot{q}_s}, \quad H(t, \boldsymbol{q}, \boldsymbol{p}) = \sum_{s=1}^{n} p_s \dot{q}_s - L_r, \quad (s = 1, \ldots, n). \tag{4}$$

We can obtain the canonical forms of Eq.(2) as follows:

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + \bar{Q}_s, \quad (s = 1, \ldots, n), \tag{5}$$

where $\bar{Q}_s = \bar{Q}_s(t, \boldsymbol{q}, \boldsymbol{p}) = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}(t, \boldsymbol{q}, \boldsymbol{p}))$.

3. Lie symmetry of the system in phase space

From Eqs.(5) we can obtain

$$\dot{q}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad \dot{p}_s = a_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad (s = 1, \ldots, n). \tag{6}$$

Introduce the infinitesimal transformations

$$\begin{aligned} t^* &= t + \Delta t, \\ q_s^*(t^*) &= q_s(t) + \Delta q_s, \\ p_s^*(t^*) &= p_s(t) + \Delta p_s, \end{aligned} \tag{7}$$

the corresponding expanding forms are

$$\begin{aligned} t^* &= t + \varepsilon \xi_0(t, \boldsymbol{q}, \boldsymbol{p}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^*(t^*) &= p_s(t) + \varepsilon \eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \tag{8}$$

where ε is the infinitesimal parameter, ξ_0, ξ_s and η_s are the generators of infinitesimal transformations.

Introduce the generator vector of infinitesimal transformations

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \sum_{s=1}^{n} \xi_s \frac{\partial}{\partial q_s} + \sum_{s=1}^{n} \eta_s \frac{\partial}{\partial p_s}, \tag{9}$$

and its first extensions

$$\begin{aligned} X^{(1)} =& X^{(0)} + \sum_{s=1}^{n} (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s} \\ &+ \sum_{s=1}^{n} (\dot{\eta}_s - \dot{p}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{p}_s}. \end{aligned} \tag{10}$$

Definition 1 If Eqs.(6) remain invariant under the infinitesimal transformations (8), then the invariance is called the Lie symmetry of the rotational relativistic system.

Criterion 1 For the rotational relativistic system (5), if the infinitesimal generators ξ_0, ξ_s and η_s satisfy the following relations:

$$\begin{aligned} &\dot{\xi}_s - \dot{q}_s \dot{\xi}_0 = X^{(0)}(h_s(t, \boldsymbol{q}, \boldsymbol{p})), \\ &\dot{\eta}_s - \dot{p}_s \dot{\xi}_0 = X^{(0)}(a_s(t, \boldsymbol{q}, \boldsymbol{p})), \end{aligned} \quad (s = 1, \ldots, n), \tag{11}$$

the corresponding symmetry is the Lie symmetry of the system.

Proof According to the invariance criterion of the differential equation, the invariance of the differential equations of motion (6) under the infinitesimal transformations (8) can be written as

$$\begin{aligned} X^{(1)}(\dot{q}_s - h_s) = 0, \\ X^{(1)}(\dot{p}_s - a_s) = 0. \end{aligned} \tag{12}$$

Using Eqs.(9) and (10), we can obtain from Eqs.(12)

$$\begin{aligned} X^{(1)}(\dot{q}_s - h_s) =& X^{(0)}(\dot{q}_s - h_s) \\ &+ \sum_{s=1}^{n} (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s} (\dot{q}_s - h_s) \\ &+ \sum_{s=1}^{n} (\dot{\eta}_s - \dot{p}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{p}_s} (\dot{q}_s - h_s) \\ =& - X^{(0)}(h_s) + (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) = 0, \end{aligned}$$

so we have

$$\dot{\xi}_s - \dot{q}_s \dot{\xi}_0 = X^{(0)}(h_s).$$

Using the same method, we can obtain

$$\dot{\eta}_s - \dot{p}_s \dot{\xi}_0 = X^{(0)}(a_s),$$

then Eqs.(11) have been proved.

4. Mei symmetry of the system in phase space

Under the infinitesimal transformations (8), $H = H(t, \boldsymbol{q}, \boldsymbol{p})$, $\bar{Q}_s = \bar{Q}_s(t, \boldsymbol{q}, \boldsymbol{p})$ become $H^* = H(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$, $\bar{Q}_s^* = \bar{Q}_s(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$.

Definition 2 If the canonical equation (5) of the rotational relativistic system keep their form invariant under the infinitesimal transformations (8), i.e.

$$\dot{q}_s^* = \frac{\partial H^*}{\partial p_s}, \quad \dot{p}_s^* = -\frac{\partial H^*}{\partial q_s} + \bar{Q}_s^*, \quad (s = 1, \ldots, n), \tag{13}$$

then the invariance is called the Mei symmetry of the rotational relativistic system in phase space.

Expanding H^* and $\bar{Q}_s^*$ under the infinitesimal transformations (8), we have

$$H^* = H + \varepsilon (X^{(0)}(H)) + O(\varepsilon^2), \tag{14}$$

$$\bar{Q}_s^* = \bar{Q}_s + \varepsilon (X^{(0)}(\bar{Q}_s)) + O(\varepsilon^2). \tag{15}$$

Criterion 2 For the rotational relativistic system (5), if the infinitesimal generators ξ_0, ξ_s and η_s satisfy the following relations:

$$\begin{aligned}\dot{\xi}_s =&\frac{\partial}{\partial p_s}(X^{(0)}(H)),\\ \dot{\eta}_s =&-\frac{\partial}{\partial q_s}(X^{(0)}(H))+X^{(0)}(\bar{Q}_s),\quad (s=1,\dots,n)\end{aligned} \tag{16}$$

then the corresponding symmetry is the Mei symmetry of the system.

Proof Computing the time derivative of q_s^*, we obtain $\dot{q}_s^*=\dot{q}_s+\varepsilon\dot{\xi}_s$, substituting this equation and Eq.(14) into the first equation in Eqs.(13), and ignoring ε^2 and higher infinitesimal terms, we have

$$\dot{q}_s+\varepsilon\dot{\xi}_s=\frac{\partial H}{\partial p_s}+\varepsilon\frac{\partial}{\partial p_s}(X^{(0)}(H)).$$

Substituting $\dot{q}_s=\dfrac{\partial H}{\partial p_s}$ into above equation, we can prove the first equation in Eqs.(16). The proof of the second equation in Eqs.(16) is similar to the first equation in Eqs.(16).

5. Relation between Lie symmetry and Mei symmetry

From Eqs.(11) and (16), it can be seen that the Lie symmetry is generally different from the Mei symmetry. Using Eqs.(5) and (6), we can write Eqs.(11) as

$$\begin{aligned}\dot{\xi}_s =&\dot{\xi}_0\frac{\partial H}{\partial p_s}+X^{(0)}\left(\frac{\partial H}{\partial p_s}\right),\\ \dot{\eta}_s =&\dot{\xi}_0\left(-\frac{\partial H}{\partial q_s}+\bar{Q}_s\right)-X^{(0)}\left(\frac{\partial H}{\partial q_s}\right)+X^{(0)}(\bar{Q}_s).\end{aligned} \tag{17}$$

We can prove

$$\begin{aligned}\frac{\partial}{\partial p_s}(X^{(0)}(H)) =&X^{(0)}\left(\frac{\partial H}{\partial p_s}\right)+\frac{\partial H}{\partial t}\frac{\partial \xi_0}{\partial p_s}\\ &+\sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\frac{\partial \xi_k}{\partial p_s}+\sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\frac{\partial \eta_k}{\partial p_s},\\ \frac{\partial}{\partial q_s}(X^{(0)}(H)) =&X^{(0)}\left(\frac{\partial H}{\partial q_s}\right)+\frac{\partial H}{\partial t}\frac{\partial \xi_0}{\partial q_s}\\ &+\sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\frac{\partial \xi_k}{\partial q_s}+\sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\frac{\partial \eta_k}{\partial q_s},\end{aligned}$$

so there is a theorem as follows.

Theorem 1 If the rotational relativistic system (5) is Lie symmetrical under the infinitesimal transformations (8), and satisfies the following equations:

$$\begin{aligned}\dot{\xi}_0\frac{\partial H}{\partial p_s}=&\frac{\partial H}{\partial t}\frac{\partial \xi_0}{\partial p_s}+\sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\frac{\partial \xi_k}{\partial p_s}+\sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\frac{\partial \eta_k}{\partial p_s},\\ \dot{\xi}_0\left(-\frac{\partial H}{\partial q_s}+\bar{Q}_s\right)=&\frac{\partial H}{\partial t}\frac{\partial \xi_0}{\partial q_s}+\sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\frac{\partial \xi_k}{\partial q_s}\\ &+\sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\frac{\partial \eta_k}{\partial q_s},\end{aligned} \tag{18}$$

then the system is also Mei symmetrical, and vice versa.

6. Conserved quantities of Lie symmetry and Mei symmetry

The Lie symmetry and Mei symmetry of the rotational relativistic system in phase space do not always have corresponding conserved quantities. The conserved quantities of the Lie symmetry and Mei symmetry are given by the following theorem.

Theorem 2 If the rotational relativistic system (5) is Lie symmetrical under the infinitesimal transformations (8), and there exists a gauge function $G_L=G_L(t,\boldsymbol{q},\boldsymbol{p})$ satisfying the structure equation

$$\begin{aligned}&\sum_{s=1}^{n}\left(\frac{\partial H}{\partial p_s}\eta_s+p_s\dot{\xi}_s\right)-H\dot{\xi}_0-X^{(0)}(H)\\ &+\sum_{s=1}^{n}(\xi_s-\dot{q}_s\xi_0)\bar{Q}_s+\dot{G}_L=0,\end{aligned} \tag{19}$$

then the system has the Lie symmetrical conserved quantity as follows:

$$I_L=\sum_{s=1}^{n}p_s\xi_s-H\xi_0+G_L=\text{const.} \tag{20}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I_L}{\mathrm{d}t}=&\sum_{s=1}^{n}(\dot{p}_s\xi_s+p_s\dot{\xi}_s)-\dot{H}\xi_0-H\dot{\xi}_0\\ &-\sum_{s=1}^{n}(\frac{\partial H}{\partial p_s}\eta_s+p_s\dot{\xi}_s)+H\dot{\xi}_0\\ &+X^{(0)}(H)-\sum_{s=1}^{n}(\xi_s-\frac{\partial H}{\partial p_s}\xi_0)\bar{Q}_s+\dot{G}_L\\ =&-\frac{\partial H}{\partial t}\xi_0-\sum_{s=1}^{n}\frac{\partial H}{\partial q_s}\xi_s\\ &-\sum_{s=1}^{n}\frac{\partial H}{\partial p_s}\eta_s+X^{(0)}(H)=0.\end{aligned}$$

Theorem 3 If the rotational relativistic system (5) is Mei symmetrical under the infinitesimal transformations (8), and there exists a gauge function $G_M = G_M(t, \boldsymbol{q}, \boldsymbol{p})$ satisfying the structure equation

$$\sum_{s=1}^{n}\left(\frac{\partial H}{\partial p_s}\eta_s + p_s\dot{\xi}_s\right) - H\dot{\xi}_0 - X^{(0)}(H) + \sum_{s=1}^{n}(\xi_s - \dot{q}_s\xi_0)\bar{Q}_s + \dot{G}_M = 0, \tag{21}$$

then the system has the Mei symmetrical conserved quantity as follows:

$$I_M = \sum_{s=1}^{n} p_s\xi_s - H\xi_0 + G_M = \text{const.} \tag{22}$$

The proof is similar to Theorem 2.

7. Illustrative example

The generalized Lagrangian of a rotational relativistic system is

$$L_r = I_0\Gamma^2(1 - \sqrt{1 - \dot{q}^2/\Gamma^2}) - V, \tag{23}$$

where the non-potential generalized force $Q = 0$, potential energy V is constant. Now study its Lie symmetry and Mei symmetry in phase space.

We can obtain

$$\begin{aligned} p =& \frac{\partial L_r}{\partial \dot{q}} = \frac{I_0\dot{q}}{\sqrt{1 - \dot{q}^2/\Gamma^2}}, \\ H =& p\dot{q} - L_r = \Gamma^2\sqrt{I_0^2 + p^2/\Gamma^2} - I_0\Gamma^2 + V, \\ \dot{q} =& \frac{\partial H}{\partial p} = \frac{p}{\sqrt{I_0^2 + p^2/\Gamma^2}}, \\ \dot{p} =& -\frac{\partial H}{\partial q} + \bar{Q} = 0. \end{aligned} \tag{24}$$

Taking

$$\xi_0 = 1, \quad \xi = 1, \quad \eta = 0, \tag{25}$$

we find that Eqs.(11) [or (17)] and (18) are all satisfied, so for generators (25) the system is both Lie symmetrical and Mei symmetrical. For generators (25), from Eqs.(19) and (21), we can obtain the gauge function $G_L = G_M = 0$. From Eqs.(20) and (22), we can obtain the same conserved quantity corresponding to the Lie symmetry and Mei symmetry

$$I = p - \Gamma^2\sqrt{I_0^2 + p^2/\Gamma^2} + I_0\Gamma^2 = \text{const.} \tag{26}$$

References

[1] Noether A E 1918 *Math. Phys.* KI **II** 235
[2] Lutzky M 1979 *J Phys. A: Math. Gen.* **12** 973
[3] Mei F X 2001 *Chin. Phys.* **10** 177
[4] Liu D 1990 *Sci. Chin.* **23** 790
[5] Zhao Y Y 1994 *Acta Mech. Sin.* **26** 380
[6] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[7] Zhao Y Y and Mei F X 1999 *Symmetries and Conserved Quantities of Mechanics Systems* (Beijing: Science Press) (in Chinese)
[8] Zhang Y and Mei F X 2000 *Appl. Math. Mech.* **21** 59
[9] Guo Y X 2001 *Chin. Phys.* **10** 181
[10] Wang S Y and Mei F X 2001 *Chin. Phys.* **10** 373
[11] Zhang H B 2002 *Chin. Phys.* **11** 1
[12] Wang S Y and Mei F X 2002 *Chin. Phys.* **11** 5
[13] Fang J H, Xue Q Z and Zhao S Q 2002 *Acta Phys. Sin.* **51** 2183 (in Chinese)
[14] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)
[15] Chen P S and Fang J H 2003 *Acta Phys. Sin.* **52** 1044 (in Chinese)
[16] Luo S K 2003 *Acta Phys. Sin.* **52** 2941 (in Chinese)
[17] Luo S K 2003 *Commun. Theor. Phys.* **40** 265
[18] Fang J H 2003 *Commun. Theor. Phys.* **40** 269
[19] Fang J H, Chen P S, Zhang J and Li H 2003 *Acta Phys. Sin.* **52** 2945 (in Chinese)
[20] Luo S K 1998 *Appl. Math. Mech.* **19** 43 (in Chinese)