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# A type of multi-component integrable hierarchy

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A new isospectral problem is established by constructing a simple interesting loop algebra. A commutation operation of the loop algebra is as straightforward as the loop algebra $\tilde{A}_1$. It follows that a type of multi-component integrable hierarchy is obtained. This can be used as a general method.

**Keywords:** loop algebra, multi-component integrable system, integrable coupling
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## 1. Introduction

How to discover new integrable hierarchies of soliton equations has been an important research topic in soliton theory.[1] Professor Tu employed some subalgebras of the loop algebra $\tilde{A}_1$ to obtain a host of interesting results on infinite integrable systems in the frame of the zero curvature equation.[2,3] The procedures proposed by Tu is known as the Tu scheme.[4] Recently, the authors of Refs.[5]–[12] further developed the Tu scheme and produced some integrable Hamiltonian hierarchies with physical background. To look for integrable hierarchies with multi-potential functions, the authors of Refs.[13] have generalized the Tu scheme to obtain the multi-component Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy. Guo[14−16] developed a few trick approaches to produce integrable hierarchies with multi-potential functions. In this paper, a new isospectral problem is established by constructing a simple loop algebra $\tilde{G}_M$,[16] and a new multi-component integrable hierarchy is worked out by adopting the Tu scheme. Thus the study of integrable couplings has become very attractive in soliton theory. Some work in this direction has been performed in Ref.[17]. A simple method for obtaining integrable couplings was proposed in Ref.[18]. Following this method, the integrable couplings of the systems, such as the AKNS system, the Kaup–Newell(KN) system and so on, were given.[19−23] In this paper, an expanded loop algebra $\tilde{F}_M$ of the loop algebra $\tilde{G}_M$ is constructed; the integrable coupling systems of the multi-component integrable hierarchy obtained above can be worked out following the procedures in Refs.[18–23].

## 2. A new loop algebra and its application

Let $G_M = \{a = (a_{ij})_{M\times 3} = (a_1, a_2, a_3)\}$ be a set of matrices, $a_i$ being the $i$th column, then $G_M$ is a linear space.

**Definition 1** A vector product with respect to two vectors $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_M)^{\mathrm{T}}, \beta = (\beta_1, \beta_2, \cdots, \beta_M)^{\mathrm{T}}$ is defined as

$$\begin{aligned}\alpha * \beta =& \beta * \alpha \\ =& (\alpha_1 * \beta_1, \alpha_2 * \beta_2, \cdots, \alpha_M * \beta_M)^{\mathrm{T}}. \end{aligned} \quad (1)$$

**Definition 2** Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be two elements in $G_M$, a commutation operation $[a, b]$ is defined as

$$\begin{aligned}[a, b] =& (a_2 * b_3 - a_3 * b_2, a_1 * b_2 \\ & - a_2 * b_1, a_3 * b_1 - a_1 * b_3). \end{aligned} \quad (2)$$

It is easy to verify that expression (2) is linear and antisymmetric and satisfies

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \forall a, b, c \in G_M. \quad (3)$$

Therefore, $G_M$ along with operation (2) is a Lie algebra.

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Denote

$$\tilde{G}_M = \{a\lambda^m, a \in G_M, m = 0, \pm 1, \pm 2, \cdots\}, \quad (4)$$

with a commutation operation

$$[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, \forall a, b \in G_M.$$

Hence, $\tilde{G}_M$ consists of a loop algebra, which has two features as follows:

(i) The commutation operation in $\tilde{G}_M$ is simple and straightforward as that in the loop algebra $\tilde{A}_1$;

(ii) By making use of $\tilde{G}_M$ and employing several clear operation steps, a host of integrable hierarchies with multi-component potential functions can be manifested.

In general, the compatibility of the following two isospectral problems:

$$\begin{cases} \varphi_x = [U, \varphi], \\ \varphi_t = [V, \varphi], \end{cases} \quad (5)$$

gives rise to

$$\begin{aligned} &[U_t, \varphi] + [U, [V, \varphi]] = [V_x, \varphi] + [V, [U, \varphi]], \\ &[U_t, \varphi] - [V_x, \varphi] + [U, [V, \varphi]] + [V, [\varphi, U]] = 0. \end{aligned} \quad (6)$$

Taking advantage of Eq.(3),Eq.(6) can be written as

$$[U_t, \varphi] - [V_x, \varphi] + [[U, V], \varphi] = 0. \quad (7)$$

Due to $\varphi$ being arbitrary, Eq.(7) holds if and only if the following holds:

$$U_t - V_x + [U, V] = 0, \quad (8)$$

i.e. the zero curvature equation holds. Therefore, an arbitrary system derived from the Lax pair (5) is Lax integrable.

For convenience, the product operation symbol $*$ in $\tilde{G}_M$ will be omitted in the following text.

In terms of Eq.(4), an isospectral problem is established as

$$\phi_x = U\phi, \lambda_t = 0, U = \left(\lambda I_M, u_1 + \frac{u_2}{\lambda}, u_3 + \frac{u_4}{\lambda}\right), \quad (9)$$

where $I_M = (1, 1, \cdots, 1)^{\mathrm{T}}_{1\times M}$, $u_i = (u_{i1}, u_{i2}, \cdots, u_{iM})^{\mathrm{T}}$, $i = 1, 2, 3, 4$.

Note that

$$\begin{aligned} V = \sum_{m=0}^{\infty} &(a(0, m) + \frac{1}{\lambda}a(1, m), b(0, m) \\ &+ \frac{1}{\lambda}b(1, m), c(0, m) + \frac{1}{\lambda}c(1, m))\lambda^{-2m}, \end{aligned}$$

where

$$a(0, m) = \begin{pmatrix} a^{(0)}_{m1} \\ a^{(0)}_{m2} \\ \vdots \\ a^{(0)}_{mM} \end{pmatrix}, a(1, m) = \begin{pmatrix} a^{(1)}_{m1} \\ a^{(1)}_{m2} \\ \vdots \\ a^{(1)}_{mM} \end{pmatrix}, \cdots$$

Solving the adjoint equation

$$V_x = [U, V] \quad (10)$$

gives

$$\begin{cases} a_x(0, m) = u_1c(0, m) - u_3b(0, m) + u_2c(1, m-1) - u_4b(1, m-1), \\ a_x(1, m) = u_2c(0, m) + u_1c(1, m) - u_4b(0, m) - u_3b(1, m), \\ b_x(0, m) = b(1, m) - u_1a(0, m) - u_2a(1, m-1), \\ b_x(1, m) = b(0, m+1) - u_2a(0, m) - u_1a(1, m), \\ c_x(1, m) = -c(0, m+1) + u_4a(0, m) + u_3a(1, m), \\ c_x(0, m) = u_3a(0, m) - c(1, m) + u_4a(1, m-1), \\ a(0, 0) = b(0, 0) = c(0, 0) = c(1, 0) = b(1, 0) = 0, \\ a(1, 0) = \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_M)^{\mathrm{T}}, \alpha_i \quad \text{are constants}, \\ b(0, 1) = \alpha u_1, c(0, 1) = \alpha u_3, a(0, 1) = 0, \\ b(1, 1) = \alpha(u_{1x} + u_2), c(1, 1) = \alpha(-u_{3x} + u_4, a(1, 1) = -\alpha u_1 u_3. \end{cases} \quad (11)$$

By means of

$$\begin{cases} V^{(n)}_+ = \sum_{m=0}^{n} (a(0, m) + \frac{1}{\lambda}a(1, m), b(0, m) + \frac{1}{\lambda}b(1, m), c(0, m) + \frac{1}{\lambda}c(1, m))\lambda^{2n-2m}, \\ V^{(n)}_- = \lambda^{2n}V - V^{(n)}_+, \end{cases}$$

then Eq.(10) can be expressed as

$$-V_{+x}^{(n)}+[U,V_{+}^{(n)}]=V_{-x}^{(n)}-[U,V_{-}^{(n)}]. \tag{12}$$

A direct calculation gives

$$\begin{aligned}&-V_{+x}^{(n)}+[U,V_{+}^{(n)}]\\ =&(a_x(0,n+1)-u_2c(0,n+1)+u_3b(0,n+1),\\ &b_x(0,n+1)-b(1,n+1)\\ &+u_1a(0,n+1)-b(0,n+1),\\ &c_x(0,n+1)-u_3a(0,n+1)\\ &+c(1,n+1)+c(0,n+1)).\end{aligned}$$

If

$$\begin{aligned}V^{(n)}=&V_{+}^{(n)}+(\partial^{-1}(a_x(0,n+1)\\ &-u_1c(0,n+1)+u_3b(0,n+1)),0,0),\end{aligned}$$

then

$$\begin{aligned}&-V_{x}^{(n)}+[U,V^{(n)}]\\ =&(0,b_x(0,n+1)-b(1,n+1)-b(0,n+1)\\ &+u_1\partial^{-1}(u_1c(0,n+1)\\ &-u_3b(0,n+1))+\frac{u_2}{\lambda}(a(0,n+1)\\ &+\partial^{-1}(-u_1c(0,n+1)+u_3b(0,n+1))),\\ &c_x(0,n+1)+c(1,n+1)+c(0,n+1)\\ &+u_3\partial^{-1}(-u_1c(0,n+1)+u_3b(0,n+1))\\ &+\frac{u_4}{\lambda}(a(0,n+1)+\partial^{-1}(-u_1c(0,n+1)\\ &+u_3b(0,n+1)))).\end{aligned}$$

Hence, the zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0 \tag{13}$$

admits

$$\begin{aligned}u_t=&\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}_t=\begin{pmatrix}b_x(0,n+1)-b(1,n+1)-b(0,n+1)+u_1\partial^{-1}(u_1c(0,n+1)-u_3b(0,n+1))\\u_2a(0,n+1)+u_2\partial^{-1}(-u_1c(0,n+1)+u_3b(0,n+1))\\c_x(0,n+1)+c(1,n+1)+c(0,n+1)+u_3\partial^{-1}(-u_1c(0,n+1)+u_3b(0,n+1))\\u_4a(0,n+1)+u_4\partial^{-1}(-u_1c(0,n+1)+u_3b(0,n+1))\end{pmatrix}\\ =&\begin{pmatrix}-\partial+u_1\partial^{-1}u_4&-u_1-u_2&-u_1\partial^{-1}u_2&-u_2\\-u_2u_4&0&u_2^2&0\\-u_3\partial^{-1}u_4&u_3+u_4&-\partial+u_3\partial^{-1}u_2&u_4\\-u_4\partial^{-1}u_4&0&u_4\partial^{-1}u_2&0\end{pmatrix}\begin{pmatrix}b(1,n)\\a(1,n)\\c(1,n)\\a(0,n)\end{pmatrix}=J\begin{pmatrix}b(1,n)\\a(1,n)\\c(1,n)\\a(0,n)\end{pmatrix}.\end{aligned} \tag{14}$$

From Eq.(11), a recurrence operator is given as

$$L=\begin{pmatrix}\partial^2+u_1\partial^{-1}(u_3u_4-u_3\partial)&\partial u_1+u_1\partial^{-1}u_2&u_1\partial^{-1}(u_2u_3-u_1\partial)&M_1\\\partial^{-1}A_1&\partial^{-1}B_1&\partial^{-1}C_1&\partial^{-1}D_1\\u_3^2u_4-u_3^2\partial&B_2&\partial^2-u_1u_3\partial-u_2u_3^2&M_2\\u_3u_4-u_3\partial&u_3^2-u_1u_3&-u_1\partial-u_2u_3&u_1u_4-u_2u_3\end{pmatrix},$$

with

$$\begin{aligned}M_1=&\partial u_2+u_1\partial^{-1}(u_1u_4-u_2u_3),\\ M_2=&\partial u_4+u_1u_3u_4-u_2u_3^2,\\ A_1=&u_3^2u_4-2u_3^2\partial-u_1u_3\partial^{-1}(u_3u_4-u_3\partial)-u_4\partial,\\ B_1=&u_2u_3+u_1(\partial u_3+u_3^3-u_1u_3^2),\\ C_1=&-u_2\partial+u_1(\partial^2-u_1u_3\partial\\ &-u_2u_3^2-u_1u_3\partial^{-1}(u_2u_3-u_1\partial)),\\ D_1=&u_1(\partial u_4+u_1u_3u_4-u_2u_3^2)\\ &-u_3(\partial u_2+u_1\partial^{-1}(u_1u_4-u_2u_3)),\\ B_2=&\partial u_3+u_3^3-u_1u_3^2+u_4,\end{aligned}$$

which meets the following relation:

$$\begin{pmatrix}b(1,n+1)\\a(1,n+1)\\c(1,n+1)\\a(0,n+1)\end{pmatrix}=L\begin{pmatrix}b(1,n)\\a(1,n)\\c(1,n)\\a(0,n)\end{pmatrix}.$$

Therefore, the multicomponent integrable hierarchy (14) can be expressed as

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}_t=JL^n\begin{pmatrix}\alpha(u_{1x}+u_2)\\-\alpha u_1u_3\\\alpha(-u_{3x}+u_4)\\0\end{pmatrix}. \tag{15}$$

## 3. An expanded loop algebra of the loop algebra $\tilde{G}_M$

A few types of expanded loop algebras of the loop algebra $\tilde{A}_1$ were constructed in Refs.[15–19] to obtain the integrable coupling systems of some known integrable hierarchies, such as the AKNS system, KN system, etc. The key idea is that if $\tilde{G}$ is an expanded loop algebra of the loop algebra $\tilde{A}_1$, then $\tilde{G}_1$ and $\tilde{G}_2$ are subalgebras of $\tilde{G}$, and satisfy

$$\begin{aligned}&\text{(i)}\tilde{G}=\tilde{G}_1\oplus\tilde{G}_2,\tilde{G}_1\simeq\tilde{A}_1,\\&\text{(ii)}[\tilde{G}_1,\tilde{G}_2]\subset\tilde{G}_2,\end{aligned}$$

where the symbol $\simeq$ stands for isomorphic relation. Then integrable couplings of some known integrable hierarchies can be produced with the help of (i) and (ii) above.

Following the above idea and taking

$$F_M=\{a=(a_{ij})_{M\times 5}=(a_1,a_2,\cdots,a_5)\},\qquad(16)$$

we find that $F_M$ is a linear space.

**Definition 3** For arbitrary $a=(a_1,a_2,\cdots,a_5)$, $b=(b_1,b_2,\cdots,b_5)\in F_M$, a commutation operation $[a,b]$ is defined as

$$\begin{aligned}[a,b]=&(a_2*b_3-a_3*b_2,\\&a_1*b_2-a_2*b_1,a_3*b_1-a_1*b_3,\\&a_1*b_4-a_4*b_1+a_2*b_5-a_5*b_2,\\&a_3*b_4-a_4*b_3+a_5*b_1-a_1*b_5).\end{aligned}\qquad(17)$$

It is easy to verify that $F_M$ along with expression (17) constitutes a Lie algebra. The corresponding loop algebra reads

$$\tilde{F}_M=\{a\lambda^n,a\in F_M,n=0,\pm1,\pm2,\cdots\},\qquad(18)$$

with a commutation operation

$$[a\lambda^m,b\lambda^n]=[a,b]\lambda^{m+n},\forall a,b\in F_M.$$

Two subalgebras $\tilde{F}_M(1)$ and $\tilde{F}_M(2)$ of the loop algebra $\tilde{F}_M$ are defined as follows, respectively:

$$\tilde{F}_M(1)=\{(a_1,a_2,a_3,0,0)\lambda^n\},$$

$$\tilde{F}_M(2)=\{(0,0,0,a_4,a_5)\lambda^n\},$$

and then

$$\text{(iii)}\tilde{F}_M=\tilde{F}_M(1)\oplus\tilde{F}_M(2),\tilde{F}_M(1)\simeq\tilde{G}_M,$$

$$\text{(iv)}[\tilde{F}_M(1),\tilde{F}_M(2)]\subset\tilde{F}_M(2).$$

By means of (iii) and (iv), we can deduce a type of integrable coupling system of the system (15) following the line given in Refs.[18–23]. Since such computing is too tedious, we shall discuss the question in another paper.

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