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The soliton-like solutions to the (2+1)-dimensional modified dispersive water-wave system*

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By a simple transformation, we reduce the (2+1)-dimensional modified dispersive water-wave system to a simple nonlinear partial differential equation. In order to solve this equation by generalized tanh-function method, we only need to solve a simple system of first-order ordinary differential equations, and by doing so we can obtain many new soliton-like solutions which include the solutions obtained by using the conventional tanh-function method.

Keywords: (2+1)-dimensional modified dispersive water-wave system, generalized tanh-function method, soliton-like solutions

PACC: 0340K, 0290, 1190

1. Introduction

The tanh method, developed some years ago, is one of the most direct and effective algebraic method for finding exact solutions of nonlinear equations.[1–3] Recently, much work has concentrated on various extensions and applications of the method.[2–13] Parkes and Duffy [3–5] have further developed a powerful automated tanh method, in which a Mathematica package can deal with the tedious algebraic computation and find required solutions. Other generalizations have been carried out by Gao and Tian [6,7] and Gudkov [8] to find more general soliton-like solutions. Recently, a generalized tanh method has been presented by Fan [9–13] for obtaining multiple travelling wave solutions. The key idea is to use the solutions of a Riccati equation to replace the tanh function in the tanh method. Some applications of the method for finding soliton-like solutions appeared in Refs.[14–16]. In Ref.[14], we can see that very tedious calculations are involved, and many complex nonlinear partial differential equations have to be dealt with when solving (2+1)-dimensional coupled nonlinear partial differential equations by using the tanh-function method directly. Although we can do it with the help of symbolic computation systems, the structure of the solutions is greatly confined by the processing capability of the computer that cannot use the tanh method effectively.

In this paper, we would consider that the following (2+1)-dimensional modified dispersive water-wave system:[17]

$$u_{yt} + u_{xxy} - 2v_{xx} - (u^2)_{xy} = 0, \tag{1}$$

$$v_t - v_{xx} - 2(uv)_x = 0, \tag{2}$$

used to model the nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow water of uniform depth, and can also be derived from the celebrated Kadomtsev–Petviashvili (KP) equation by the symmetry constraint.[18] It is worthwhile mentioning that this system has been widely applied to many branches of physics such as plasma physics, fluid dynamics, nonlinear optics, etc. So a good understanding of more solutions of Eqs.(1) and (2) is very helpful, especially for coastal and civil engineers to apply the nonlinear water model in harbour and coastal design. Meanwhile, finding more types of solutions to the system (1) and (2) is of fundamental interest in fluid dynamics.

In Refs.[19, 20], many significant localized coherent structures have been obtained by using the stan-

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dard truncated Painlevé expansion and the variable separation method. In this paper, in order to avoid tedious calculations and the treatment of a large number of nonlinear partial differential equations, we reduce the (2+1)-dimensional modified dispersive water-wave system to a simple nonlinear partial differential equation by a simple transformation. To solve this equation by the generalized tanh-function method, we only need to solve a simple system of first-order ordinary differential equations, and by doing so we can obtain many new soliton-like solutions, including the solutions obtained by using the conventional tanh method.

2. Soliton-like solutions

In order to apply the generalized tanh-function method, we first introduce the transformation

$$v = u_y. \tag{3}$$

Substituting Eq.(3) into Eqs.(1)–(2), we obtain the following single equation:

$$u_{yt} - u_{xxy} - (u^2)_{xy} = 0. \tag{4}$$

It is obvious that Eq.(4) is equivalent to

$$\partial_y(u_t - u_{xx} - 2uu_x) = 0, \tag{5}$$

or

$$u_t - u_{xx} - 2uu_x = \alpha(x,t), \tag{6}$$

where $\alpha(x,t)$ is an arbitrary function of the indicated variables. In what follows, we only consider $\alpha(x,t) = \alpha(t)$.

According to the generalized tanh-function method, we assume that Eq.(6) has the solution of the form

$$\begin{aligned} u(x,y,t) =& f_1(y) + f_2(t) + g(y)\phi(\xi) \\ &+ h(y)\sqrt{\delta + \phi^2(\xi)} \\ =& f_1 + f_2 + g\phi + h\sqrt{\delta + \phi^2}, \end{aligned} \tag{7}$$

and

$$\phi' = \delta + \phi^2, \tag{8}$$

where f_1, f_2, g, h, p, q are functions to be determined, $\phi = \phi(\xi)$, $\xi = p(y)x + q(y,t)$, $\phi' = \mathrm{d}\phi/\mathrm{d}\xi$.

The Riccati equation (8) has the general solutions

$$\phi = -\sqrt{-\delta}\tanh(\sqrt{-\delta}\xi), \quad \delta < 0 \tag{9}$$

$$\phi = -\sqrt{-\delta}\coth(\sqrt{-\delta}\xi), \quad \delta < 0 \tag{10}$$

$$\phi = -\frac{1}{\xi}, \qquad \delta = 0 \tag{11}$$

$$\phi = \sqrt{\delta}\tan(\sqrt{\delta}\xi), \qquad \delta > 0 \tag{12}$$

$$\phi = -\sqrt{\delta}\cot(\sqrt{\delta}\xi), \qquad \delta > 0. \tag{13}$$

Substituting Eq.(7) into Eq.(6) and making use of Eq.(8), we obtain

$$\begin{aligned} & u_t - u_{xx} - 2uu_x - \alpha(t) \\ =& f_{2t}(t) + \delta g(y) q_t(y,t) - 2\delta f_2(t) g(y) p(y) \\ & -2\delta f_1(y) g(y) p(y) \\ & -\alpha(t) - [2\delta h(y) g(y) p(y) + \delta h(y) p^2(y)]\sqrt{\delta + \phi^2} \\ & -[4g(y)h(y)p(y) + 2h(y)p^2(y)]\phi^2\sqrt{\delta + \phi^2} \\ & +[2f_2(t)h(y)p(y) + h(y)q_t(y,t) \\ & -2f_1(y)h(y)p(y)]\phi\sqrt{\delta + \phi^2} \\ & -[2g(y)p^2(y) + 2g^2(y)p(y) + 2h^2(y)p(y)]\phi^3 \\ & +[g(y)q_t(y,t) - 2f_1(y)g(y)p(y) \\ & -2f_2(t)g(y)p(y)]\phi^2 \\ & -[2\delta g^2(y)p(y) + 2\delta g(y)p^2(y) \\ & +2\delta h^2(y)p(y)]\phi = 0. \end{aligned} \tag{14}$$

Collecting all terms with the same power in $\phi^i(\sqrt{\delta + \phi^2})^j$ $(i = 0,1,2,3,; j = 0,1)$ and setting the coefficients of $\phi^i(\sqrt{\delta + \phi^2})^j$ $(i = 0,1,2,3,; j = 0,1)$ to zero yields a simple system of first-order ordinary differential equations:

$$\begin{aligned} & f_{2t}(t) + \delta g(y) q_t(y,t) - 2\delta f_2(t) g(y) p(y) \\ & - 2\delta f_1(y) g(y) p(y) - \alpha(t) = 0, \end{aligned} \tag{15}$$

$$2\delta h(y) g(y) p(y) + \delta h(y) p^2(y) = 0, \tag{16}$$

$$4g(y)h(y)p(y) + 2h(y)p^2(y) = 0, \tag{17}$$

$$2f_2(t)h(y)p(y) + h(y)q_t(y,t) - 2f_1(y)h(y)p(y) = 0, \tag{18}$$

$$2g(y)p^2(y) + 2g^2(y)p(y) + 2h^2(y)p(y) = 0, \tag{19}$$

$$g(y)q_t(y,t) - 2f_1(y)g(y)p(y) - 2f_2(t)g(y)p(y) = 0, \tag{20}$$

$$2\delta g^2(y)p(y) + 2\delta g(y)p^2(y) + 2\delta h^2(y)p(y) = 0. \tag{21}$$

Solving Eq.(16) or Eq. (17), we obtain

$$g(y) = -\frac{p(y)}{2}. \tag{22}$$

Substituting Eq.(22) into Eq.(19) or Eq.(21), we obtain

$$h(y) = \pm\frac{p(y)}{2}. \tag{23}$$

Substituting Eqs.(22), (23) into Eqs.(15), (18), (20), we obtain

$$f_{2t}(t) = \alpha(t), \tag{24}$$

$$q_t(y,t) = 2p(y)[f_1(y) + f_2(t)]. \tag{25}$$

Solving Eqs.(24) and (25), we have

$$f_2(t) = \int \alpha(t)\mathrm{d}t + c, \tag{26}$$

$$q(y,t) = \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y), \tag{27}$$

where c is an arbitrary constant, f_1, p, f_3 are arbitrary functions of y, and α is an arbitrary function of t.

Substituting Eqs.(22), (23), (26) and (27) into Eqs.(3) and (7) gives the following solutions for Eqs.(1) and (2):

(a) Soliton-like solution $(\delta < 0)$

$$\begin{aligned}
u_1(x,y,t) =& f_1(y) + f_2(t) + \frac{p(y)}{2}\sqrt{-\delta}\tanh(\sqrt{-\delta}\xi) \\
& \pm \mathrm{i}\frac{p(y)}{2}\sqrt{-\delta}\mathrm{sech}(\sqrt{-\delta}\xi), \\
v_1(x,y,t) =& f_{1y}(y) + \frac{p_y(y)}{2}\sqrt{-\delta}\tanh(\sqrt{-\delta}\xi) \\
& - \frac{p(y)\xi_y}{2}\delta\mathrm{sech}^2(\sqrt{-\delta}\xi) \pm \mathrm{i}\frac{p_y(y)}{2} \\
& \times \sqrt{-\delta}\mathrm{sech}(\sqrt{-\delta}\xi) \mp \mathrm{i}\frac{p(y)\xi_y}{2}\delta \\
& \times \mathrm{sech}(\sqrt{-\delta}\xi)\tanh(\sqrt{-\delta}\xi), \\
u_2(x,y,t) =& f_1(y) + f_2(t) + \frac{p(y)}{2}\sqrt{-\delta}\coth(\sqrt{-\delta}\xi) \\
& \pm \frac{p(y)}{2}\sqrt{-\delta}\mathrm{csch}(\sqrt{-\delta}\xi), \\
v_2(x,y,t) =& f_{1y}(y) + \frac{p_y(y)}{2}\sqrt{-\delta}\coth(\sqrt{-\delta}\xi) \\
& - \frac{p(y)\xi_y}{2}\delta\mathrm{csch}^2(\sqrt{-\delta}\xi) \pm \frac{p_y(y)}{2} \\
& \times \sqrt{-\delta}\mathrm{csch}(\sqrt{-\delta}\xi) \mp \frac{p(y)\xi_y}{2}\delta \\
& \times \mathrm{csch}(\sqrt{-\delta}\xi)\coth(\sqrt{-\delta}\xi),
\end{aligned}$$

where $\xi = p(y)x + \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y)$.

(b) Rational function solution $(\delta = 0)$

$$u_3(x,y,t) = f_1(y) + f_2(t),$$

or

$$u_3(x,y,t) = f_1(y) + f_2(t) + \frac{p(y)}{\xi},$$

$$v_3(x,y,t) = f_{1y}(y),$$

or

$$v_3(x,y,t) = f_{1y}(y) + \frac{p_y(y)}{\xi} - \frac{p(y)\xi_y}{\xi^2},$$

where $\xi = p(y)x + \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y)$.

(c) Trigonometric function solution $(\delta > 0)$

$$\begin{aligned}
u_4(x,y,t) =& f_1(y) + f_2(t) - \frac{p(y)}{2}\sqrt{\delta}\tan(\sqrt{\delta}\xi) \\
& \pm \frac{p(y)}{2}\sqrt{\delta}\sec(\sqrt{\delta}\xi), \\
v_4(x,y,t) =& f_{1y}(y) - \frac{p_y(y)}{2}\sqrt{\delta}\tan(\sqrt{\delta}\xi) \\
& - \frac{p(y)\xi_y}{2}\delta\sec^2(\sqrt{\delta}\xi) \\
& \pm \frac{p_y(y)}{2}\sqrt{\delta}\sec(\sqrt{\delta}\xi) \\
& \pm \frac{p(y)\xi_y}{2}\delta\sec(\sqrt{\delta}\xi)\tan(\sqrt{\delta}\xi), \\
u_5(x,y,t) =& f_1(y) + f_2(t) + \frac{p(y)}{2}\sqrt{\delta}\cot(\sqrt{\delta}\xi) \\
& \pm \frac{p(y)}{2}\sqrt{\delta}\csc(\sqrt{\delta}\xi), \\
v_5(x,y,t) =& f_{1y}(y) + \frac{p_y(y)}{2}\sqrt{\delta}\cot(\sqrt{\delta}\xi) \\
& - \frac{p(y)\xi_y}{2}\delta\csc^2(\sqrt{\delta}\xi) \\
& \pm \frac{p_y(y)}{2}\sqrt{\delta}\csc(\sqrt{\delta}\xi) \\
& \mp \frac{p(y)\xi_y}{2}\delta\csc(\sqrt{\delta}\xi)\cot(\sqrt{\delta}\xi),
\end{aligned}$$

where $\xi = p(y)x + \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y)$.

Similarly, when we take $h(y) = 0$ in Eq.(7), we can obtain

$$g(y) = -p(y), \tag{28}$$

$$f_{2t}(t) = \alpha(t), \tag{29}$$

$$q_t(y,t) = 2p(y)(f_1(y) + f_2(t)). \tag{30}$$

Solving Eqs.(29) and (30), we have

$$f_2(t) = \int \alpha(t)\mathrm{d}t + c, \tag{31}$$

$$q(y,t) = \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y), \tag{32}$$

where c is an arbitrary constant; f_1, p, f_3 are arbitrary functions of y and α is an arbitrary function of t.

Again using Eqs.(3) and (7), we obtain the following solutions:

(d) Soliton-like solution $(\delta < 0)$

$$\begin{aligned}
u_6(x,y,t) =& f_1(y) + f_2(t) + p(y)\sqrt{-\delta}\tanh(\sqrt{-\delta}\xi), \\
v_6(x,y,t) =& f_{1y}(y) + p_y(y)\sqrt{-\delta} \\
& \times \tanh(\sqrt{-\delta}\xi) - p(y)\xi_y\delta\mathrm{sech}^2(\sqrt{-\delta}\xi), \\
u_7(x,y,t) =& f_1(y) + f_2(t) + p(y)\sqrt{-\delta}\coth(\sqrt{-\delta}\xi), \\
v_7(x,y,t) =& f_{1y}(y) + p_y(y)\sqrt{-\delta} \\
& \times \coth(\sqrt{-\delta}\xi) - p(y)\xi_y\delta\mathrm{csch}^2(\sqrt{-\delta}\xi),
\end{aligned}$$

where $\xi = p(y)x + \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y)$.

(e) Rational function solution $(\delta = 0)$

$$u_8(x,y,t) = f_1(y) + f_2(t) + \frac{p(y)}{\xi},$$

$$v_8(x,y,t) = f_{1y}(y) + \frac{p_y(y)}{\xi} - \frac{p(y)\xi_y}{\xi^2},$$

where $\xi = p(y)x + \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y)$.

(f) Trigonometric function solution $(\delta > 0)$

$$\begin{aligned} u_9(x,y,t) =& f_1(y) + f_2(t) - p(y)\sqrt{\delta}\tan(\sqrt{\delta}\xi), \\ v_9(x,y,t) =& f_{1y}(y) - p_y(y)\sqrt{\delta}\tan(\sqrt{\delta}\xi) \\ & - p(y)\xi_y\delta\sec^2(\sqrt{\delta}\xi), \\ u_{10}(x,y,t) =& f_1(y) + f_2(t) + p(y)\sqrt{\delta}\cot(\sqrt{\delta}\xi), \\ v_{10}(x,y,t) =& f_{1y}(y) + p_y(y)\sqrt{\delta}\cot(\sqrt{\delta}\xi) \\ & - p(y)\xi_y\delta\csc^2(\sqrt{\delta}\xi), \end{aligned}$$

where $\xi = p(y)x + \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y)$.

Remark. In this paper, the tanh and coth type solutions obtained above are called *soliton-like solutions,* while the tan and cot type solutions are called *trigonometric function solutions.* Solitary wave solutions in which $\xi = p(y)x + q(y,t) = p(y)x + \int [2p(y)(f_1(y) + f_2(t))]\mathrm{d}t + f_3(y)$ is the linear combination of x, y and t are special soliton-like solutions.

It is obvious that we have obtained some new exact solutions for the (2+1)-dimensional modified dispersive water-wave system. To our knowledge, these new solutions to the (2+1)-dimensional modified dispersive water-wave system have not been reported.

3. Conclusion and discussion

In summary, a (2+1)-dimensional modified dispersive water-wave system with two dependent variables can be reduced to a simple (1+1)-dimensional nonlinear evolution equation through a simple transformation. Since there is one less independent variable in the new nonlinear evolution equation, we can solve it through the generalized tanh-function method to gain the exact solutions for the original one in the present way.

It is obvious that by using this technique, the amount of computation necessary to solve the nonlinear evolution equations through the generalized tanh-function method is greatly reduced. Meanwhile, this technique leads to some new solutions which we cannot find using the conventional tanh-function method. It is worthy of special mention that we do not need the help of the symbolic computation system of the computer in our solving procedure, but only need to solve a simple system of first order ordinary differential equations.

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