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Characteristic functional structure of infinitesimal symmetry transformations of Birkhoffian systems*

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In this paper, it is shown that infinitesimal symmetry transformations of Birkhoffian systems have a characteristic functional structure, which is formulated by means of an auxiliary symmetry transformation function $Z_\mu\ (t,a)$ (introduced by the relation $\xi_\mu\ (t,a) = Z_\mu(t,a) + \dot{a}^\mu \xi_0(t,\boldsymbol{a})$) and is manifestly dependent upon the constants of motion of the system. At the end of the paper, an example is given to illustrate the application of the results.

Keywords: analytical mechanics, Birkhoffian system, constant of motion, characteristic functional structure, infinitesimal symmetry transformation

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1. Introduction

The symmetry of a mechanical system has a close relation with the conserved quantity. In 1985, Katzin and Levine [1] showed that all (Noether and non-Noether) infinitesimal velocity-dependent symmetry transformations ($\Delta q_s = \varepsilon \xi_s(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$, $\Delta t = \varepsilon \xi_0(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$, $s = 1, \ldots, n$) of second-order dynamical systems were expressible in a form with a characteristic functional structure which was the same for all dynamical systems and was manifestly dependent upon the constants of motion of the system. This characteristic structure was formulated by means of an auxiliary symmetry transformation function $Z_s(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$ (introduced by the relation $\xi_s(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) = Z_s(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) + \dot{\boldsymbol{q}}_s \xi_0(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$). In later accompanying papers,[2,3] Katzin and Levine further investigated the characteristic functional structure for systems of first-order differential equations and dynamical systems with cyclic coordinates.

The dynamics of the Birkhoffian system [4] is an important branch of classical mechanics in modern time. Santilli [5] made a primary research on the Birkhoffian system and obtained some results. Later, the frame of Birkhoff dynamics was constructed.[6] In recent years, the study on the Birkhoff dynamics has become a popular subject and many important results have been obtained.[7–15] In this paper, we further study the characteristic functional structure of the Birkhoffian system. The result shows that infinitesimal symmetry transformations ($\Delta a^\mu = \varepsilon \xi_\mu(\boldsymbol{a}, t)$, $\Delta t = \varepsilon \xi_0(\boldsymbol{a}, t)$, $\mu = 1, 2, \ldots, 2n$) of Birkhoffian systems are also expressible in a form with a characteristic functional structure. At the end of the paper, an example is given to illustrate the application of the results.

2. Characteristic functional structure of infinitesimal symmetry transformations for Birkhoffian systems

The Birkhoffian equations can be written in the contravariant form

$$G(\dot{\boldsymbol{a}}, \boldsymbol{a}, t) = \dot{a}^\mu - \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) = 0, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{1}$$

where the function $B = B(t, \boldsymbol{a})$ is called a Birkhoffian, $R_\mu = R_\mu(t, \boldsymbol{a})$ are called Birkhoff's functions, $\Omega^{\mu\nu}$ is called the contravariant Birkhoff tensor, and a^μ are variables. The system described by Eqs.(1) is called a Birkhoffian system.

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A complete solution of Eqs.(1) is denoted by

$$a^\mu = \Phi^\mu(\gamma^1, \ldots, \gamma^{2n}, t) \equiv \Phi^\mu(\gamma, t), \quad \gamma^\nu = \text{const.}$$
$$(\mu, \nu = 1, 2, \ldots, 2n). \tag{2}$$

Inversion of the solution (2) leads to the $2n$ functionally independent constants of motion

$$I^\nu(\boldsymbol{a}, t) \overset{t}{=} \gamma^\nu, \quad (\nu = 1, 2, \ldots, 2n), \tag{3}$$

where the notation '$\overset{t}{=}$' indicates that the finite solution (2) is used to express a function completely in terms of the path parameter t.

The reciprocity between a complete solution and the constants of motion obtained by its inversion is essential in the determination of the characteristic functional structure of symmetry transformations of Birkhoffian systems.

Introduce the infinitesimal transformations

$$t^* = t + \Delta t, \tag{4}$$

$$a^{\mu^*}(t^*) = a^\mu(t) + \Delta a^\mu, \quad (\mu = 1, 2, \ldots, 2n); \tag{5}$$

or their expansion formulae

$$t^* = t + \varepsilon \xi_0(t, \boldsymbol{a}), \tag{6}$$

$$a^{\mu^*}(t^*) = a^\mu(t) + \varepsilon \xi_\mu(t, \boldsymbol{a}), \quad (\mu = 1, 2, \ldots, 2n). \tag{7}$$

If the infinitesimal transformations (6) and (7) transform the set of all solution curves of Eq.(1) into itself it is customarily said to be a Lie symmetry transformation of the Birkhoffian system (1).[6] Such a transformation is determined by the following condition:

$$\Delta G(\dot{\boldsymbol{a}}, \boldsymbol{a}, t) \overset{\circ}{=} 0, \tag{8}$$

where the'$\overset{\circ}{=}$' notation implies that the differential equation (1) is to be used to eliminate all dot derivatives of the variable from the expressions or equations.

We now find for differential equation (1) that the symmetry condition (8) leads to

$$\dot{\xi}_\mu - \dot{a}^\mu \dot{\xi}_0 - \frac{\partial}{\partial a^\rho}\left[\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\right]\xi_\rho - \frac{\partial}{\partial t}\left[\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\right]\xi_0 = 0,$$
$$(\mu, \nu, \rho = 1, 2, \ldots, 2n). \tag{9}$$

Introduce a decomposition of the infinitesimal transformations (7) by expressing ξ_μ in the form

$$\xi_\mu(\boldsymbol{a}, t) = Z_\mu(\boldsymbol{a}, t) + \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\xi_0(\boldsymbol{a}, t),$$
$$(\mu, \nu = 1, 2, \ldots, 2n). \tag{10}$$

By use of Eqs.(1) and (10) the symmetry condition (9) may be expressed in the form

$$\dot{Z}_\mu(\boldsymbol{a}, t) + K^\mu_\rho Z_\rho(\boldsymbol{a}, t) \overset{\circ}{=} 0,$$
$$(\mu, \rho = 1, 2, \ldots, 2n), \tag{11}$$

where

$$K^\mu_\rho \equiv \frac{\partial}{\partial a^\rho}\left[\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\right],$$
$$(\mu, \nu, \rho = 1, 2, \ldots, 2n). \tag{12}$$

It is of significance that the elimination of ξ_μ from Eq.(9) by the transformation (10) also results in the simultaneous elimination of ξ_0. Hence for the infinitesimal transformations (6) and (7), the solutions $Z_\mu(\boldsymbol{a}, t)$ of Eq.(11) can be immediately used in Eq.(10) to obtain the functions $\xi_\mu(\boldsymbol{a}, t)$ with the function $\xi_0(\boldsymbol{a}, t)$ arbitrarily chosen. Thus for a given $Z_\mu(\boldsymbol{a}, t)$ solution, any choice of $\xi_0(\boldsymbol{a}, t)$ will lead to an associated $\xi_\mu(\boldsymbol{a}, t)$ such that ξ_μ and ξ_0 will satisfy Eq.(9). Then ξ_μ and ξ_0 is the infinitesimal symmetry transformations of Birkhoffian system (1).

For finding ξ_μ and ξ_0 which satisfy Eq.(9), substituting Eqs.(2) into Eqs.(12), we obtain

$$\dot{z}_\mu(\gamma, t) + k^\mu_\rho(\gamma, t) z_\rho(\gamma, t) \overset{t}{=} 0,$$
$$(\mu, \rho = 1, 2, \ldots, 2n), \tag{13}$$

where

$$Z_\mu(\boldsymbol{a}, t) \overset{t}{=} Z_\mu[\Phi(\gamma, t), t] \equiv z_\mu(\gamma, t),$$
$$(\mu = 1, 2, \ldots, 2n), \tag{14}$$

$$K^\mu_\rho(\boldsymbol{a}, t) \overset{t}{=} K^\mu_\rho[\Phi(\gamma, t), t] \equiv k^\mu_\rho(\gamma, t),$$
$$(\mu, \rho = 1, 2, \ldots, 2n). \tag{15}$$

We note that the associated system of Eqs.(13) which must be satisfied by $z_\mu(\gamma, t)$ of (14) is always a system of linear equations, irrespective of whether the original dynamical system (1) is linear or nonlinear. Solutions of Eqs.(13) are expressible in the form

$$z_\mu(\gamma, t) = c^\sigma g^\sigma_\mu(\gamma, t),$$
$$(\mu, \sigma = 1, 2, \ldots, 2n), \tag{16}$$

where the c^σ $(\sigma = 1, \ldots, 2n)$ are arbitrary constants. Since on a trajectory every solution $Z_\mu(\boldsymbol{a}, t)$ of Eq.(11)

has the form of Eq.(16), we are led to assume (to be proved below) that the functions $Z^*_\mu(\boldsymbol{a},t)$ defined by

$$\begin{aligned} Z^*_\mu(\boldsymbol{a},t) \equiv & C^{*\sigma}(\boldsymbol{a},t) g^\sigma_\mu \left[\gamma^1(\boldsymbol{a},t),\ldots,\gamma^{2n}(\boldsymbol{a},t),t\right] \\ \equiv & C^{*\sigma} g^\sigma_\mu(\gamma,t), \end{aligned}$$

$$(\mu,\sigma=1,2,\ldots,2n), \tag{17}$$

will be a solution of the (partial differential) symmetry equation (11). In Eq.(17) the $C^{*\sigma}(\boldsymbol{a},t)$ are arbitrarily chosen constants of motion. Hence, they can always be expressed as functions of $2n$ functionally independent constants of motion $\gamma^\nu(\boldsymbol{a},t)$ (3), which are associated with the inversion of the dynamical solution (2). We may thus write

$$C^{*\sigma}(\boldsymbol{a},t) = c^{*\sigma}\left[\gamma(\boldsymbol{a},t)\right] \overset{t}{=} c^{*\sigma}(\gamma), \quad c^{*\sigma}(\gamma) = \text{const},$$

$$(\sigma=1,2,\ldots,2n). \tag{18}$$

In Eq.(17) the functions $g^\sigma_\mu\left[\gamma(\boldsymbol{a},t)\right]$ are obtained by replacing the constants γ^μ appearing in $g^\sigma_\mu(\gamma,t)$ of Eq.(16) by the respective above-mentioned constants of motion γ^μ by means of Eq.(3).

It will now be verified that the assumed solution $Z^*_\mu(\boldsymbol{a},t)$ (17) satisfies the (partial differential) symmetry equation (11). To do so, it is notably convenient to first define

$$Q_\mu(Z) \equiv \dot{Z}_\mu + K^\mu_\rho Z_\rho,$$

$$(\mu,\rho=1,2,\ldots,2n), \tag{19}$$

so that it is required to prove $Q_\mu(Z^*) \overset{\circ}{=} 0$.

We first form from Eq.(16)

$$\dot{Z}^{*\mu} = \dot{C}^{*\sigma} g^\sigma_\mu + C^{*\sigma}\left(\frac{\partial g^\sigma_\mu}{\partial\gamma^\nu}\dot{\gamma}^\nu + \frac{\partial g^\sigma_\mu}{\partial t}\right),$$

$$(\mu,\nu,\sigma=1,2,\ldots,2n). \tag{20}$$

Since $C^{*\sigma}(\boldsymbol{a},t)$ and γ^ν are constants of motion, we have by use of Eq.(1) $\dot{C}^{*\sigma} \overset{\circ}{=} 0$, $\dot{\gamma}^\nu \overset{\circ}{=} 0$, and we thereby obtain from Eq.(20)

$$\dot{Z}^{*\mu} \overset{\circ}{=} C^{*\sigma}\frac{\partial g^\sigma_\mu}{\partial t},$$

$$(\mu,\sigma=1,2,\ldots,2n). \tag{21}$$

Use of Eqs.(17), (19) and (20) leads to

$$Q_\mu(Z^*) \overset{\circ}{=} C^{*\sigma}\left[\frac{\partial g^\sigma_\mu}{\partial t} + K^\mu_\rho g^\rho_\mu(\gamma,t)\right],$$

$$(\mu,\rho,\sigma=1,2,\ldots,2n). \tag{22}$$

It is noted that as a consequence of the '$\overset{\circ}{=}$' in Eq.(22), $Q_\mu(Z^*)$ is now a function of a^μ and t, so that we may write

$$Q_\mu(Z^*) \overset{\circ}{=} P_\mu(\boldsymbol{a},t), \tag{23}$$

where

$$P_\mu(\boldsymbol{a},t) \equiv C^{*\sigma}\left[\frac{\partial g^\sigma_\mu}{\partial t} + K^\mu_\rho g^\rho_\mu(\gamma,t)\right]. \tag{24}$$

We note that in the derivatives $\partial g^\sigma_\mu(\gamma,t)/\partial t$ which appear in Eq.(24) the functions $\gamma^\nu(\boldsymbol{a},t)$ are held fixed. Therefore, if we evaluate the expression $P_\mu(\boldsymbol{a},t)$ of Eq.(24) on the dynamical paths by means of Eqs.(2), (14), (15) and (18), we may write the resulting expression in the form

$$P_\mu(\boldsymbol{a},t) \overset{t}{=} c^{*\sigma}(\gamma)\left[\dot{g}^\sigma_\mu(\gamma,t) + k^\mu_\rho(\gamma,t) g^\rho_\sigma(\gamma,t)\right]. \tag{25}$$

The functions $g^\sigma_\mu(\gamma,t)$ in Eq.(25) are solutions of Eqs.(13) for all values of γ^ν. It follows that $P_\mu(\boldsymbol{a},t) \overset{t}{=} 0$. Since the functions $P_\mu(\boldsymbol{a},t)$ vanish at each point of every trajectory, they vanish identically for variables a^μ and t. From Eq.(23) we thereby conclude that $Q_\mu(Z^*) \overset{\circ}{=} 0$. Hence the symmetry condition (11) is satisfied by $Z^{*\mu}(\boldsymbol{a},t)$ (17).

As a result, we may state the following theorem.

Theorem: Consider a Birkhoffian system

$$G(\dot{\boldsymbol{a}},\boldsymbol{a},t) = \dot{a}^\mu - \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) = 0,$$

$$(\mu,\nu=1,2,\ldots,2n), \tag{1}'$$

with a complete solution given by

$$a^\mu = \Phi^\mu(\gamma^1,\ldots,\gamma^{2n},t) \equiv \Phi^\mu(\gamma,t), \quad \gamma^\nu = \text{const},$$

$$(\mu,\nu=1,2,\ldots,2n). \tag{2}'$$

Let

$$I^\nu(\boldsymbol{a},t) \overset{t}{=} \gamma^\nu,$$

$$(\nu=1,2,\ldots,2n), \tag{3}'$$

be the specific set of $2n$ functionally independent constants of motion obtained by inversion of Eq.(2)$'$ for the constants γ^ν.

A solution $Z_\mu(\boldsymbol{a},t)$ to the (partial differential) symmetry equations

$$\delta G \overset{\circ}{=} \dot{Z}_\mu(\boldsymbol{a},t) + K^\mu_\rho Z_\rho(\boldsymbol{a},t) \overset{\circ}{=} 0,$$

$$(\mu,\rho=1,2,\ldots,2n), \tag{11}'$$

where

$$K^\mu_\rho \equiv \frac{\partial}{\partial a^\rho}\left[\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\right],$$

$$(\mu,\nu,\rho=1,2,\ldots,2n), \tag{12}'$$

is

$$Z_\mu(\boldsymbol{a},t) \equiv C^\sigma(\boldsymbol{a},t) g^\sigma_\mu \left[\gamma^1(\boldsymbol{a},t),\ldots,\gamma^{2n}(\boldsymbol{a},t),t\right] \equiv C^\sigma g^\sigma_\mu(\gamma,t),$$

$$(\mu,\sigma = 1,2,\ldots,2n), \qquad (17)'$$

where $C^\sigma(\boldsymbol{a},t)$ are arbitrarily chosen constants of motion in Eq.(1)$'$. Hence, they can always be expressed as functions of $2n$ functionally independent constants of motion $I^\nu(\boldsymbol{a},t)$ (3)$'$, and the functions $g^\sigma_\mu[I(\boldsymbol{a},t),t]$ are obtained by replacing the constants γ^ν appearing in the functions $g^\sigma_\mu(\gamma,t)$ of Eq.(16) by the respective constants of motion $I^\nu(\boldsymbol{a},t)$ of Eq.(3)$'$.

Thus, for every solution $Z_\mu(\boldsymbol{a},t)$ any choice of $\xi_0(\boldsymbol{a},t)$ will lead to an associated $\xi_\mu(\boldsymbol{a},t)$ such that ξ_0 and ξ_μ satisfy Eq.(8). Then $\xi_\mu(\boldsymbol{a},t)$ and $\xi_0(\boldsymbol{a},t)$ are infinitesimal symmetry transformations of Birkhoffian systems (1).

3. Example

To illustrate the various aspects of the theory developed in the preceding sections we now give an example as follows.

Example: Consider a fourth-order Birkhoffian system defined by the Birkhoffian B and Birkhoff functions R_μ

$$B = \frac{1}{2}\left[a^3 - \frac{1}{b}\mathrm{arctg}(bt)\right]^2 + \frac{1}{2}\left[a^4 - \frac{1}{2b}\ln(1+b^2t^2)\right]^2; \qquad (26a)$$

$$R_1 = a^3, \quad R_2 = a^4, \quad R_3 = 0, \quad R_4 = 0. \qquad (26b)$$

Then the equation of motion of the Birkhoffian system is

$$\begin{pmatrix} \dot{a}^1 \\ \dot{a}^2 \\ \dot{a}^3 \\ \dot{a}^4 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ a^3 - \frac{1}{b}\mathrm{arctg}(bt) \\ a^4 - \frac{1}{2b}\ln(1+b^2t^2) \end{pmatrix} = 0, \qquad (27)$$

with a complete solution given by

$$a^1 = -\frac{t}{b}\mathrm{arctg}(bt) + \frac{1}{2b^2}\ln\left[1+b^2t^2\right] + \gamma^1 t + \gamma^2, \qquad (28)$$

$$a^2 = \frac{t}{b} - \frac{1}{b^2}\mathrm{arctg}(bt) - \frac{t}{2b}\ln\left[1+b^2t^2\right] + \gamma^3 t + \gamma^4, \qquad (29)$$

$$a^3 = \gamma^1, \qquad (30)$$

$$a^4 = \gamma^3. \qquad (31)$$

Equations (28)–(31) can be solved for γ^1, γ^2, γ^3 and γ^4; this inversion procedure leads to four functionally independent constants of motion

$$I^1 = a^3, \qquad (32)$$

$$I^2 = a^4, \qquad (33)$$

$$I^3 = a^1 - a^3 t + \frac{1}{b}\int \mathrm{arctg}(bt)\mathrm{d}t, \qquad (34)$$

$$I^4 = a^2 - a^4 t + \frac{1}{2b}\int \ln(1+b^2t^2)\mathrm{d}t. \qquad (35)$$

By using of Eq.(27) we find

$$\begin{array}{llll} K^1_1 = 0, & K^1_2 = 0, & K^1_3 = -1, & K^1_4 = 0; \\ K^2_1 = 0, & K^2_2 = 0, & K^2_3 = 0, & K^2_4 = -1; \\ K^3_1 = 0, & K^3_2 = 0, & K^3_3 = 0, & K^3_4 = 0; \\ K^4_1 = 0, & K^4_2 = 0, & K^4_3 = 0, & K^4_4 = 0. \end{array} \qquad (36)$$

It now follows that the symmetry condition (11) takes the form

$$\dot{Z}_1 - Z_3 = 0, \qquad (37)$$

$$\dot{Z}_2 - Z_4 = 0, \qquad (38)$$

$$\dot{Z}_3 = 0, \qquad (39)$$

$$\dot{Z}_4 = 0. \qquad (40)$$

The equation may be evaluated on the dynamical trajectory by means of Eqs.(28)–(31):

$$\dot{z}_1 - z_3 = 0, \qquad (41)$$

$$\dot{z}_2 - z_4 = 0, \qquad (42)$$

$$\dot{z}_3 = 0, \qquad (43)$$

$$\dot{z}_4 = 0. \qquad (44)$$

The solution to Eqs.(41)–(44) is

$$z_1 = c^3 t + c^1, \qquad (45)$$

$$z_2 = c^4 t + c^2, \qquad (46)$$

$$z_3 = c^3, \qquad (47)$$

$$z_4 = c^4. \qquad (48)$$

By the procedure described in the theorem in Sec.2, we may now construct the solutions $Z_1(\boldsymbol{a},t)$, $Z_2(\boldsymbol{a},t)$, $Z_3(\boldsymbol{a},t)$ and $Z_4(\boldsymbol{a},t)$ to the partial differential equations (37)–(40) obtained from the expansion of the symmetry condition (11). Hence from Eqs.(45)–(48) the solutions Z_1, Z_2, Z_3 and Z_4 as given by Eq.(25) take the form

$$Z_1(\boldsymbol{a},t) = C^3(\boldsymbol{a},t)t + C^1(\boldsymbol{a},t), \tag{49}$$

$$Z_2(\boldsymbol{a},t) = C^4(\boldsymbol{a},t)t + C^2(\boldsymbol{a},t), \tag{50}$$

$$Z_3(\boldsymbol{a},t) = C^3, \tag{51}$$

$$Z_4(\boldsymbol{a},t) = C^4, \tag{52}$$

where $C^1(\boldsymbol{a},t)$, $C^2(\boldsymbol{a},t)$, $C^3(\boldsymbol{a},t)$ and $C^4(\boldsymbol{a},t)$ are arbitrary constants for motion for the Birkhoffian system (26) and hence may be regarded as arbitrary functions of the functionally independent constants of motion I^1, I^2, I^3 and I^4. If in the general symmetry solutions (49)–(52) we choose

$$C^1 = 1, \quad C^2 = 0, \quad C^3 = 0, \quad C^4 = 1, \tag{53}$$

and let $\xi_0 = 0$, then we have

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = t, \quad \xi_3 = 0, \quad \xi_4 = 1. \tag{54}$$

It is easily verified that Eq.(54) satisfies (8). Hence, the generator (54) is the infinitesimal symmetry transformation of the Birkhoffian system (26).

4. Conclusion

For a Birkhoffian system, all (Noether or non-Noether) infinitesimal symmetry transformation have a characteristic functional structure, which is manifestly dependent upon constants of motion of the system.

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