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The dynamical feature of transition of a Hamiltonian system to a dissipative system*

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The mechanism of generation and annihilation of attractors during transition from a Hamiltonian system to a dissipative system is studied numerically using the dissipative standard map. The transient process related to the formation of attracting basins of periodic attractors is studied by discussing the evolution of the KAM tori of the standard map. The result shows that as damping increases, attractors are mainly generated from elliptic orbits of the Hamiltonian system and annihilated by colliding with unstable periodic orbits originating from the corresponding hyperbolic orbits of the Hamiltonian system. The transient process also exhibits the general feature of bifurcation.

Keywords: dynamic system, dissipative standard map, KAM torus
PACC: 0500

1. Introduction

The fundamental physical laws are described by conservative equations that are time reversible.[1] From the viewpoint of the statistics ensemble, the distribution function in a phase space satisfies the Liouville equation and the flux in the phase space is impressible. However, the macroscopic system described by phenomenological laws constantly exhibits time-irreversible properties. The connection between microscopic and macroscopic descriptions is that a large number of the variables of the microscopic system are reduced to a few macroscopic variables by statistical averaging. The well-known 'time arrow' problem extends debate to research fields abroad. With the development of microscopic technology, interest in the control of microscopic phenomena arises, for example in relation to the cooling of particles in traps by laser,[2,3] or eliminating dusts in plasmas. The issue of the transition from a conservative system to a dissipative system is also related to the pattern selection and competition,[4] an important problem when a weakly dissipative interaction is applied to a space-extended system.

The conservative system and dissipative system have completely different dynamical properties. Two-dimensional (2D) near-integral area-preserving maps are used to model conservative physical phenomena in such fields as celestial mechanics, cosmic-ray physics, and accelerator theory. The phase-plane structure in near-integral measure-preserving maps is well known.[5] The model for a periodically kicked mechanical rotor yields a 2D map which can be taken as a prototype, and it is the area-preserving standard map introduced by Chirikov to a Hamiltonian system that has been widely studied.[6,7] For a moderate kick strength, the typical feature of the system is the coexistence of elliptic periodic orbits, quasi-periodic orbits or Kolmogorov–Arnold–Moser (KAM) orbits, hyperbolic periodic orbits and the chaotic motions nearby. Regions of persistent chaotic motion are densely interwoven with these regular regions. For an intensive kick strength, the chaotic regions are dominant. The measurements of the regular and chaotic regions in the system can vary widely within the phase-space plane and the system parameters.

The KAM structure of the Hamiltonian system is not maintained under dissipative perturbation, and infinitesimal damping can destroy any KAM curves. Some of the elliptic periodic orbits may turn into stable periodic orbits, and others (especially for the large periodic elliptic orbit) are smeared away. Although there are transient chaotic motions, in general the system, starting from an arbitrary initial condition,

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will thread its way through a complex chaotic region and ultimately be attracted to one of the sinks; the motion eventually becomes periodic. For some dissipative systems such as the dissipative standard map, the most eminent dynamical property is the coexistence of multiple attractors. This behaviour is found in a variety of systems in different disciplines of science, like semiconductor physics,[8] chemistry,[9] and laser physics.[10] The application studies are closely related to the problem of steering the system to a desired attractor.[11,12]

Schmidt and Wang [13] studied the damping standard map and the bifurcation of its periodic orbit with a nonlinear parameter. Feudel *et al*[14] studied the qualitative behaviour of a single mechanical rotor with weak damping, and found that for small dissipation the system can have as many as 100 attractors. The attracting basin has also been studied; it has a complex structure of closely interwoven basins of attraction. The bifurcations include both double bifurcation and tangent bifurcation. Recently, Kraut *et al*[15] studied the damping standard map model under the influence of noise, and found the phenomenon of noise-induced preference of attractors.

In this paper we study the dynamical feature of transition from a conservative system to a dissipative system. We find a general rule under which attractors are generated and annihilated, i.e. the attractive orbits originate from an elliptic periodic orbit of the Hamiltonian system and vanish by colliding with unstable orbits. The transient process exhibits a fascinating nomadic trajectory to its destination. This can be explained by the bifurcation mechanism.

2. The bifurcation of attractors

The simplest model for studying the transition from a conservative system to a dissipative system is the kicked single rotor. The variables that describe the rotor are its phase and angular velocity. The rotor rotates with a linear friction except that after each periodic time, a nonlinear force kicks it. After each kick, the phase of the rotor increases by an amount appropriate to the angular velocity, while its angular velocity gains a certain amount depending on the angle and the angular velocity itself. The expressions read as

$$\begin{aligned} x_{n+1} &= x_n + y_n, \qquad \mod 2\pi, \\ y_{n+1} &= (1-\nu)y_n + k\sin(x_n + y_n), \end{aligned} \tag{1}$$

where x denotes the angle, y denotes the angular velocity and ν denotes the friction coefficient (damping parameter). The damping is added by changing the damping parameter ν from 0 (conservative system) to 1 (extremely strong dissipative system).

For the conservative system of $\nu = 0$, the dynamical character depends strongly on the kick intensity k. For $k = 0$, it is a trivial motion, and whether the trajectories are periodic or quasi-periodic depends on its angular velocity. For $0 < k < k_c$ (6.10497), the feature of the dynamic system can be illustrated by KAM tori. As k increases, the measure of KAM tori decreases while the chaotic region increases, until $k > k_c$, the last remaining KAM torus which stretches horizontally from $x = 0$ to $x = 2\pi$ is broken, and what is left is the Cantor tori embedded in the chaotic region.

The dynamic behaviour for $\nu \to 0$ can be easily analysed from the viewpoint of continuity. For $\nu = 0$, near each elliptic periodic orbit, the KAM tori are encircled. The Jacobian matrix of the map at the periodic orbit can be diagonalized as

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}. \tag{2}$$

For the case of small ν, from the viewpoint of the perturbation method we can easily find the periodic orbits corresponding to the elliptic and hyperbolic orbits of the original Hamiltonian system. Based on continuity, the eigenvalues of the Jacobian in the orbit also change slightly. On the periodic orbit corresponding to the original elliptic orbit, the eigenvalues of the Jacobian must be near $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$. As the system is 2D and dissipative, the product of the two eigenvalues must be real and less than 1, therefore the eigenvalues of the Jacobian must be $r\mathrm{e}^{\mathrm{i}\theta'}$ and $r\mathrm{e}^{-\mathrm{i}\theta'}$, where $r = \sqrt{1-\nu}$ approaches 1 and less than 1, and θ' approaches θ. In this way, the damping stabilizes the elliptic orbits of the original system. The largest Lyapunov exponent of a hyperbolic orbit cannot change from a positive value to a negative one by infinitesimal perturbation. The hyperbolic periodic orbit is not singular and cannot vanish due to a small perturbation. It still exists and is unstable.

The bifurcation of attractors of the system over damping parameter ν can be analysed numerically; both stable and unstable orbits can be obtained by a target method. Figure 1 is the bifurcation diagram of the orbits with lower periods for a fixed driving force $k = 0.8$. It is interesting that the bifurcations for all

periodic orbits according to parameter ν are tangent bifurcations. It is clear that the number of attractors decreases as the dissipative parameter ν increases. Formally the critical ν for the bifurcation decreases with increasing period of the orbit. As ν decreases, the number of orbits increases rapidly and approaches infinity as $\nu \to 0$; in this way the attractors change to elliptic orbits of the standard map system. Figures 2(a)–(c) are the detailed bifurcations of five-periodic orbits, it is clear that the stable orbit stems from the elliptic orbit of the original Hamiltonian system (solid line), where the elliptic KAM curves encompass it; the unstable orbit stems from the corresponding hyperbolic orbit of the original system (dotted line), as ν increases; the distance between these two orbits decreases until $\nu = 0.001105$, where these two orbits collide, and at that point tangent bifurcation occurs. Coincidentally, the numerical result shows that this is a common process for all periodic orbits.

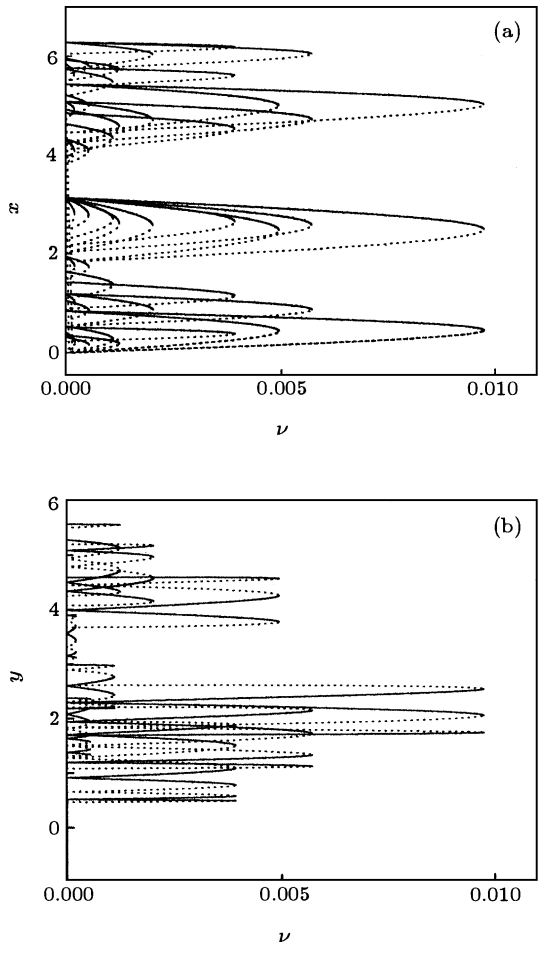


Fig.1. Periodic attractors (solid lines) and unstable periodic orbits (dotted line) under a fixed driving force (k=0.8) depending on damping ν: (a) phase space of x; (b) phase space of y.

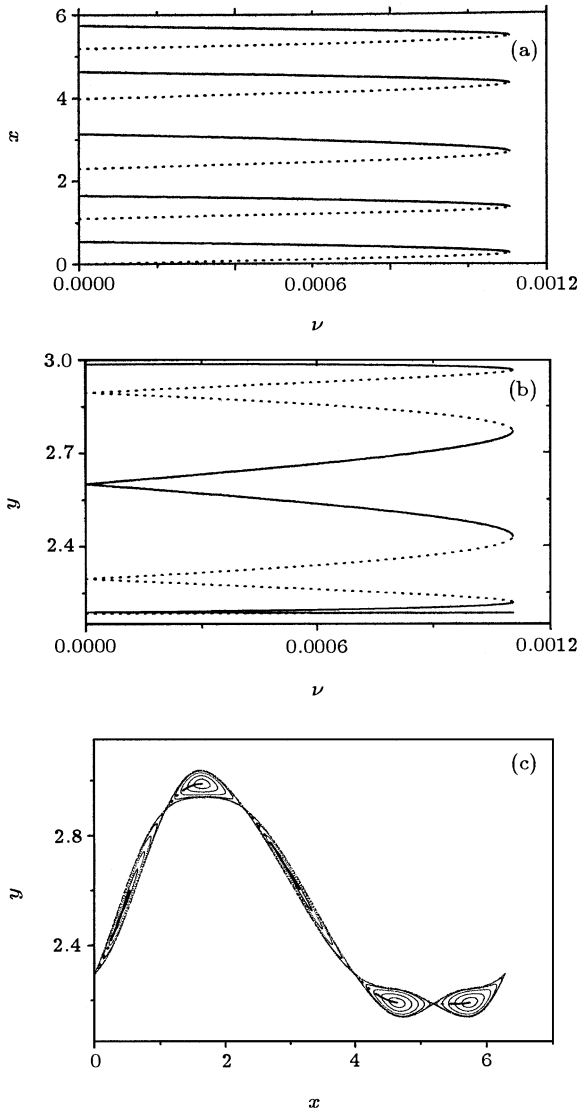


Fig.2. The KAM curves, the stable and unstable five-periodic orbit: (a) phase space x; (b) phase space y; (c) phase space (x, y).

3. The transient processes

The transient process of a system is closely related to the attractive basin of attractors and the structure of the stochastic web near separatrices. Because of the interweaving of attractive basins of different attractors in a multi-attractor system, an unambiguous description of the boundaries of these basins is difficult even for simple systems. However, the consecutive processes of transition from a standard map to a dissipative map provide the special view of the formation of attractive basins.

Consider the case of an infinitesimal damping coefficient $\nu \to 0$: the flux in the phase space (x, y) is similar to that in a standard map due to continuity of ν, and the changes of trajectory in the phase space (x, y) cannot be discerned locally. The fundamental difference is that the KAM tori, which are the main feature of a weakly nonlinear Hamiltonian system, do not exist any longer. Instead, they change into spiral trajectories because of shrinkage of the closed area for a damping system. The trajectories in the phase space do not close except for the original elliptic periodic orbit which continues its circulation. Thus, the attractive basins of these periodic attractors are the region of continuous KAM circles separated by chaotic separatrices. The chaotic separatrices consist of homoclinic or heteroclinic interwoven states of delicate structures. The basins of attractors in the infinitesimal-damping system are not regular. In fact, the number of elliptic periodic orbits in a standard map is infinite, and there are infinite numbers of attractors in the corresponding infinitesimal system. The attracting basins of attractors with high periods are so small that they cannot be discerned. In that case the eternal chaos exists in the homoclinic or heteroclinic interweaving region.

The analysis leads us to a new idea about the formation of attracting basins. By studying the evolution of the curves of the original KAM tori during the iteration of a damping map, some features of the attracting basin boundary can be discussed and analysed. Firstly, the KAM torus is a closed one-dimensional curve in the phase space, and it will shrink due to the effect of damping. Secondly, because of the nonlinear effect, it will stretch and fold. Because the damping coefficient ν of our interest is relatively small, the KAM torus changes little in a few steps of iteration.

For a small ν, the initial points start on the KAM curves and shrink inside until the hyperbolic orbits of a standard map are encountered. If the largest Lyapunov exponent of the original hyperbolic orbits is large enough, the shrinking circle is ripped apart into several parts and the different parts shrink around different elliptic orbits. This procedure continues until it sinks into one of the periodic orbits and becomes an attractor. The process is clearly shown in Fig.3(a), where the damping coefficient is relatively small ($\nu = 0.0005$), therefore the original KAM torus shrinks during the initial several hundred iterations and is ripped apart by a six-periodic hyperbolic orbit. One part is attracted to the six-periodic stable orbit and another to a one-periodic stable orbit.

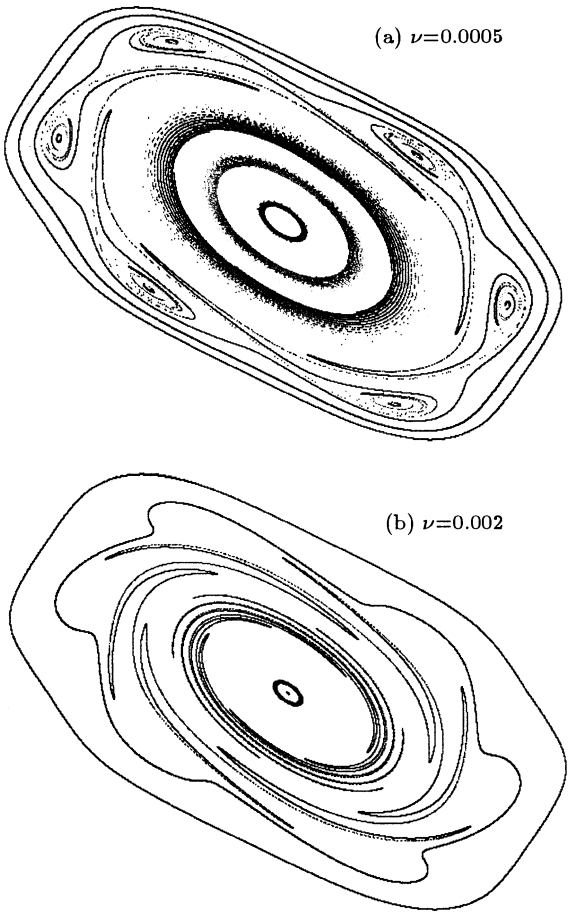


Fig.3. The evolution of the original KAM torus iterating in the damping standard map. Panels (a) and (b) correspond to the damping coefficients ν=0.0005 and 0.0020 respectively. Curves of different shade correspond to 0 (KAM torus), 30, 50, 150, 220, 500 iterations for (a) and 0, 15, 30, 50, 100, 500 iterations for (b).

For a relatively larger ν, or if the largest Lyapunov exponent of the hyperbolic orbit encountered in the transient process is not very large, the shrinking can overcome the divergent tendency of the unstable manifold in the neighbourhood of the original hyperbolic periodic orbit, as shown in Fig.3(b). The attracting basins of the system are shown in Fig.4. For a large damping coefficient, the hyperbolic orbit and elliptic orbit enclosed in the shrinking loop will be squeezed together. Therefore, the transient evolution depends on these two tendencies. Figure 5 shows the iteration process starting from an unstable five-periodic orbit. It first moves along the unstable manifold and then is attracted to the five-periodic orbit. Figure 6 shows the phase diagram of a dissipative standard map near its stable fixed point. It should be noted that the chaotic motion of the area-preserving map (for $\nu = 0$) no longer exists. Instead, the transient chaotic process

persists much longer for small damping. Two kinds of mechanism affect the transient process: the stochastic webs of the Hamiltonian system and the damping. The former is manifested by the hyperbolic manifold near an unstable periodic orbit and the latter is manifested by the attracting tendency of stable periodic orbits. If the former is dominant, the system diffuses in a wide range of chaotic webs; if the latter is dominant, the system is attracted and converges to a stable periodic orbit. These two mechanisms determine the critical value of ν needed to annihilate a pair of stable and unstable periodic orbits.

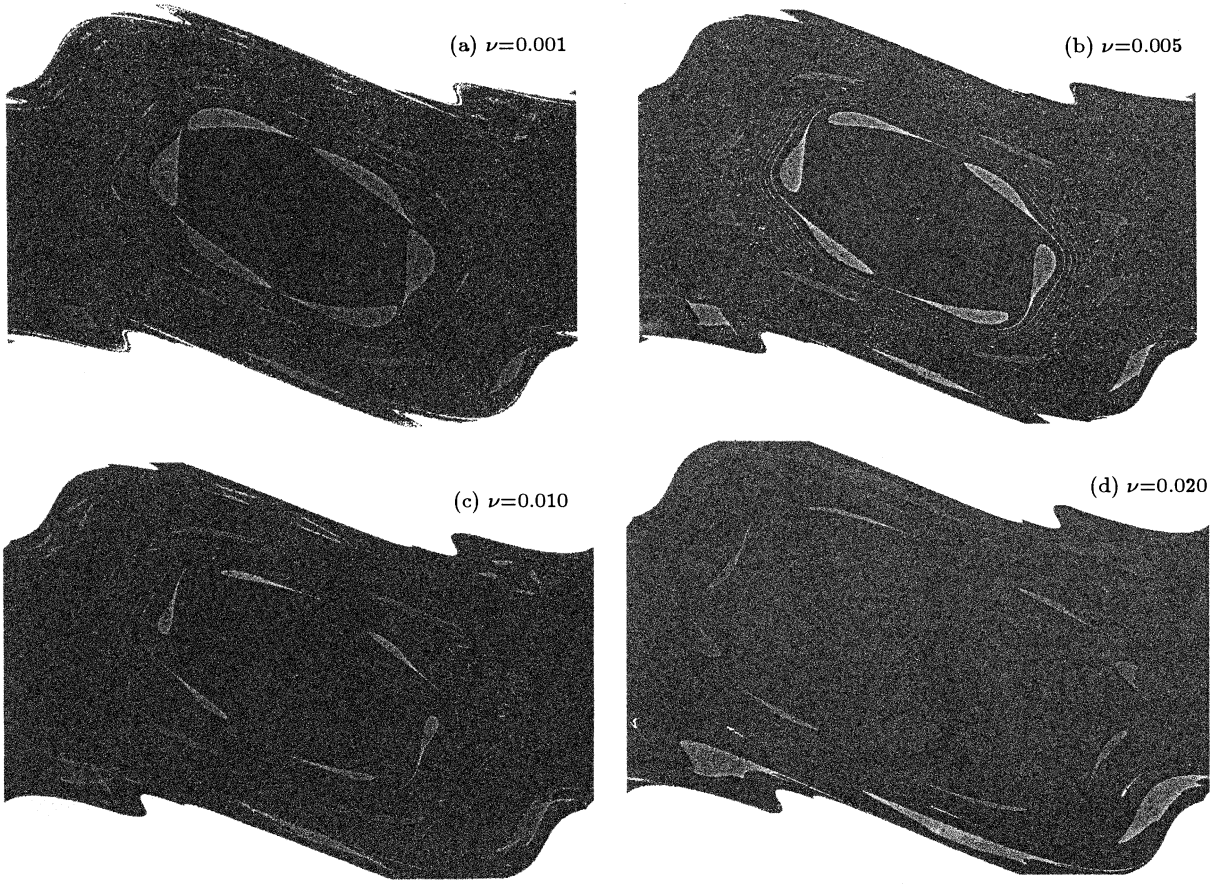


Fig.4. The attracting basins of periodic attractors in the damping standard map. Figures (a)–(d) correspond to the damping coefficients ν=0.001, 0.005, 0.010, 0.020 (k=1.20), respectively.

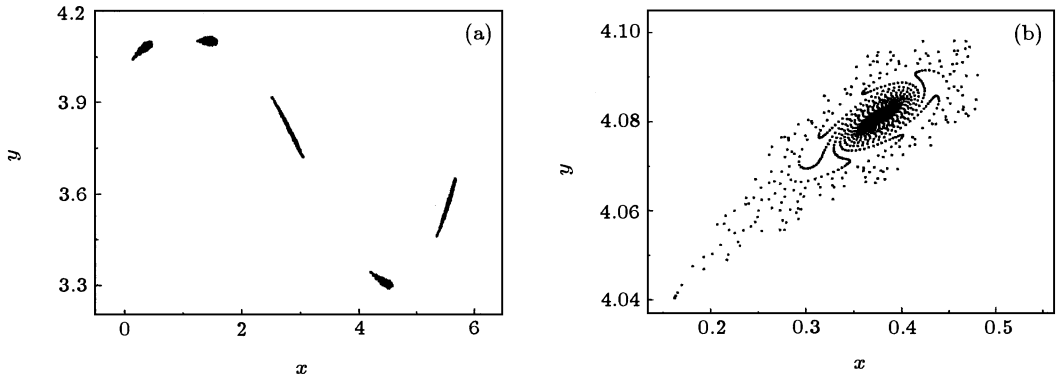


Fig.5. Trajectory in space (x, y) (a) and local enlargement with ν=0.0005 (b). The iteration process starts from the hyperbolic orbit of a conservative system (ν=0).

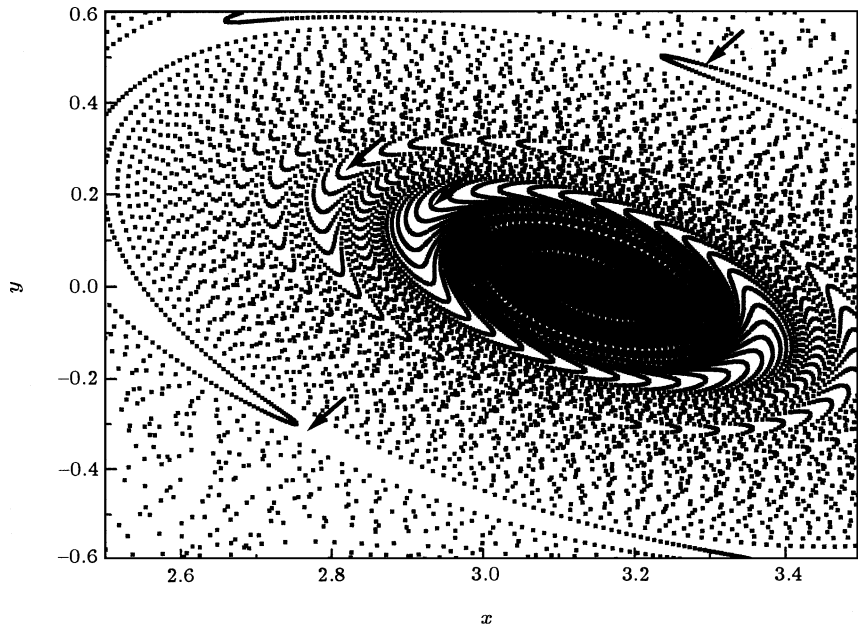


Fig.6. Phase diagram of a dissipative standard map with parameters of k=0.8, ν=0.0003.

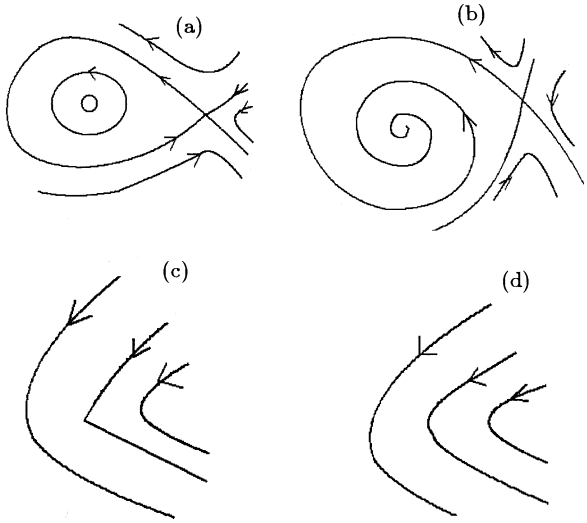


Fig.7. Schematic diagram of trajectories for (a) ν=0, (b) $0 < \nu < \nu_{\rm bif}$, (c) $\nu = \nu_{\rm bif}$ and (d) $\nu > \nu_{\rm bif}$.

Although the annihilated orbits are not attractors of the system, they influence the trajectories of the transient processes. When a trajectory passes by the uneven region where a pair of periodic orbits are annihilated, it moves more slowly and the tendency of convergence is exhibited. The swerving curve in the trajectory as marked by arrows in Fig.6 represents the trajectory of a transient process that passes near the region of the annihilated pairs. This picture reminds us of the trajectory of an over-damped oscillator. Figures 7(a)–(d) schematically illustrate the flow in the phase space for $\nu = 0$, $0 < \nu < \nu_{\rm bif}$, $\nu = \nu_{\rm bif}$ and $\nu > \nu_{\rm bif}$ respectively. For small ν, the KAM curves no longer close. In fact, they turn into inward-revolving curves. As ν increases further, the elliptic and hyperbolic orbits get nearer and the revolving gets slower; as ν reaches a certain value $\nu \to \nu_{\rm bif}$, the revolving stops, the two orbits coincide and the tangent bifurcation takes place. Then the trajectory moves slowly and approaches the region of the annihilated pairs; the swerving line can be seen in Fig.6.

4. Conclusion and discussion

We have used a dissipative standard map to study the general features of the transition from a conservative system (especially the Hamiltonian system) to a dissipative system. In this system, all the attractors are generated from the elliptic orbits of the conservative system and they vanish by colliding with the corresponding hyperbolic orbits as the damping parameter increases. The universality of the bifurcation reflects the special characters of damping. By simply computing the determining Jacobian of the map, one can find $\mathrm{Det}(J) = 1 - \nu < 1$, and we know that the area of the phase space shrinks at any place. It is evident that this is not correct for a general dissipative problem because one can construct another kind of bifurcation if one selects damping in a different way. It need not be correct for a system of which the Jacobian is less than 1 at any place of the phase space, because a high-dimensional system can be constructed by simply combining two irrelevant

subsystems so that the condition is satisfied by itself, whereas its subsystem does not satisfy the condition. A less general proposition is that the law is true for a uniformly damping Hamiltonian system. By numerical testing we find that it is indeed true in a laser-cooled trapped ion system, though so far we cannot prove the proposition. The conjecture that the attractors of a uniformly damped Hamiltonian system originate from its elliptic orbits, and vanish by collision with the hyperbolic orbits, is most likely correct. The symplectic structure of the Hamiltonian system ensures that any area-element in the phase space keeps its area. As uniform damping is applied to such a system, any area in the phase space shrinks. This greatly restricts its dynamical properties.

A useful application of our theory is that we can cool the two particles in a Paul trap by feeding a suitable damping feedback to the driving potential. We only consider the relative motions; the scaled equation of two degrees of freedoms is

$$\begin{aligned} \mathrm{d}^2z/\mathrm{d}t^2 + \Gamma \mathrm{d}z/\mathrm{d}t + z\left[-\alpha/\left(r^2+z^2\right)^{3/2} + 2u\right] = 0, \\ \mathrm{d}^2r/\mathrm{d}t^2 + \Gamma \mathrm{d}r/\mathrm{d}t - r\left[\alpha/\left(r^2+z^2\right)^{3/2} + u\right] = 0, \quad (3) \end{aligned}$$

where r and z are the trap's axial and radial coordinates respectively, and $u = \cos(2t)$ is the driving electrical potential. By applying a feedback $u = \cos(2t) + \varepsilon(2z\mathrm{d}z/\mathrm{d}t - r\mathrm{d}r/\mathrm{d}t)/\sqrt{r^2+z^2}$ to the driving electric potential, a uniform damping is added to the system. The bifurcation diagram for the damping intensity ε is shown in Fig.8. Once more all the bifurcations of stable periodic orbits are tangent bifurcations.

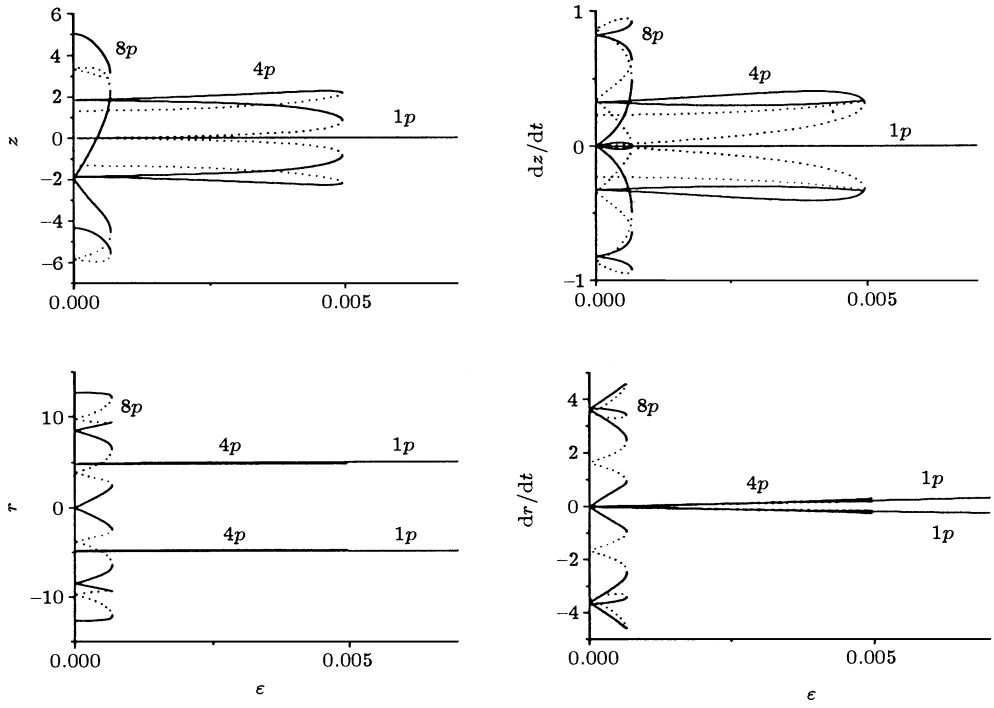


Fig.8. The bifurcation in a controlled-particle system. ϵ is the control intensity.

The conclusion is also useful for understanding the general character of pattern formation; the selection between different modes depends crucially on the dissipative coefficient. If the system has a large dissipative coefficient, one can expect a unique pattern for any initial pattern, though different kinds of pattern may still exist due to configuration symmetry.

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