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Euclidean non-vacuum wormholes in Brans–Dicke theory*

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The Brans–Dicke theory is investigated in which the Pauli metric is identified to be a physical spacetime metric. The solutions of a wormhole are obtained in Brans–Dicke theory with a relativistic radiation field for $\omega > -3/2$. However, it is found that one cannot construct a wormhole in the presence of a 3-form axion field.

Keywords: Brans–Dicke theory, Jordan frame, Einstein frame
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1. Introduction

Recently, several authors [1−11] have studied the Brans–Dicke scalar–tensor theory of gravitation in which the background spacetime is described by the metric with a dilaton field. This is of interest for the following reasons. No experiment has been able to rule out the existence of Brans–Dicke theory with a dilaton, which is an essential element of string theories. The Brans–Dicke theory can be derived from the Kaluza–Klein theories and the supersymmetric string theories. The Brans–Dicke theory has an important cosmological application in the extended inflation theory. The Lagrangian of Brans–Dicke theory is allowed to have two spacetime metrics (the Jordan metric and the Pauli metric) which can describe two different physics. Although both metrics are logically acceptable, they are physically distinct. Reference [12] showed that the unitarity of the Lagrangian can be assured with a simple conformal transformation. The original set of variables (the Jordan metric and the Brans–Dicke scalar field) is usually called the Jordan frame, while the transformed set (the Pauli metric and the dilaton field) is called the Einstein frame.

Recently, several authors have studied inflation cosmology in terms of Brans–Dicke theory with Fermi fields.[1,8,9] It is found that Euclidean wormholes in quantum gravity are possibly useful and the vacuum wormhole solution cannot be constructed in the Einstein frame. We have investigated the quantum cosmology in the charged Brans–Dicke theory.[13] In this paper, we study the nonvacuum wormholes in the Einstein frame. It is found that the existence of wormholes is dependent on matter sources. The solutions of wormholes are obtained in the Einstein frame with a relativistic radiation field for $\omega > -2/3$. It is shown that a wormhole cannot be constructed in the Einstein frame with a 3-form axion field.

The present paper is organized as follows. In Section 2, we find the wormhole solution in the Einstein frame with a relativistic radiation field. In Section 3, we discuss the wormhole solution in the Einstein frame with a 3-form axion field. Finally, in Section 4, a summary and conclusions about the results are given.

2. Wormhole solution with a relativistic radiation field

The Lagrangians in the Jordan frame and Einstein frame are, respectively, given by

$$L_{\rm J} = -\left[\phi\widetilde{R}(\widetilde{g}) + \omega\widetilde{g}^{\mu\nu}\frac{\partial_\mu\phi\partial_\nu\phi}{\phi}\right] - \left[\frac{1}{2}\widetilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + V(\psi)\right], \quad (1)$$

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and

$$L_{\rm E} = -\left[R(g) + g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right] - {\rm e}^{-\alpha\sigma}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + {\rm e}^{-\alpha\sigma}V(\psi)\right], \quad (2)$$

where $\widetilde{R}$ and R are the scalar curvatures, $\widetilde{g} = \det(\widetilde{g}_{\mu\nu})$, $\widetilde{g}_{\mu\nu}$ is the Jordan metric, ϕ is the Brans–Dicke scalar field, ω is the dimensionless Brans–Dicke parameter, ψ is the relativistic particle field, $g = \det(g_{\mu\nu})$, $g_{\mu\nu}$ is the Pauli metric, $8\pi G = 1$ for convenience, $\alpha = (3+2\omega)^{1/2}$ and σ is the dilaton scalar field. Equation (1) can be transformed into Eq.(2) under the following Weyl conformal transformation:

$$g_{\mu\nu} = {\rm e}^{\alpha\sigma}\widetilde{g}_{\mu\nu}, \qquad \phi = \frac{1}{2}{\rm e}^{\alpha\sigma}. \quad (3)$$

It is shown by Cho [12] that only the Pauli metric can represent the massless spin-two and thus identify it to be physical. Lagrangian (2) yields the following field equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -(T_{(\sigma)\mu\nu} + {\rm e}^{-\alpha\sigma}T_{(m)\mu\nu}), \quad (4)$$

$$\Box\sigma = \frac{1}{2}\alpha{\rm e}^{-\alpha\sigma}T_{(m)}, \quad (5)$$

$$\Box\,\Psi = \alpha g^{\mu\nu}\partial_\mu\Psi\partial_\nu\sigma - {\rm e}^{-\alpha\sigma}V'(\Psi), \quad (6)$$

where

$$T_{(\sigma)\mu\nu} = \partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma, \quad (7)$$

$$T_{(m)\mu\nu} = \partial_\mu\Psi\partial_\nu\Psi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\Psi\partial_\beta\Psi - g_{\mu\nu}{\rm e}^{-\alpha\sigma}V(\Psi), \quad (8)$$

$$T_{(m)} = g^{\mu\nu}T_{(m)\mu\nu} = -g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - 4{\rm e}^{-\alpha\sigma}V(\Psi), \quad (9)$$

where $\Box \equiv \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$, $V'(\Psi) = {\rm d}V(\Psi)/{\rm d}\Psi$. For a perfect fluid

$$T_{(m)\mu\nu} = p_m g_{\mu\nu} + (\rho_m + p_m)U_\mu U_\nu, \quad (10)$$

where

$$\rho_m = \frac{1}{2}\dot{\psi}^2 + {\rm e}^{-\alpha\sigma}V(\psi), \qquad p_m = \frac{1}{2}\dot{\psi}^2 - {\rm e}^{\alpha\sigma}V(\psi). \quad (11)$$

Now let us consider the case of the Einstein frame. The Euclidean metric is

$${\rm d}s^2 = {\rm d}\tau^2 + a^2(\tau)\left[\frac{{\rm d}r^2}{1-r^2} + r^2{\rm d}\Omega\right]. \quad (12)$$

Equations (4)–(6) become

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = \frac{1}{3}\left(\frac{1}{2}\dot{\sigma}^2 - {\rm e}^{-\alpha\sigma}\rho_m\right), \quad (13)$$

$$\ddot{\sigma} + 3\frac{\dot{a}}{a}\dot{\sigma} = \frac{1}{2}\alpha{\rm e}^{-\alpha\sigma}(\rho_m - 3p_m), \quad (14)$$

and

$$\dot{\rho_m} + 3\frac{\dot{a}}{a}(\rho_m + p_m) = \frac{1}{2}\alpha\dot{\sigma}(\rho_m + 3p_m). \quad (15)$$

Here ρ_m and p_m are the energy density and the pressure of relativistic radiation field respectively; the upper dot denotes the derivative with respect to τ. For a relativistic radiation field we can obtain

$$\rho_m = 3p_m. \quad (16)$$

Equation (16) indicates that the condition $\dot{\Psi}^2/2 = 2{\rm e}^{-\alpha\sigma}V(\Psi)$ is required. Integrating Eq.(15) by using Eq.(16), we can obtain

$$\rho_m = \frac{A{\rm e}^{\alpha\sigma}}{a^4}, \quad (17)$$

where A is an integration constant. From Eq.(17) and the relation $\dot{\Psi}^2/2 = 2{\rm e}^{-\alpha\sigma}V(\Psi)$, we can obtain

$$V(\Psi) = \frac{A{\rm e}^{2\alpha\sigma}}{3a^4}. \quad (18)$$

Integrating Eq.(14), we have

$$\dot{\sigma} = \frac{C_0}{a^3}, \quad (19)$$

where C_0 is another integration constant. We now consider the field equation (13) in two cases: $\dot{\sigma} = 0$ and $\dot{\sigma} \neq 0$.

In the first case, $\dot{\sigma} = 0$ means that $\sigma = \sigma_0$=constant. We put this result and Eq.(17) into Eq.(13) and obtain

$$\dot{a}^2 = 1 - \frac{A}{3a^2}. \quad (20)$$

Therefore, the solution of Eq.(20) is

$$a = \sqrt{\tau^2 + \frac{A}{3}}, \quad (21)$$

and the throat of the wormhole has a size

$$a_0 \equiv a(\tau_0) = \sqrt{\frac{A}{3}}. \quad (22)$$

This is the Tolman wormhole.

In the latter case, $\dot{\sigma} \neq 0$, substituting Eqs.(17) and (19) into Eq.(13) yields

$$\dot{a}^2 = 1 - \frac{A}{3a^2} + \frac{C_0^2}{6a^4} \equiv 1 - \frac{B^2}{a^2} + \frac{D^4}{a^4}. \tag{23}$$

The boundary conditions for a wormhole solution are $\dot{a}(\tau_0) = 0, \ddot{a}(\tau_0) > 0, a(\tau_0) \neq 0$. Setting $\dot{a}(\tau_0) = 0$ in Eq.(23), one can obtain

$$\begin{aligned} a_+ &= \sqrt{\frac{B^2 + \sqrt{B^4 - 4D^4}}{2}}, \\ a_- &= \sqrt{\frac{B^2 - \sqrt{B^4 - 4D^4}}{2}}. \end{aligned} \tag{24}$$

To determine which one is the true size of the wormhole throat, we differentiate Eq.(23) with respect to τ and obtain

$$\begin{aligned} \ddot{a}|_{a=a_+} &= \frac{2\sqrt{2(B^4 - 4D^4)}}{B^2 + \sqrt{B^4 - 4D^4}} > 0, \\ \ddot{a}|_{a=a_-} &= -\frac{2\sqrt{2(B^4 - 4D^4)}}{B^2 - \sqrt{B^4 - 4D^4}} < 0. \end{aligned} \tag{25}$$

Therefore, the size of wormhole throat is written as $a_0 = a(\tau_0) = a_+$, if the relation $B^2 - 4D^2 > 0$ is satisfied. Referring to Ref.[15], we can derive the solution of Eq.(23) as

$$\begin{aligned} \tau =& a_0 \left[F\left(\arcsin\sqrt{\frac{a^2 - a_0^2}{a^2 - a_-^2}}, \frac{a_-}{a_0} \right) \right. \\ & \left. - E\left(\arcsin\sqrt{\frac{a^2 - a_0^2}{a^2 - a_-^2}}, \frac{a_-}{a_0} \right) \right] \\ & + a\sqrt{\frac{a^2 - a_0^2}{a^2 - a_-^2}}, \end{aligned} \tag{26}$$

where F and E are the first and second kind of elliptic integrals respectively. Finally, by combining Eqs.(19) and (23), we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}a} = \frac{C_0}{a\sqrt{a^4 - B^2a^2 + D}}. \tag{27}$$

Then the solution of Eq.(27) reads as

$$\sigma = \sigma_0 + \frac{C_0}{2a_0a_-} \ln \left(\frac{\sqrt{\dfrac{a^2 - a_-^2}{a^2 - a_0^2}} - \dfrac{a_-}{a_0}}{\sqrt{\dfrac{a^2 - a_-^2}{a^2 - a_0^2}} + \dfrac{a_-}{a_0}} \right), \tag{28}$$

where σ_0 is an integration constant. As a conclusion, when $\tau \to \tau_0$, it is easy to find that $a \to a_0$, $\dot{\sigma} \to \dfrac{C_0}{a_0^3}$, $\sigma \to \sigma_0$, and $\rho_m \to \dfrac{A\mathrm{e}^{\alpha\sigma_0}}{a_0^4}$; while for $\tau \to \infty$, one can have $a \to \tau, \dot{\sigma}$ and $\rho_m \to 0$ and $\sigma \to \sigma_0 + \dfrac{C_0}{2a_0a_-} \ln\left(\dfrac{a_0 - a_-}{a_0 + a_-}\right)$. Equation (26) is a wormhole solution.

3. Wormhole with an axion field

In the Jordan frame, the Lagrangian with a 3-form axion field is given by

$$L_J = -\left[\phi\widetilde{R}(\widetilde{g}) + \omega\widetilde{g}^{\mu\nu}\frac{\partial_\mu\phi\partial_\nu\phi}{\phi} + \widetilde{\beta}^2\widetilde{H}_{\mu\nu\lambda}\widetilde{H}^{\mu\nu\lambda}\right], \tag{29}$$

where $\widetilde{\beta}$ is the Peccei–Quinu scale coupling constant. Under the Weyl conformal transformation, Eqs.(3) and (29) can be transformed into

$$L_E = -[R + g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \mathrm{e}^{-\alpha\sigma}\beta^2 H_{\mu\nu\lambda}H^{\mu\nu\lambda}], \tag{30}$$

where

$$\beta^2 = \frac{1}{4\omega}\widetilde{\beta}^2. \tag{31}$$

Equation (30) is the Lagrangian with a 3-form field in the Einstein frame. From Eq.(30), we can obtain the field equations

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \\ =& (\nabla_\mu\sigma\nabla_\nu\sigma - \frac{1}{2}g_{\mu\nu}\partial_\lambda\sigma\partial^\lambda\sigma) \\ & + 6\beta^2\mathrm{e}^{-\alpha\sigma}[H_{\mu\lambda\rho}H_\nu^{\lambda\rho} - \frac{1}{6}g_{\mu\nu}H_{\mu\lambda\rho}H^{\mu\lambda\rho}], \end{aligned} \tag{32}$$

and

$$\Box\sigma + 6\alpha\beta^2\mathrm{e}^{-\alpha\sigma}H_{\mu\nu\gamma}H^{\mu\nu\gamma} = 0. \tag{33}$$

The conformal Euclidean R–W metric is Eq.(12). It is assumed that an axion field is presented by

$$H_{jke} = h(\tau)\varepsilon_{jke}, \qquad h(\tau) = \frac{m}{\beta^2a^3}, \tag{34}$$

where m is the axion charge. Using Eqs.(12) and (34), Eqs.(32) and (33) become

$$H^2 - \frac{1}{a^2} = \frac{1}{3}\left(\frac{\dot{\sigma}^2}{2} - 6\mathrm{e}^{-\alpha\sigma}\frac{m^2}{\beta^2a^6}\right), \tag{35}$$

$$\dot{H} + \frac{1}{a^2} = -\frac{1}{2}(\dot{\sigma}^2 - 12\mathrm{e}^{-\alpha\sigma}\frac{m^2}{\beta^2a^6}), \tag{36}$$

$$\ddot{\sigma} + 3H\dot{\sigma} + 6\alpha\mathrm{e}^{-\alpha\sigma}\frac{m^2}{\beta^2a^6} = 0, \tag{37}$$

where $H = \dfrac{\dot{a}}{a}$. The upper dot denotes the derivative with respect to τ. It is clear that Eq.(36) can be derived from Eqs.(35) and (37). In order to solve Eq.(37), we need make a transformation: $\mathrm{d}\rho = (\mathrm{d}\tau/a^3)(\sqrt{6}m/\beta)$, then Eq.(37) becomes

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}\rho^2} + \alpha\mathrm{e}^{-\alpha\sigma}. \tag{38}$$

By solving Eq.(38), we obtain solutions which depend on the sign of the first integration constant as follows:

$$\sigma = \begin{cases} \frac{1}{\alpha}\ln\left\{\frac{2}{k_0}\sin h^2\left[\frac{\alpha\sqrt{k_0}}{2}(\rho+k_1)\right]\right\}, & k_0>0; \\ \frac{1}{\alpha}\ln\left[\frac{\alpha^2}{2}(\rho+k_1)^2\right], & k_0=0; \\ \frac{1}{\alpha}\ln\left\{-\frac{2}{k_0}\cos^2\left[\frac{\alpha\sqrt{-k_0}}{2}(\rho+k_1)\right]\right\}, & k_0<0, \end{cases} \tag{39}$$

where k_0 and k_1 are the first and second integration constants. By substituting the solutions (39) into Eq.(35), we obtain

$$\dot{a}^2 = \begin{cases} 1+\frac{2m^2k_0}{\beta a^4}, & k_0>0; \\ 1, & k_0=0; \\ 1-\frac{2m^2k_0}{\beta^2a^4}, & k_0<0. \end{cases} \tag{40}$$

The boundary conditions, $\frac{\mathrm{d}a(\tau_0)}{\mathrm{d}\tau}=0, \frac{\mathrm{d}^2a(\tau_0)}{\mathrm{d}\tau^2}>0, a(\tau_0)\neq 0$, must be satisfied by the wormhole solution of Eq.(40). It is obvious that $\frac{\mathrm{d}a}{\mathrm{d}\tau}$ cannot vanish in Eq.(40). This means that a wormhole cannot be created in the Einstein frame with a 3-form axion field.

4. Conclusion

In the Einstein frame, we cannot construct a vacuum wormhole solution.[1] However, when matter sources are relativistic radiation fields, we obtain an analytical wormhole solution. Solution (26) is a combination of the first and second kind of elliptic integrals for $\omega > -\frac{3}{2}$. When matter sources are axion fields, one cannot obtain wormhole solutions. The existence of a wormhole depends on matter sources. Some authors have showed that the Jordan frame and Einstein frame are different at the classical level and almost also at the quantum level.[4,16,17] We will investigate nonvacuum wormhole solutions of a Dirac field and a massive scalar field [18,19] in future work.

Wormholes have been studied mainly in the semiclassical approximation as solutions of the classical Euclidean field equations. However, such solutions are special, and exist only for certain kinds of matter. On the other hand, one can represent wormholes in a more general manner as solutions of the Wheeler–Dewitt equation with appropriate boundary conditions. Hawking and Page gained, through a tedious calculation, solutions of Wheeler–Dewitt equation with massless minimal or conformal scalar field.[20] It was found that Giddings–Strominger instanton solution corresponds to a sum of an infinite number of Hawking's solutions. We will investigate further wormhole wavefunctions as solutions of the Wheeler–Dewitt equation with appropriate boundary conditions in the Brans–Dicke theory in future work.

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