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# Periodic solutions of nonlinear Schrödinger type equations*

Liu Guan-Ting(刘官厅)[a)b)†] and Fan Tian-You(范天佑)[a)]

[a)] *Department of Physics, Beijing Institute of Technology, Beijing* 100081, *China*

[b)] *Department of Mathematics, Inner Mongolia Normal University, Huhhot* 010022, *China*

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By using the solutions of an auxiliary elliptic equation, a direct algebraic method is proposed to construct the exact solutions of nonlinear Schrödinger type equations. It is shown that many exact periodic solutions of some nonlinear Schrödinger type equations are explicitly obtained with the aid of symbolic computation, including corresponding envelope solitary and shock wave solutions.

**Keywords:** auxiliary equation method, Schrödinger type equation, periodic solution, symbolic computation

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## 1. Introduction

Much effort has been made on the construction of exact solutions of nonlinear evolution equations in view of their important role in understanding the nonlinear problems. In recent years, many methods have been proposed, such as the hyperbolic function method,[1−4] the homogeneous balance method,[5−8] the mixed exponential method,[9,10] auxiliary equation method,[11−13] variable separation method [14] and so on. In connection with finding the periodic solutions of nonlinear evolution equations, there are the Weierstrass elliptic function expansion method [15−17] and Jacobi elliptic sine (or cosine or the third kind of Jacobi elliptic) function expansion method,[18−22] and so on. Many exact solutions of nonlinear equations have been obtained. The main idea of the above methods is that the solution of the nonlinear evolution equation is expanded as a series of finite terms of functions, resulting in a set of algebraic equations. In this paper, along with this idea, we propose a new method to construct the exact periodic solutions of nonlinear Schrödinger type equations based on an auxiliary elliptic equation. It is shown that many exact periodic solutions of some nonlinear Schrödinger type equations are explicitly obtained with the aid of symbolic computation (Mathematica). Under a limiting condition, they give the corresponding envelope solitary and shock wave solutions.

## 2. Auxiliary elliptic equation method

Consider a given nonlinear Schrödinger type equation

$$N(u, |u|, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \qquad (1)$$

where $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xt} = \partial^2 u/\partial x \partial t, \cdots$, and $|u|$ denotes the modulus of $u$, as hereafter.

Since nonlinear Schrödinger type equations describe a nonlinear wavepacket, we can seek their envelope wave solutions in the following form:

$$u = \phi(\xi) \mathrm{e}^{\mathrm{i}(kx - \omega t)}, \quad \xi = x - 2\alpha k t \qquad (2)$$

with $\mathrm{i} = \sqrt{-1}$, where $\phi(\xi)$ is a real value function and $k$ and $\omega$ denote the wavenumber and the circular frequency, respectively. Substituting Eq.(2) into Eq.(1) leads to an ordinary differential equation

$$F\left(\phi, \frac{\mathrm{d}\phi}{\mathrm{d}\xi}, \frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}, \frac{\mathrm{d}^3\phi}{\mathrm{d}\xi^3}, \ldots\right) = 0. \qquad (3)$$

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†E-mail: guantingliu@163.com

Introducing an auxiliary elliptic equation as follows:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\xi^2} = bz + 2cz^3, \tag{4}$$

where $b$ and $c$ are two constant parameters, it is found that the elliptic equation (4) admits the following several types of solutions:

$$z_1 = \mathrm{sn}(\xi, m),$$
$$\text{as} \quad b = -(1+m^2),\ c = m^2, \tag{5}$$
$$z_2 = \mathrm{cn}(\xi, m),$$
$$\text{as} \quad b = 2m^2 - 1,\ c = -m^2, \tag{6}$$
$$z_3 = \mathrm{dn}(\xi, m),$$
$$\text{as} \quad b = 2 - m^2,\ c = -1, \tag{7}$$
$$z_4 = \mathrm{ns}(\xi, m) \equiv \frac{1}{\mathrm{sn}(\xi, m)},$$
$$\text{as} \quad b = -(1+m^2),\ c = 1, \tag{8}$$
$$z_5 = \mathrm{nc}(\xi, m) \equiv \frac{1}{\mathrm{cn}(\xi, m)},$$
$$\text{as} \quad b = 2m^2 - 1,\ c = 1 - m^2, \tag{9}$$
$$z_6 = \mathrm{nd}(\xi, m) \equiv \frac{1}{\mathrm{dn}(\xi, m)},$$
$$\text{as} \quad b = 2 - m^2,\ c = -(1 - m^2), \tag{10}$$
$$z_7 = \mathrm{sc}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)},$$
$$\text{as} \quad b = 2 - m^2,\ c = 1 - m^2, \tag{11}$$
$$z_8 = \mathrm{sd}(\xi, m) \equiv \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)},$$
$$\text{as} \quad b = 2m^2 - 1,\ c = -m^2(1 - m^2), \tag{12}$$
$$z_9 = \mathrm{cs}(\xi, m) \equiv \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)},$$
$$\text{as} \quad b = 2 - m^2,\ c = 1, \tag{13}$$
$$z_{10} = \mathrm{ds}(\xi, m) \equiv \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)},$$
$$\text{as} \quad b = 2m^2 - 1,\ c = 1, \tag{14}$$

where $\mathrm{sn}(\xi, m)$, $\mathrm{cn}(\xi, m)$ and $\mathrm{dn}(\xi, m)$ are Jacobi elliptic sine, cosine and the third kind of Jacobi elliptic function, respectively, and $m$ denotes the modulus. For detailed information about Jacobi elliptic functions see Refs.[23,24].

Now suppose that $\phi(\xi)$ is expressed as a finite series of function $z(\xi)$, i.e. the ansatz

$$\phi(\xi) = \sum_{j=0}^{n} a_j\ z(\xi)^j \tag{15}$$

is made and its highest degree is

$$O(\phi(\xi)) = n. \tag{16}$$

From Eq.(4), the highest degree of $\phi^q \dfrac{\mathrm{d}^p \phi}{\mathrm{d}\xi^p}$ is taken as

$$O\left(\phi^q \frac{\mathrm{d}^p \phi}{\mathrm{d}\xi^p}\right) = (q+1)n + p, \quad p, q = 0, 1, 2, 3, \ldots \tag{17}$$

Thus we can choose $n$ in Eq.(15) to balance the derivative term of the highest order and the nonlinear term in Eq.(3). Substituting ansatz Eq.(15) and elliptic equation (4) into Eq.(3) with computerized symbolic computation and equating to zero the coefficients of all powers of $z(\xi)$ yields a set of algebraic equations for $a_i$ $(i = 0, 1, 2, \ldots, n)$. Finally, by inserting each solution of this set of algebraic equations and elliptic equation (4) into Eq.(15), then into Eq.(2) and setting $\xi = x - 2\alpha k t$, the exact periodic solutions of Eq.(1) are obtained. We observe that some solutions of the algebraic equations lead to the trivial solutions of the equation, which will not be considered here.

# 3. Applications

In this section, we will solve three important nonlinear Schrödinger type equations to illustrate our method.

### 3.1. Nonlinear Schrödinger equation

We consider the nonlinear Schrödinger (NLS) equation in the following form:[24]

$$\mathrm{i}u_t + \alpha u_{xx} + \beta |u|^2 u = 0 \tag{18}$$

with $\mathrm{i} = \sqrt{-1}$, in which $\alpha$ and $\beta$ are two constant parameters that denote the dispersion frequency coefficient and Landau coefficient, respectively. An NLS equation describes a nonlinear wavepacket. Let

$$u = \phi(\xi) \mathrm{e}^{\mathrm{i}(kx - \omega t)}, \quad \xi = x - 2\alpha k t, \tag{19}$$

where $\phi(\xi)$ is a real value function and $k$ and $\omega$ denote the wavenumber and the circular frequency, respectively. Substituting Eq.(19) into Eq.(18), we get

$$\alpha \frac{\mathrm{d}^2 \phi}{\mathrm{d}\xi^2} + (\omega - \alpha k^2)\phi + \beta \phi^3 = 0. \tag{20}$$

Suppose

$$\phi(\xi) = \sum_{j=0}^{n} a_j z(\xi)^j, \tag{21}$$

from Eq.(17), we have $O\left(\dfrac{\mathrm{d}^2 \phi}{\mathrm{d}\xi^2}\right) = n + 2,\ O(\phi^3) = 3n$, so $n = 1$ and $\phi(\xi)$ may have the following form:

$$\phi(\xi) = a_0 + a_1 z(\xi), \tag{22}$$

where $a_0$ and $a_1$ are constants to be determined.

Substituting Eq.(22) into Eq.(20), then substitution of Eq.(4) and setting the coefficients of $z(\xi)^j (j =$

$0, 1, 2, 3)$ to zero leads to a set of algebraic equations for $a_0$, $a_1$ and $\omega$,

$$\begin{aligned}
&a_0(\omega - k^2\alpha) + a_0^3\beta = 0,\\
&a_1 b\alpha + a_1(\omega - k^2\alpha) + 3a_0^2 a_1\beta = 0,\\
&3a_0 a_1^2\beta = 0,\\
&2a_1 c\alpha + a_1^3\beta = 0.
\end{aligned}$$

Solving this set of algebraic equations, we can determine the parameters:

$$\begin{aligned}
&a_0 = 0,\\
&a_1 = \pm\sqrt{\frac{-2c\alpha}{\beta}},\\
&\omega = -(b - k^2)\alpha.
\end{aligned} \tag{23}$$

Substituting Eq.(23) into Eq.(22), then into Eq.(19) and using Eqs.(5)–(14), the exact periodic solutions of the NLS equation (18) are obtained as follows:

$$u_1(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{sn}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = -(1+m^2),\ c = m^2, \tag{24}$$

$$u_2(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{cn}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2m^2 - 1,\ c = -m^2, \tag{25}$$

$$u_3(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{dn}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2 - m^2,\ c = -1, \tag{26}$$

$$u_4(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{ns}(x - 2\alpha kt)\ \mathrm{e}^{i(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = -(1+m^2),\ c = 1, \tag{27}$$

$$u_5(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{nc}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2m^2 - 1,\ c = 1 - m^2, \tag{28}$$

$$u_6(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{nd}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2 - m^2,\ c = -(1 - m^2), \tag{29}$$

$$u_7(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{sc}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2 - m^2,\ c = 1 - m^2, \tag{30}$$

$$u_8(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{sd}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2m^2 - 1,\ c = -m^2(1 - m^2), \tag{31}$$

$$u_9(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{cs}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2 - m^2,\ c = 1, \tag{32}$$

$$u_{10}(x,t) = \pm\sqrt{\frac{-2c\alpha}{\beta}}\mathrm{ds}(x - 2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\quad b = 2m^2 - 1,\ c = 1. \tag{33}$$

Notice that when $m \to 1$, $\mathrm{sn}\,\xi \to \tanh\xi$, $\mathrm{cn}\,\xi \to \mathrm{sech}\,\xi$, and $\mathrm{dn}\,\xi \to \mathrm{sech}\,\xi$, solutions $u_1 - u_{10}$ are degenerated as the corresponding envelope solitary or shock wave solutions.

### 3.2. Zakharov equations

Consider the Zakharov equations in the following form:[24]

$$u_{tt} - c_s^2 u_{xx} = \beta\frac{\partial^2|v|^2}{\partial x^2}, \tag{34}$$

$$\mathrm{i}v_t + \alpha v_{xx} - \delta uv = 0 \tag{35}$$

with $\mathrm{i} = \sqrt{-1}$, in which $\alpha$, $\beta$ and $\delta$ are three constant parameters, and $c_s$ denotes the velocity of thermal motion for the electron-ion, $u$ the partial increment of number density, $v$ the strength of electric field with slow-changing amplitude. Zakharov equations describe the high-frequency motion of plasma or nonlinear light wave, which are a pair of coupled nonlinear evolution equations. Take

$$\begin{aligned}
&u = u(\xi),\\
&v = \phi(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t)},\\
&\xi = x - 2\alpha kt,
\end{aligned} \tag{36}$$

where $\phi(\xi)$ is a real value function and $k$ and $\omega$ denote the wavenumber and the circular frequency, respectively.

Substituting Eq.(36) into Eqs.(34) and (35), we have

$$(4\alpha^2k^2 - c_s^2)\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = \beta\frac{\mathrm{d}^2(\phi^2)}{\mathrm{d}\xi^2}, \tag{37}$$

$$\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + (\omega - \alpha k^2)\phi - \delta u\phi = 0. \tag{38}$$

From Eq.(37), we can deduce that

$$(4\alpha^2k^2 - c_s^2)u = \beta\phi^2, \tag{39}$$

in which some constant terms are omitted.

Substitution of Eq.(39) in Eq.(38) yields

$$\alpha\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + (\omega - \alpha k^2)\phi - \frac{\delta\beta}{4\alpha^2k^2 - c_s^2}\phi^3 = 0. \tag{40}$$

It is easy to find that Eq.(40) is similar to Eq.(20). From Eq.(23), we get

$$\phi(\xi) = \pm\sqrt{\frac{2c\alpha(4\alpha^2k^2 - c_s^2)}{\delta\beta}}z(\xi). \tag{41}$$

Substituting Eq.(41) into Eqs.(39) and (36) and using Eqs.(5)–(14), ten pairs of exact periodic solutions of Zakharov equations are obtained as follows:

$$u_1(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{sn}^2(x-2\alpha kt),\quad v_1(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{sn}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=-(1+m^2),\ c=m^2; \tag{42}$$

$$u_2(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{cn}^2(x-2\alpha kt),\quad v_2(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{cn}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2m^2-1,\ c=-m^2; \tag{43}$$

$$u_3(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{dn}^2(x-2\alpha kt),\quad v_2(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{dn}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2-m^2,\ c=-1; \tag{44}$$

$$u_4(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{ns}^2(x-2\alpha kt),\quad v_4(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{ns}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=-(1+m^2),\ c=1; \tag{45}$$

$$u_5(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{nc}^2(x-2\alpha kt),\quad v_5(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{nc}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2m^2-1,\ c=1-m^2; \tag{46}$$

$$u_6(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{nd}^2(x-2\alpha kt),\quad v_6(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{nd}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2-m^2,\ c=-(1-m^2); \tag{47}$$

$$u_7(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{sc}^2(x-2\alpha kt),\quad v_7(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{sc}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2-m^2,\ c=1-m^2; \tag{48}$$

$$u_8(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{sd}^2(x-2\alpha kt),\quad v_8(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{sd}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2m^2-1,\ c=-m^2(1-m^2); \tag{49}$$

$$u_9(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{cs}^2(x-2\alpha kt),\quad v_9(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{cs}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2-m^2,\ c=1; \tag{50}$$

$$u_{10}(x,t)=\frac{2c\alpha}{\delta}\ \mathrm{ds}^2(x-2\alpha kt),\quad v_{10}(x,t)=\pm\sqrt{\frac{2c\alpha(4\alpha^2k^2-c_s^2)}{\delta\beta}}\ \mathrm{sn}(x-2\alpha kt)\ \mathrm{e}^{\mathrm{i}(kx+\alpha(b-k^2)t)},$$
$$\text{for}\ \ b=2m^2-1,\ c=1. \tag{51}$$

When $m\to 1$, the solutions $u_1$–$u_{10}$ are degenerated as the corresponding solitary wave solutions; $v_1$–$v_{10}$ are degenerated as the corresponding envelope solitary or shock wave solutions of Zakharov equations.

### 3.3. Coupled nonlinear Schrödinger equations

We consider coupled nonlinear Schrödinger equations in the following form:[25,26]

$$\mathrm{i}u_t+u_{xx}+2|u|^2+|v|^2u=0, \tag{52}$$

$$\mathrm{i}v_t+v_{xx}+2|u|^2+|v|^2v=0 \tag{53}$$

with $\mathrm{i}=\sqrt{-1}$. Let

$$u=\phi(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$
$$v=\varphi(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$
$$\xi=x-2kt, \tag{54}$$

where $\phi(\xi)$ and $\varphi(\xi)$ are two real value functions and $k$ and $\omega$ denote the wavenumber and the circular frequency, respectively.

Substituting Eq.(54) into Eqs.(52) and (53), we get

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}+(\omega-k^2)\phi+2(\phi^2+\varphi^2)\phi=0, \tag{55}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}+(\omega-k^2)\varphi+2(\phi^2+\varphi^2)\varphi=0. \tag{56}$$

Suppose

$$\phi(\xi) = \sum_{j=0}^{n} a_j z(\xi)^j,$$
$$\varphi(\xi) = \sum_{j=0}^{l} b_j z(\xi)^j, \tag{57}$$

by virtue of the principle of balance of the highest derivative terms yields $n = l = 1$. So $\phi(\xi)$ and $\varphi(\xi)$ may have the following form:

$$\phi(\xi) = a_0 + a_1 z(\xi),$$
$$\varphi(\xi) = b_0 + b_1 z(\xi), \tag{58}$$

where $a_0$, $a_1$, $b_0$ and $b_1$ are constants to be determined.

Substituting Eq.(58) into Eqs.(55) and (56), using Eqs.(4)–(14) and setting the coefficients of $z(\xi)^j (j = 0, 1, 2, 3)$ to zero leads to a set of algebraic equations for $a_0$, $a_1$, $b_0$, $b_1$ and $\omega$:

$$\begin{aligned}
&a_0(\omega - k^2) + 2a_0^3 + 2a_0 b_0^2 = 0,\\
&6a_0^2 a_1 + a_1 b + 2a_1 b_0^2 + 4a_0 b_0 b_1 + a_1(\omega - k^2) = 0,\\
&6a_0 a_1^2 + 4a_1 b_0 b_1 + 2a_0 b_1^2 = 0,\\
&2a_1^3 + 2a_1 b_1^2 + 2a_1 c = 0,\\
&2a_0^2 b_0 + 2b_0^3 + b_0(\omega - k^2) = 0,\\
&4a_0 a_1 b_0 + 2a_0^2 b_1 + b b_1 + 6b_0^2 b_1 + b_1(\omega - k^2) = 0,\\
&2a_1^2 b_0 + 4a_0 a_1 b_1 + 6b_0 b_1^2 = 0,\\
&2a_1^2 b_1 + 2b_1^3 + 2b_1 c = 0.
\end{aligned}$$

Solving this set of algebraic equations with the aid of Mathematica, we can determine the parameters:

$$a_0 = 0, \quad b_0 = 0, \quad a_1 = \pm\sqrt{-b_1^2 - c}, \quad \omega = -b + k^2. \tag{59}$$

Substituting Eq.(59) into Eq.(58), then into Eq.(54) and using Eqs.(5)–(14), the ten pairs of exact periodic solutions of coupled nonlinear Schrödinger equations (52) and (53) are obtained as follows:

$$\begin{aligned}
u_1(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{sn}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_1(x,t) &= b_1 \mathrm{sn}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = -(1 + m^2), \quad c = m^2;
\end{aligned} \tag{60}$$

$$\begin{aligned}
u_2(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{cn}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_2(x,t) &= b_1 \mathrm{cn}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2m^2 - 1, \quad c = -m^2;
\end{aligned} \tag{61}$$

$$\begin{aligned}
u_3(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{dn}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_3(x,t) &= b_1 \mathrm{dn}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2 - m^2, \quad c = -1;
\end{aligned} \tag{62}$$

$$\begin{aligned}
u_4(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{ns}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_4(x,t) &= b_1 \mathrm{ns}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = -(1 + m^2), \quad c = 1;
\end{aligned} \tag{63}$$

$$\begin{aligned}
u_5(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{nc}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_5(x,t) &= b_1 \mathrm{nc}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2m^2 - 1, \quad c = 1 - m^2;
\end{aligned} \tag{64}$$

$$\begin{aligned}
u_6(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{nd}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_6(x,t) &= b_1 \mathrm{nd}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2 - m^2, \quad c = -(1 - m^2);
\end{aligned} \tag{65}$$

$$\begin{aligned}
u_7(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{sc}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_7(x,t) &= b_1 \mathrm{sc}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2 - m^2, \quad c = 1 - m^2;
\end{aligned} \tag{66}$$

$$\begin{aligned}
u_8(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{sd}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_8(x,t) &= b_1 \mathrm{sd}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2m^2 - 1, \quad c = -m^2(1 - m^2);
\end{aligned} \tag{67}$$

$$\begin{aligned}
u_9(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{cs}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_9(x,t) &= b_1 \mathrm{cs}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2 - m^2, \quad c = 1;
\end{aligned} \tag{68}$$

$$\begin{aligned}
u_{10}(x,t) &= \pm\sqrt{-b_1^2 - c}\ \mathrm{ds}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
v_{10}(x,t) &= b_1 \mathrm{ds}(x - 2kt)\ \mathrm{e}^{\mathrm{i}(kx + (b - k^2)t)},\\
&\text{for} \quad b = 2m^2 - 1, \quad c = 1;
\end{aligned} \tag{69}$$

where $b_1$ is a free constant.

When $m \to 1$, the solutions (60)–(69) are degenerated as the corresponding envelope solitary or shock wave solutions of coupled nonlinear Schrödinger equations (52) and (53).

## 4. Conclusions and discussion

Now we summarize the overall conclusions of this paper.

1) An auxiliary elliptic equation method is proposed to construct the exact periodic solutions of some nonlinear Schrödinger type equations. It is shown that this method can be used to construct more periodic solutions of nonlinear Schrödinger type equations in a unified way.

2) Since Jacobi elliptic functions are more gen-

eral than the hyperbolic tangent and secant functions, so under the limiting condition ($m \to 1$), our results give the corresponding envelope solitary or shock wave solutions.

3) This method is also a computerized method, which allows us to perform complicated and tedious algebraic calculation with the aid of symbolic computation (using Mathematica) on the computer.

## References


[1] Li Z B and Yao R X 2001 *Acta Phys. Sin.* **50** 2062 (in Chinese)

[2] Zhang G R, Li Z B and Duan Y S 2000 *Chin. Sci.* (Ser. A) **30** 1103 (in Chinese)

[3] Yang L, Liu J and Yang K 2001 *Phys. Lett.* A **278** 267

[4] Guo G P and Zhang J F 2002 *Acta Phys. Sin.* **51** 1159 (in Chinese)

[5] Wang M L 1995 *Phys. Lett.* A **199** 169

[6] Wang M L, Zhou Y B and Li Z B 1996 *Phys. Lett.* A **216** 67

[7] Fan E G and Zhang H Q 1998 *Acta Phys. Sin.* **47** 353 (in Chinese)

[8] Zhang J L, Wang M L and Wang Y M 2003 *Chin. Phys.* **12** 245

[9] Xu G Q and Li Z B 2002 *Acta Phys. Sin.* **51** 946 (in Chinese)

[10] Xu G Q and Li Z B 2002 *Acta Phys. Sin.* **51** 1424 (in Chinese)

[11] Fan E G 2000 *Phys. Lett.* A **277** 212

[12] Sirendaoreji and Sun J 2003 *Phys. Lett.* A **309** 387

[13] Chen Y, Li B and Zhang H Q 2003 *Chin. Phys.* **12** 940

[14] Ruan H Y and Chen Y X 2003 *Acta Phys. Sin.* **52** 1313 (in Chinese)

[15] Porubov A V 1996 *Phys. Lett.* A **221** 391

[16] Porubov A V and Velarde M G 1999 *J. Math. Phys.* **40** 884

[17] Porubov A V and Parker D F 1999 *Wave Motion* **29** 97

[18] Liu S K, Fu Z T, Liu S D *et al* 2001 *Acta Phys. Sin.* **50** 2068 (in Chinese)

[19] Fu Z T, Liu S K, Liu S D *et al* 2002 *Acta Phys. Sin.* **51** 10 (in Chinese)

[20] Liu S K, Fu Z T, Liu S D *et al* 2001 *Phys. Lett.* A **289** 69

[21] Fu Z T, Liu S K, Liu S D *et al* 2001 *Phys. Lett.* A **290** 72

[22] Zhang S Q, Li Z B 2003 *Acta Phys. Sin.* **52** 1066 (in Chinese)

[23] Prasolov V and Solovyev Y 1997 *Elliptic Functions and Elliptic Integrals* (Providence, RI: American Mathematical Society)

[24] Liu S K and Liu S D 2000 *Nonlinear Equations in Physics* (Beijing: Peking University Press) (in Chinese)

[25] Ablowitz M J and Clarkson P A 1991 *Soliton, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[26] Zakharov V E and Schulman E I 1982 *Physica* D **4** 270