Vol 13 No 6, June 2004
1009-1963/2004/13(06)/0798-07

Chinese Physics

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General Jacobian elliptic function expansion method and its applications*

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(Received 19 March 2003; revised manuscript received 29 December 2003)

An extended Jacobian elliptic function expansion method is presented and successfully applied to the nonlinear Schrödinger (NLS) equation and Zakharov equation. We obtain some new solutions besides Fu *et al*'s results. The results show that our method is more powerful to construct Jacobian elliptic function and can be applied to other nonlinear physics systems.

Keywords: nonlinear Schrödinger equation, Zakharov equation, Jacobian elliptic function, solitary wave solution

PACC: 0340, 0365, 0545

1. Introduction

Seeking exact solutions of nonlinear evolution equations (NEEs) is of important significance in mathematical physics. Many powerful methods have been developed to construct many types of exact solutions of NEEs such as solitary solution,[1–5] dromion solution,[6,7] compaction solution,[8] rational solution,[9] etc. Recently Fu *et al*[10,11] presented a Jacobian elliptic function expansion method and constructed Jacobian elliptic function envelope periodic solutions of NEEs including the KdV equation, mKdV equation, nonlinear Schrödinger (NLS) equation and Zakharov equation, but they only obtained four Jacobian elliptic function solutions. Yan[12–14] further developed this method and proposed an extended Jacobian elliptic function method so that he could obtain more types of Jacobian elliptic function solutions. In this paper, we consider and analyse the Jacobian elliptic function expansion method further. The key idea is that we extend the range of j from $-n$ to n. Moreover, we also notice that the dn-function method is redundant because there is a Jacobian transformation so that one can convert a dn function to a cn function (see Section 2 for more details).[15] We discuss the NLS equation and Zakharov equation using the new method. It can be seen that more types of Jacobian function solutions including Fu *et al*'s results are obtained.

2. A general Jacobian elliptic function expansion method

For a given nonlinear evolution equation, say in two variables x and t,

$$F(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, \cdots) = 0. \tag{1}$$

We look for its travelling wave ansatz as follows:

$$\begin{aligned} u(x,t) &= u(\xi), \\ \xi &= p(x - ct), \end{aligned} \tag{2}$$

where p and c are respectively wave number and wave speed to be determined later. Substituting Eq.(2) into Eq.(1) yields a nonlinear ordinary differential equation

$$K\left(u, \frac{\mathrm{d}u}{\mathrm{d}\xi}, \frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}, \cdots\right) = 0. \tag{3}$$

Making good use of the Jacobian elliptic functions, we

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assume that Eq.(3) has the solutions in the form

$$u(\xi) = a_0 + \sum_{j=-n}^{n} [a_i f_k^j(\xi) + b_i h_k^j(\xi)], \quad (k = 1, 2, 3), \tag{4}$$

with

$$\begin{aligned} &f_1(\xi) = \mathrm{sn}(\xi), \quad h_1(\xi) = \mathrm{cn}(\xi); \\ &f_2(\xi) = \mathrm{sn}(\xi), \quad h_2(\xi) = \mathrm{cs}(\xi) \equiv \frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)}; \\ &f_3(\xi) = \mathrm{cn}(\xi), \quad h_3(\xi) = \mathrm{cs}(\xi), \end{aligned} \tag{5}$$

where a_0, a_i, b_i are constants to be determined, $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$, $\mathrm{dn}(\xi)$ are the Jacobian elliptic sine function, Jacobian elliptic cosine function and Jacobian elliptic function of the third kind and other Jacobian functions which are denoted by Glaisher symbols and are generated by these three kinds of functions,[12−14] and there are the relations

$$\begin{aligned} &\mathrm{cn}^2(\xi) + \mathrm{sn}^2(\xi) = 1, \\ &\mathrm{dn}^2(\xi) + m^2\mathrm{sn}^2(\xi) = 1, \\ &\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{sn}(\xi) = \mathrm{cn}(\xi)\mathrm{dn}(\xi), \\ &\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{cn}(\xi) = -\mathrm{sn}(\xi)\mathrm{dn}(\xi), \\ &\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{dn}(\xi) = -m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi), \end{aligned} \tag{6}$$

where m $(0 < m < 1)$ is a modulus of the Jacobian elliptic function.

To balance the highest-order linear term with the nonlinear term we define the degree of u as $D[u] = n$, which gives rise to the degrees of other expressions as

$$\begin{aligned} &D\left[\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right] = n + q, \\ &D\left[u^p\left(\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right)^s\right] = np + s(n + q). \end{aligned} \tag{7}$$

According to the rules, we can balance the highest-order linear term and nonlinear term in Eq.(3) so that n can be determined.

In addition, we see that when $m \to 1$, $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ degenerate into $\tanh(\xi)$, $\mathrm{sech}(\xi)$, $\mathrm{sech}(\xi)$, respectively, while for $m \to 0$, $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$ and $\mathrm{dn}(\xi)$ degenerate into $\sin(\xi)$, $\cos(\xi), 1$, respectively. Finally, we note that the Jacobian transformation

$$\mathrm{dn}(k\xi, m) = \mathrm{cn}(\sqrt{m}k\xi, m^{-1}) \tag{8}$$

implies that any solution found by the dn-function method may be transformed into an equivalent one that can be obtained by the cn-function method, therefore the dn-function method is redundant.

3. Envelope periodic solutions for NLS equation

The NLS equation reads as

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta|u|^2 u = 0. \tag{9}$$

We seek the following wavepacket solutions:

$$\begin{aligned} &u = u(\xi)\mathrm{e}^{\mathrm{i}(kx - \omega t)}, \\ &\xi = p(x - c_{\mathrm{g}}t), \end{aligned} \tag{10}$$

where k and ω are the wavenumber and phase speed respectively, c_{g} is the group speed.

Substituting Eq.(10) into Eq.(9) yields

$$\begin{aligned} &\alpha p^2\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \mathrm{i}p(2\alpha k - c_{\mathrm{g}})\frac{\mathrm{d}u}{\mathrm{d}\xi} \\ &+ (\omega - \alpha k^2)u + \beta u^3 = 0. \end{aligned} \tag{11}$$

Taking $2\alpha k = c_{\mathrm{g}}$, $\omega - \alpha k^2 = -\gamma$ $(\gamma > 0)$, we have

$$\alpha p^2\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} - \gamma u + \beta u^3 = 0. \tag{12}$$

Thus we can obtain

$$\begin{aligned} &D\left[\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2}\right] = n + 2, \\ &D[u^3] = 3n, \end{aligned} \tag{13}$$

which leads to $n = 1$. Next, we solve Eq.(12) using Jacobian elliptic function expansions.

3.1. Jacobian elliptic function $\mathrm{sn}(\xi)$-$\mathrm{cn}(\xi)$ expansion

Along with the above ideas we assume that Eq.(12) has the solution

$$u(\xi) = a_0 + a_1\mathrm{sn}(\xi) + a_2\mathrm{sn}^{-1}(\xi) + a_3\mathrm{cn}(\xi) + a_4\mathrm{cn}^{-1}(\xi). \tag{14}$$

Substituting Eq.(14) into Eq.(12) yields a system of nonlinear algebraic equations

$$\begin{aligned} &2\alpha p^2 a_1 m^2 + 3\beta a_1 a_3^2 - \beta a_1^3 = 0, \\ &-2\alpha p^2 a_3 m^2 + 3\beta a_1^2 a_3 - \beta a_3^3 = 0, \\ &6\beta a_0 a_1 a_3 = 0, \\ &-3\beta a_0 a_3^2 + 3\beta a_0 a_1^2 = 0, \\ &-3\beta a_2 a_3^2 + 3\beta a_0^2 a_1 - \gamma a_1 + 3\beta a_1^2 a_2 - 4\alpha p^2 a_1 m^2 \\ &-6\beta a_1 a_3^2 + 6\beta a_1 a_3 a_4 - \alpha p^2 a_1 + 3\beta a_1^3 = 0, \\ &3\beta a_1^2 a_4 - 6\beta a_1^2 a_3 + \alpha p^2 a_3 - 6\beta a_1 a_2 a_3 + 3\alpha p^2 a_3 m^2 \\ &-3\beta a_0^2 a_3 + \beta a_3^3 - 3\beta a_3^2 a_4 + \gamma a_3 = 0, \end{aligned}$$

$$
\begin{aligned}
&6\beta a_0a_1a_4-12\beta a_0a_1a_3-6\beta a_0a_2a_3=0,\\
&-6\beta a_0a_1a_2+\gamma a_0-6\beta a_0a_3a_4-6\beta a_0a_1^2+3\beta a_0a_3^3-\beta a_0^3=0,\\
&-3\beta a_0^2a_2+2\alpha p^2a_1m^2+3\beta a_2a_3^2+2\alpha p^2a_1+2\gamma a_1-6\beta a_0^2a_1-12\beta a_1a_3a_4-2\alpha p^2a_2m^2\\
&+3\beta a_1a_3^2-6\beta a_2a_3a_4-3\beta a_1a_2^2+3\beta a_1a_4^2-6\beta a_1^2a_2-3\beta a_1^3+\gamma a_2+\alpha p^2a_2=0,\\
&\beta a_4^3+2\alpha p^2a_4m^2=0,\\
&3\beta a_2a_4^2+3\beta a_1a_4^2=0,\\
&3\beta a_0a_4^2=0,\\
&6\beta a_0a_1a_4+6\beta a_0a_2a_4=0,\\
&3\beta a_2^2a_4-\alpha p^2a_4m^2-\gamma a_4+6\beta a_1a_2a_4+3\beta a_1^2a_4-\beta a_4^3-3\alpha p^2a_4+3\beta a_3a_4^2+3\beta a_0^2a_4=0,\\
&3\beta a_1^2a_2-\alpha p^2a_1+6\beta a_1a_3a_4-6\beta a_1a_4^2+\gamma a_2+\beta a_2^3+3\beta a_0^2a_1+\beta a_1^3-3\beta a_2a_4^2+3\beta a_0^2a_2\\
&+3\beta a_1a_2^2+\alpha p^2a_2-\gamma a_1+6\beta a_2a_3a_4=0,\\
&-\gamma a_0+\beta a_0^3+6\beta a_0a_3a_4+3\beta a_0a_1^2+6\beta a_0a_1a_2+3\beta a_0a_2^2-3\beta a_0a_4^2,\\
&6\beta a_0a_1a_3-12\beta a_0a_1a_4+6\beta a_0a_2a_3-6\beta a_0a_2a_4=0,\\
&3\beta a_1^2a_3+6\beta a_1a_2a_3-3\beta a_3a_4^2+3\beta a_2^2a_3-6\beta a_1^2a_4-\alpha p^2a_3m^2-\gamma a_3\\
&-6\beta a_1a_2a_4-\alpha p^2a_3+\alpha p^2a_4+\gamma a_4+3\beta a_3^2a_4+3\beta a_0^2a_3+\alpha p^2a_4m^2-3\beta a_0^2a_4=0. \qquad (15)
\end{aligned}
$$

According to Eq.(15), we can derive a_0, a_1, a_2, a_3 and a_4; substituting a_0, a_1, a_2, a_3, a_4 into Eq.(14) yields the following Jacobian elliptic function solutions (doubly periodic solutions) of Eq.(11):

$$u_{1,2}=\pm m\sqrt{\frac{2\gamma}{\beta(1+m^2)}}\ \mathrm{sn}\left(\sqrt{\frac{-\gamma}{\alpha(1+m^2)}}(x-c_gt)\right), \qquad (16)$$

$$u_{3,4}=\pm\sqrt{\frac{2\gamma}{\beta(1+m^2)}}\ \mathrm{sn}^{-1}\left(\sqrt{\frac{-\gamma}{\alpha(1+m^2)}}(x-c_gt)\right), \qquad (17)$$

where $\alpha<0$, $\beta>0$ (see Fig.1). When $m=1$, we obtain the corresponding envelope solitary shock wave solutions

$$u'_{1,2}=\pm\sqrt{\frac{2\gamma}{2\beta}}\ \tanh\left(\sqrt{\frac{-\gamma}{2\alpha}}(x-c_gt)\right), \qquad (18)$$

$$u_{5,6}=\pm\sqrt{\frac{2\gamma}{\beta(2m^2-1)}}m\ \mathrm{cn}\left(\sqrt{\frac{\gamma}{\alpha(2m^2-1)}}(x-c_gt)\right), \qquad (19)$$

$$u_{7,8}=\pm\sqrt{\frac{2\gamma(1-m^2)}{\beta(1-2m^2)}}\ \mathrm{cn}^{-1}\left(\sqrt{\frac{\gamma}{\alpha(2m^2-1)}}(x-c_gt)\right), \qquad (20)$$

where $(2m^2-1)\alpha>0$, $(2m^2-1)\beta>0$ (see Fig.2). When $m=1$, we obtain the corresponding envelope solitary wave solutions

$$u'_{7,8}=\pm\sqrt{\frac{2\gamma}{\beta}}m\ \mathrm{sech}\left(\sqrt{\frac{\gamma}{\alpha}}(x-c_gt)\right), \qquad (21)$$

$$
\begin{aligned}
u_{9,10}=&\pm\sqrt{\frac{2\gamma}{\beta(1+m^2+6m)}}m\ \mathrm{sn}\left(\sqrt{\frac{-\gamma}{\alpha(1+m^2+6m)}}(x-c_gt)\right)\\
&\mp\sqrt{\frac{2\gamma}{\beta(1+6m+m^2)}}\ \mathrm{sn}^{-1}\left(\sqrt{\frac{-\gamma}{\alpha(1+m^2+6m)}}(x-c_gt)\right), \qquad (22)\\
u_{11,12}=&\pm\sqrt{\frac{2\gamma}{\beta(1+m^2-6m)}}m\ \mathrm{sn}\left(\sqrt{\frac{-\gamma}{\alpha(1+m^2-6m)}}(x-c_gt)\right)\\
&\mp\sqrt{\frac{2\gamma}{\beta(1-6m+m^2)}}\ \mathrm{sn}^{-1}\left(\sqrt{\frac{-\gamma}{\alpha(1+m^2-6m)}}(x-c_gt)\right), \qquad (23)
\end{aligned}
$$

$$u_{13,14} = \pm\sqrt{\frac{\gamma}{\beta(m^2-1)}}\,m\ \mathrm{cn}\left(\sqrt{\frac{2\gamma}{\alpha(m^2-2)}}(x-c_{\mathrm{g}}t)\right) \mp \sqrt{\frac{\gamma}{\beta(m^2-2)}}\,m\ \mathrm{sn}\left(\sqrt{\frac{2\gamma}{\alpha(m^2-2)}}(x-c_{\mathrm{g}}t)\right), \quad (24)$$

$$u_{15,16} = \pm\sqrt{\frac{2\gamma(1-2m^2+6m\sqrt{m^2-1})}{\beta(32m^4-1-32m^2)}}\,m\ \mathrm{cn}\left(\sqrt{\frac{\gamma(2m^2-1-6m\sqrt{m^2-1})}{\alpha(32m^4-32m^2-1)}}(x-c_{\mathrm{g}}t)\right)$$

$$\pm\frac{\dfrac{(2m^2-1)(1-2m^2+6m\sqrt{m^2-1})}{32m^4-32m^2-1}-1}{3m\sqrt{\dfrac{2\beta\gamma(1-2m^2+6m\sqrt{m^2-1})}{32m^4-32m^2-1}}}\ \mathrm{cn}^{-1}\left(\sqrt{\frac{\gamma(2m^2-1-6m\sqrt{m^2-1})}{\alpha(32m^4-32m^2-1)}}(x-c_{\mathrm{g}}t)\right), \quad (25)$$

where $u_{1,2}, u_{5,6}$ have been obtained by Fu *et al.*

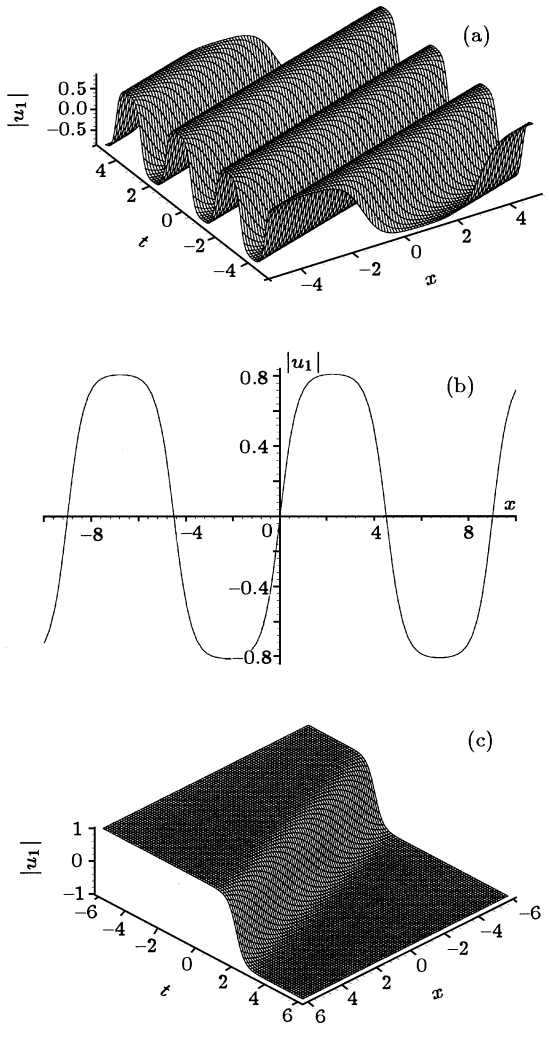


Fig.1. The modulus of the Jacobian doubly periodic solution u_1 with parameters $\gamma = 1, \beta = 1, \alpha = 1, c_{\mathrm{g}} = 3, m = 0.7$ (a). The modulus of the Jacobian doubly periodic solution u_1 at t=0 (b). The corresponding kink waves related to u_1 for $m = 1$ (c).

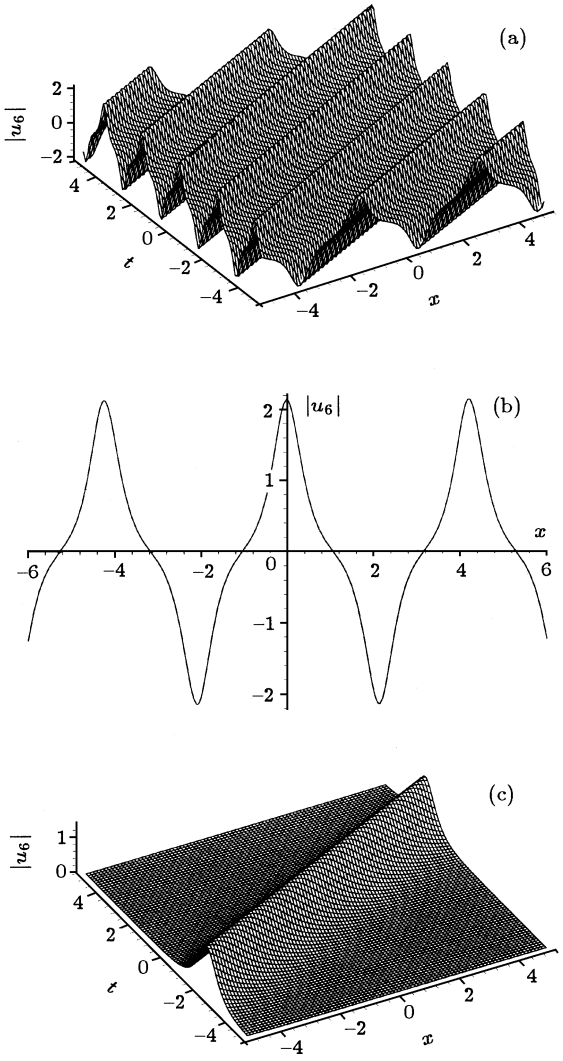


Fig.2. The modulus of the Jacobian doubly periodic solution u_6 with parameters $\gamma = 1, \beta = 1, \alpha = 1, c_{\mathrm{g}} = 2, m = 0.8(a)$. The modulus of the Jacobian doubly periodic solution u_6 at t=0 (b). The corresponding bell waves related to u_6 for $m = 1$ (c).

3.2. Jacobian elliptic function $\mathrm{sn}(\xi)$-$\mathrm{cs}(\xi)$ expansion

Now we assume that Eq.(12) has the solution

$$u(\xi) = a_0 + a_1\,\mathrm{sn}(\xi) + a_2\,\mathrm{sn}^{-1}(\xi) + a_3\,\mathrm{cs}(\xi) + a_4\,\mathrm{cs}^{-1}(\xi). \tag{26}$$

Substituting Eq.(26) into Eq.(12) yields a system of nonlinear algebraic equations. Correspondingly we have the new solutions

$$u_{17,18} = \pm\sqrt{\frac{2\gamma(1-m^2)}{\beta(m^2-2)}} \times \mathrm{cs}^{-1}\left(\sqrt{\frac{\gamma}{\alpha(2-m^2)}}(x-c_\mathrm{g}t)\right), \tag{27}$$

$$u_{19,20} = \pm\sqrt{\frac{2\gamma}{\beta(m^2-2)}} \times \mathrm{cs}\left(\sqrt{\frac{\gamma}{\alpha(2-m^2)}}(x-c_\mathrm{g}t)\right), \tag{28}$$

where $\alpha > 0$, $\beta < 0$. When $m = 1$, we have

$$u'_{19,20} = \pm\sqrt{\frac{-2\gamma}{\beta}}\,\mathrm{csch}\left(\sqrt{\frac{\gamma}{\alpha}}(x-c_\mathrm{g}t)\right)\mathrm{e}^{(kx-\omega t)},$$

$$u_{21,22} = \pm\sqrt{\frac{2\gamma(m^2-1)-12\gamma\sqrt{1-m^2}}{\beta(m^4+32m^2-32)}} \times \mathrm{cs}\left(\sqrt{\frac{\gamma(2-m^2-6\sqrt{1-m^2})}{\alpha(m^4+32m^2-32)}}(x-c_\mathrm{g}t)\right) \pm\left[\frac{\gamma\sqrt{m^2-2-6\sqrt{1-m^2}}(2-m^2)}{3\beta\sqrt{32m^2+m^4-32}} + \frac{\gamma\sqrt{32m^2+m^4-32}}{3\beta\sqrt{m^2-2-6\sqrt{1-m^2}}}\right] \times \mathrm{cs}^{-1}\left(\sqrt{\frac{\gamma(2-m^2-6\sqrt{1-m^2})}{\alpha(m^4+32m^2-32)}}(x-c_\mathrm{g}t)\right), \tag{29}$$

$$u_{23,24} = \pm\sqrt{\frac{\gamma}{\beta(2m^2-1)}} \times \mathrm{sn}^{-1}\left(\sqrt{\frac{2\gamma}{\alpha(1-2m^2)}}(x-c_\mathrm{g}t)\right) \pm\sqrt{\frac{\gamma}{\beta(2m^2-1)}}\,\mathrm{cs}\left(\sqrt{\frac{2\gamma}{\alpha(1-2m^2)}}(x-c_\mathrm{g}t)\right), \tag{30}$$

where $u_{19,20}$ has been obtained by Fu *et al.*

3.3. Jacobian elliptic functions $\mathrm{cn}(\xi)$-$\mathrm{cs}(\xi)$ expansion

Similarly, we further assume that Eq.(12) has the solution

$$u(\xi) = a_0 + a_1\,\mathrm{cn}(\xi) + a_2\,\mathrm{cn}^{-1}(\xi) + a_3\,\mathrm{cs}(\xi) + a_4\,\mathrm{cs}^{-1}(\xi). \tag{31}$$

Substituting Eq.(31) into Eq.(12) also yields a system of nonlinear algebraic equations, from which we can derive the following new solutions:

$$u_{25,26} = \pm\sqrt{\frac{\gamma(m^2-1)}{\beta(m^2+1)}} \times \mathrm{cn}^{-1}\left(\sqrt{\frac{2\gamma}{\alpha(m^2+1)}}(x-c_\mathrm{g}t)\right) \pm\sqrt{\frac{\gamma(m^2-1)}{\beta(m^2+1)}}\,\mathrm{cs}^{-1}\left(\sqrt{\frac{2\gamma}{\alpha(m^2+1)}}(x-c_\mathrm{g}t)\right), \tag{32}$$

where $\beta < 0$, $\alpha > 0$.

4. Envelope periodic solutions for Zakharov equation

The nonlinear Zakharov equation reads as

$$\frac{\partial^2 u}{\partial t^2} - c_\mathrm{s}^2\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial^2|v|^2}{\partial x^2} = 0,$$
$$\mathrm{i}\frac{\partial v}{\partial t} + \alpha\frac{\partial^2 v}{\partial x^2} - \delta uv = 0, \tag{33}$$

where c_s is the speed. We seek the following wavepacket solutions:

$$u = u(\xi),$$
$$v = v(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$
$$\xi = p(x-c_\mathrm{g}t), \tag{34}$$

where k and ω are respectively wavenumber and phase speed, c_g is the group speed. Substituting Eq.(34) into Eq.(33) yields

$$(c_\mathrm{g}^2 - c_\mathrm{s}^2)\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} - \beta\frac{\mathrm{d}^2 v^2}{\mathrm{d}\xi^2} = 0,$$
$$\alpha p^2\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} + \mathrm{i}p(2\alpha k - c_\mathrm{g})\frac{\mathrm{d}v}{\mathrm{d}\xi} + (\omega - \alpha k^2)v - \delta uv = 0. \tag{35}$$

Taking

$$2\alpha k = c_\mathrm{g},$$
$$\omega - \alpha k^2 = -\gamma, \quad (\gamma > 0), \tag{36}$$

and considering Eq.(35) to balance the highest order of derivative and nonlinear term, we can obtain

$$n = 2, \qquad m = 1. \tag{37}$$

4.1. Jacobian elliptic $\mathrm{sn}(\xi)$-$\mathrm{cn}(\xi)$ function expansion

Based on Eq.(37), Eq.(35) may possess the following solution:

$$\begin{aligned} u(\xi) =& a_0 + a_1\ \mathrm{sn}(\xi) + a_2\ \mathrm{sn}^2(\xi) \\ &+ a_3\ \mathrm{cn}(\xi) + a_4\ \mathrm{sn}(\xi)\ \mathrm{cn}(\xi) + a_5\ \mathrm{sn}^{-1}(\xi) \\ &+ a_6\ \mathrm{sn}^{-2}(\xi) + a_7\ \mathrm{sn}^{-1}(\xi)\ \mathrm{cn}^{-1}(\xi), \\ v(\xi) =& b_0 + b_1\ \mathrm{sn}(\xi) + b_2\ \mathrm{sn}^{-1}(\xi) \\ &+ b_3\ \mathrm{cn}(\xi) + b_4\ \mathrm{cn}^{-1}(\xi). \end{aligned} \tag{38}$$

Substituting Eq.(38) into Eq.(35) yields a system of nonlinear algebraic equations, from which the following Jacobian elliptic function solutions (doubly periodic solutions) of Eq.(35) can be obtained:

$$\begin{aligned} u_{1,2} &= -\frac{\alpha p^2 + \alpha p^2 m^2 + \gamma}{\delta} + \frac{2\alpha p^2 m^2\ \mathrm{sn}^2(\xi)}{\delta}, \\ v_{1,2} &= \pm\sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, pm\ \mathrm{sn}(\xi); \end{aligned} \tag{39}$$

if $c_\mathrm{g} > c_\mathrm{s}$, we then have $\alpha < 0$, $\beta > 0$, $\delta < 0$, or $\alpha < 0$, $\beta < 0$, $\delta > 0$; if $c_\mathrm{g} < c_\mathrm{s}$, then we have $\alpha < 0$, $\beta > 0$, $\delta > 0$ or $\alpha < 0$, $\beta < 0$, $\delta < 0$. When $m = 1$,

$$\begin{aligned} u'_{1,2} &= -\frac{\alpha p^2 + \alpha p^2 + \gamma}{\delta} + \frac{2\alpha p^2\ \tanh^2(\xi)}{\delta}, \\ v'_{1,2} &= \pm\sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, p\ \tanh(\xi); \end{aligned} \tag{40}$$

$$\begin{aligned} u_{3,4} &= -\frac{\alpha p^2 + \gamma}{\delta} + \frac{2\alpha p^2 m^2\ \mathrm{sn}^2(\xi)}{\delta}, \\ v_{3,4} &= \pm\sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, mp\ \mathrm{cn}(\xi). \end{aligned} \tag{41}$$

If $c_\mathrm{g} > c_\mathrm{s}$, then $\alpha > 0$, $\beta > 0$, $\delta > 0$, or $\alpha > 0$, $\beta < 0$, $\delta < 0$; if $c_\mathrm{g} < c_\mathrm{s}$, then $\alpha > 0$, $\beta > 0$, $\delta < 0$ or $\alpha > 0$, $\beta < 0$, $\delta > 0$. When $m = 1$,

$$\begin{aligned} u'_{3,4} &= -\frac{\alpha p^2 + \gamma}{\delta} + \frac{2\alpha p^2\ \tanh^2(\xi)}{\delta}, \\ v'_{3,4} &= \pm\sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, mp\ \mathrm{sech}(\xi); \end{aligned} \tag{42}$$

$$\begin{aligned} u_{5,6} =& \pm\frac{\alpha p^2 + \gamma}{\delta} + \frac{\alpha p^2 m^2\ \mathrm{sn}^2(\xi)}{\delta} \\ &+ \frac{I\alpha p^2 m^2\ \mathrm{sn}(\xi)\ \mathrm{cn}(\xi)}{\delta}, \\ v_{5,6} =& \pm\sqrt{\frac{(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{2\beta\delta}}\, mp\ \mathrm{sn}(\xi) \\ &\pm\sqrt{\frac{(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{2\beta\delta}}\, mp\ \mathrm{cn}(\xi); \end{aligned} \tag{43}$$

$$\begin{aligned} u_{7,8} &= -\frac{\alpha p^2 + \gamma + m^2 p^2 \alpha}{\delta} + \frac{2\alpha p^2\ \mathrm{sn}^{-2}(\xi)}{\delta}, \\ v_{7,8} &= \pm\sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, p\ \mathrm{sn}^{-1}(\xi); \end{aligned} \tag{44}$$

$$\begin{aligned} u_{9,10} =& -\frac{\alpha p^2 + \gamma - 2\alpha m p^2 + m^2 p^2 \alpha}{\delta} \\ &+ \frac{2\alpha p^2 m^2\ \mathrm{sn}^2(\xi)}{\delta} + \frac{2\alpha p^2\ \mathrm{sn}^{-2}(\xi)}{\delta}, \\ v_{9,10} =& \pm\sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, mp\ \mathrm{sn}(\xi) \\ &+ \sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, mp\ \mathrm{sn}^{-1}(\xi). \end{aligned} \tag{45}$$

Here we take the solution u_9, v_9 as an example to further analyse the properties by figures (see Figs.3 and 4). The modulus of solution u_9, v_9 is a line solitary wave solution.

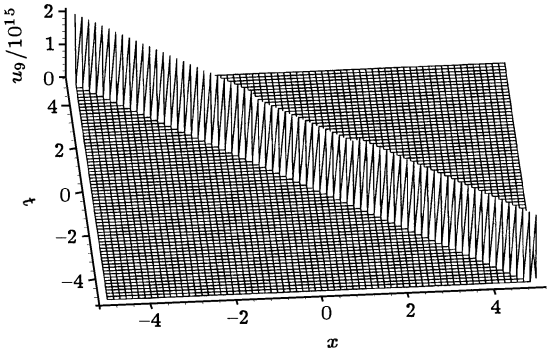


Fig.3. The modulus of the Jacobian doubly periodic solution u_9 with $\gamma = 1$, $\alpha = 1$, $p = 1$, $m = 0.3$, $\delta = 1$, c_g=1.2, c_s=1.

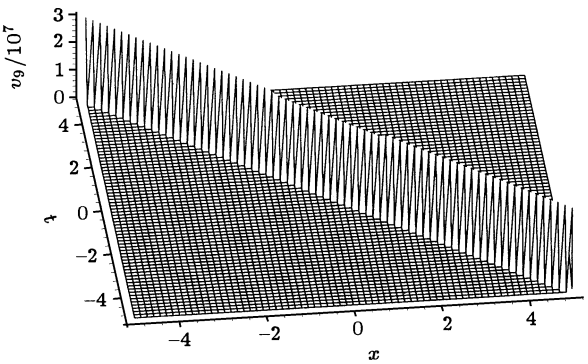


Fig.4. The modulus of the Jacobian doubly periodic solution v_9 with $\gamma = 1$, $\alpha = 1$, $\beta = 1$, $p = 1$, m=0.3, δ=1, c_g=1.2, c_s=1.

4.2. Jacobian elliptic function $\mathrm{cs}(\xi)$ expansion

Now we assume that Eq.(35) has the following solutions:

$$\begin{aligned} u(\xi) =& a_0 + a_1\ \mathrm{cs}(\xi) + a_2\ \mathrm{cs}^{-1}(\xi) \\ &+ a_3\ \mathrm{cs}^2(\xi) + a_4\ \mathrm{cs}^{-2}(\xi), \\ v(\xi) =& b_0 + b_1\ \mathrm{cs}(\xi) + b_2\ \mathrm{cs}^{-1}(\xi). \end{aligned} \tag{46}$$

Substituting Eq.(35) into Eq.(27) yields a system of nonlinear algebraic equations, from which we can obtain the following Jacobian elliptic function solutions (doubly periodic solutions) of Eq.(27):

$$\begin{aligned} u_{11,12} =& -\frac{\alpha p^2 m^2 - 2\alpha p^2 + \gamma}{\delta} \\ &- \frac{2\alpha p^2 (m^2-1)\ \mathrm{cs}^{-2}(\xi)}{\delta}, \\ v_{11,12} =& \pm \sqrt{\frac{2\alpha(1-m^2)(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, p\ \mathrm{cs}^{-1}(\xi); \end{aligned} \tag{47}$$

$$\begin{aligned} u_{13,14} =& -\frac{\alpha p^2 m^2 - 2\alpha p^2 + \gamma}{\delta} + \frac{2\alpha p^2\ \mathrm{cs}^2(\xi)}{\delta}, \\ v_{13,14} =& \pm \sqrt{\frac{2\alpha(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, p\ \mathrm{cs}(\xi); \end{aligned} \tag{48}$$

$$\begin{aligned} u_{15,16} =& \frac{2\alpha p^2 - \alpha p^2 m^2 - \gamma + 2\alpha p^2\sqrt{1-m^2}}{\delta} \\ &+ \frac{2\alpha p^2\ \mathrm{cs}^2(\xi)}{\delta} - \frac{2\alpha p^2 (m^2-1)\ \mathrm{cs}^{-2}(\xi)}{\delta}, \\ v_{15,16} =& \pm \sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, p\ \mathrm{cs}(\xi) \\ &\pm \sqrt{\frac{2(c_\mathrm{g}^2 - c_\mathrm{s}^2)\alpha}{\beta\delta}}\, p\ \mathrm{cs}^{-1}(\xi). \end{aligned} \tag{49}$$

If $c_\mathrm{g} > c_\mathrm{s}$, then $\alpha > 0$, $\beta > 0$, $\delta > 0$, or $\alpha > 0$, $\beta < 0$, $\delta < 0$; if $c_\mathrm{g} < c_\mathrm{s}$, then $\alpha > 0$, $\beta > 0$, $\delta < 0$ or $\alpha > 0$, $\beta < 0$, $\delta > 0$.

In these solutions, $u_{1,2}$, $u_{3,4}$, $u_{9,10}$, $u_{15,16}$ have been obtained by Fu *et al*, the others are new solutions.

5. Summary

In summary, with the guidance of Painlevé truncation theory, we extend the Jacobian elliptic function expansion method and apply it to the NLS and Zakharov systems. We have obtained many new solutions. We expect that this approach could be applied to other nonlinear physics systems.

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