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Intense beam nonlinear transport in solenoid lenses*

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Lie algebraic methods are used in the analysis of the nonlinear transport of charged particle beams in solenoid lenses. Space charge effects are included. The particle distribution we used is a K–V type. Similar procedures can also be used for magnetic dipoles, quadrupoles, sextupoles and other optic elements.

Keywords: solenoid lenses, intense beams, Lie map, nonlinear
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1. Introduction

The nonlinear effects of space charge forces can cause beam emittance growth, beam instability and beam loss. In the periodic transport or accelerating structures, space charge effects can cause nonlinear resonance, halo and chaos. So, to design intense beam accelerators, space charge nonlinear effects should be taken into account. The Lie algebraic method [1] provides a good tool to study nonlinear transport of intense beams. Using the Lie method, we analyse the nonlinear transport of intense beams in solenoid lenses.

2. The Hamiltonian and its expansion [2]

When a charged particle moves in electromagnetic fields, the Hamiltonian H_t with time t as independent variable is

$$H_t = [m_0^2c^4 + c^2(p_x - qA_x)^2 + c^2(p_y - qA_y)^2 + c^2(p_z - qA_z)^2 +]^{\frac{1}{2}} + q\phi, \quad (1)$$

where m_0 is the particle rest energy, q is the charge, p_x, p_y and p_z are the x, y, and z components of the particle momentum respectively, A_x, A_y and A_z are the x, y, and z components of the magnetic vector potential respectively, ϕ is the electric potential, and c is the light velocity. Here, the canonical variables are η=(x, y, z, p_x, p_y, p_z).

Introduce a new variable p_t:

$$p_t = -H_t(x, y, z, p_x, p_y, p_z; t).$$

Solving p_z from p_t, one obtains

$$K = -p_z = -[(p_t + q\phi)^2/c^2 - m_0^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2]^{\frac{1}{2}} - qA_z. \quad (2)$$

Define another set of canonical variables ζ= (x, y, τ, x', y', p_τ):

$$x = x, y = y, t = T - z/\beta_0,$$
$$x' = p_x/p_0, y' = p_y/p_0, p_\tau = p_T - p_T^0, \quad (3)$$

where $T = ct$, $\beta_0 = c/v_0$ (v_0 is the velocity of the reference particle); p_0 is the momentum of the reference particle; $p_T = p_t/(p_0c)$; p_T^0 is the value of p_T for the reference particle. Under the canonical transformation expressed by Eq.(3), the new Hamiltonian is

$$H = -\left[\left(p_\tau + p_T^0 + \frac{q\phi}{p_0c}\right)^2 - \frac{1}{\beta_0^2\gamma^2} - (x' - qA_x/p_0)^2 - (y' - qA_y/p_0)^2\right]^{\frac{1}{2}} - qA_z/p_0 - (p_\tau + p_T^0)/\beta_0. \quad (4)$$

For the solenoid magnetic lenses,

$$A_x = -\frac{1}{2}By + \frac{1}{16}B''(X^2 + y^2)y,$$
$$A_y = \frac{1}{2}Bx - \frac{1}{16}B''(X^2 + y^2)x,$$
$$A_z = 0, \quad (5)$$

where B is the magnetic field distribution along the z axis. The electric potential excited by the charged particle beams in the case of K–V distribution is

$$\phi = -\frac{3IT_{\rm rf}}{8\pi\varepsilon_0\gamma_0XYZ}(\mu_x x^2 + \mu_y y^2 + \mu_z\beta_0^2\tau^2), \quad (6)$$

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where I is the average beam current of the pulsed beams; $T_{\rm rf}$ is the beam repetition period; X, Y and Z are the pulsed beam dimensions; μ_x, μ_y and μ_z are the beam shape factors of the bundles, expressed as

$$\mu_x=\frac{XYZ\gamma}{2}\int_0^\infty\frac{\mathrm{d}\xi}{(x^2+\xi)\sqrt{(x^2+\xi)(y^2+\xi)(z^2\gamma^2+\xi)}},$$
$$\mu_y=\frac{XYZ\gamma}{2}\int_0^\infty\frac{\mathrm{d}\xi}{(y^2+\xi)\sqrt{(x^2+\xi)(y^2+\xi)(z^2\gamma^2+\xi)}},$$
$$\mu_z=\frac{XYZ\gamma}{2}\int_0^\infty\frac{\mathrm{d}\xi}{(z^2\gamma^2+\xi)\sqrt{(x^2+\xi)(y^2+\xi)(z^2\gamma^2+\xi)}}. \tag{7}$$

Substituting Eq.(5) into Eq.(4), one obtains

$$\begin{aligned}H=&-\left\{\left[(p_\tau+p_T^0-Q(\mu_x x^2+\mu_y y^2+\mu_z\beta_0^2\tau^2))^2\right.\right.\\&\left.\left.-\frac{1}{\beta_0^2\gamma^2}-x'^2-y'^2\right]\right\}^{\frac{1}{2}}+\frac{qG}{2p_0}(x^2+y^2)\\&-(p_\tau+p_T^0)/\beta_0,\end{aligned} \tag{8}$$

where

$$Q=\frac{3qIT_{\rm rf}}{8\pi\varepsilon_0\gamma_0 p_0 cXYZ}. \tag{9}$$

Expanding the Hamiltonian (4) about the equilibrium orbit, we have

$$H=\sum_{n=0}^{\infty}H_n, \tag{10}$$

where

$$\begin{aligned}H_0=&\frac{1}{\beta_0^2\gamma_0^2},\\H_1=&0,\\H_2=&\left(\frac{qG}{2p_0}-Q\mu_x/\beta_0\right)x^2\\&-\left(\frac{qG}{2p_0}+Q\mu_x/\beta_0\right)y^2+\frac{1}{2}(x'^2+y'^2)\\&-Q\mu_z\beta_0\tau^2+\frac{1}{2\beta_0^2\gamma_0^2}p_\tau^2,\\H_3=&-\frac{Q\mu_x}{\beta_0^2\gamma_0^2}x^2p_\tau-\frac{Q\mu_y}{\beta_0^2\gamma_0^2}y^2p_\tau\\&+\frac{1}{2\beta_0}(x'^2p_\tau+y'^2p_\tau)-\frac{Q\mu_z}{\gamma_0^2}\tau^2p_\tau+\frac{1}{2\beta_0^3\gamma_0^2}p_\tau^3,\\&\cdots\cdots\end{aligned} \tag{11}$$

3. First-order approximation

The linear map $\boldsymbol{M}_2$ is expressed as

$$\boldsymbol{M}_2=\exp(-:f_2:), \tag{12}$$

where $:f_2:$ is a Lie operator; when acting on another function, it performs Poisson bracket operation, and

$$f_2=-lH_2\quad(l\text{ is the length of the lens}). \tag{13}$$

Let the subscript '1' express the first-order terms of the map, and $\boldsymbol{M}_2$ act on the components of the canonical variable ζ; one obtains the first-order approximation solutions of the particle trajectories. Expressed in matrix form, they are

$$\begin{bmatrix}x_1\\x_1'\\y_1\\y_1'\\\tau_1\\p_{\tau 1}\end{bmatrix}=\begin{bmatrix}\cos(k_x l)\cos(k_m l) & \frac{1}{k_x}\sin(k_x l)\cos(k_m l) & \cos(k_y l)\sin(k_m l) & \frac{1}{k_y}\sin(k_y l)\sin(k_m l) & 0 & 0\\-k_x\sin(k_x l)\cos(k_m l) & \cos(k_x l)\cos(k_m l) & -k_y\sin(k_y l)\sin(k_m l) & \cos(k_y l)\sin(k_m l) & 0 & 0\\-\cos(k_x l)\sin(k_m l) & -\frac{1}{k_x}\sin(k_x l)\sin(k_m l) & \cos(k_y l)\cos(k_m l) & \frac{1}{k_y}\sin(k_y l)\cos(k_m l) & 0 & 0\\k_x\sin(k_x l)\sin(k_m l) & -\cos(k_x l)\sin(k_m l) & -k_y\sin(k_y l)\cos(k_m l) & \cos(k_y l)\cos(k_m l) & 0 & 0\\0 & 0 & 0 & 0 & \cosh(k_z l) & \frac{\sinh(k_z l)}{k_z\beta_0^2\gamma_0^2}\\0 & 0 & 0 & 0 & k_z\beta_0^2\gamma_0^2\sinh(k_z l) & \cosh(k_z l)\end{bmatrix}\cdot\begin{bmatrix}x\\x'\\y\\y'\\\tau\\p_\tau\end{bmatrix} \tag{14}$$

where

$$\begin{aligned}k_x^2&=\frac{qB}{2p_0}-2Q\mu_x/\beta_0,\\k_y^2&=\frac{qB}{2p_0}-2Q\mu_y/\beta_0,\\k_m^2&=\frac{qB}{2p_0},\\k_z^2&=\frac{2Q\mu_z}{\beta_0\gamma_0^2}.\end{aligned} \tag{15}$$

4. Second-order approximation

The second-order map $\boldsymbol{M}_3$ can be expressed as $\boldsymbol{M}_3=:f_3:$, where

$$f_3=-\int_0^l h_3^{\rm int}(\zeta,z_1)\mathrm{d}z_1=-\int_0^l\boldsymbol{M}_2H_3(\zeta,z_1)\mathrm{d}z_1. \tag{16}$$

Let the map $\boldsymbol{M}_3$ act on the linear solution $\zeta_1=(x_1,x_1',y_1,y_1',\tau_1,p_{\tau 1}',)$. One obtains the second-order solutions ζ_2 (the subscript '2' expresses second order) of the map. The results are listed as the following:

$$
\begin{aligned}
x_2 =& x\tau\beta_0\gamma_0^2 k_z\left[\frac{k_x^2}{k_y}(I_3\sin(k_xl)-I_7\cos(k_xl))\cos(k_ml)+\frac{k_y^2}{k_x}(I_9\sin(k_yl)-I_{15}\cos(k_yl))\sin(k_ml)\right.\\
&+k_x((I_1+I_2)\sin(k_xl)+I_7\cos(k_yl))\cos(k_ml)+k_y(I_{15}\cos(k_yl)-I_9\sin(k_yl))\sin(k_ml)\\
&\left.+\frac{k_xk_m}{k_y}(I_{17}\cos(k_yl)-I_{10}\sin(k_yl))\sin(k_ml)+k_m(I_{11}\sin(k_yl)+I_{16}\cos(k_yl))\sin(k_ml)\right]\\
&+xp_\tau\frac{1}{\beta_0}\left[\frac{k_x^2}{k_y}(I_{12}\sin(k_yl)-I_{18}\cos(k_yl))\sin(k_ml)+\frac{k_y^2}{k_x}(I_6\sin(k_xl)-I_8\cos(k_xl))\cos(k_ml)\right.\\
&+k_x((I_4+I_5)\sin(k_xl)+I_8\cos(k_xl))\cos(k_ml)+k_y(I_{18}\cos(k_yl)-I_{12}\sin(k_yl))\sin(k_ml)\\
&\left.+\frac{k_xk_m}{k_y}(I_{20}\cos(k_yl)-I_{13}\sin(k_yl))\sin(k_ml)+k_m(I_{14}\sin(k_yl)+I_{19}\cos(k_yl))\sin(k_ml)\right]\\
&+x'\tau\beta_0\gamma_0^2 k_z\left[\left(\frac{k_x}{k_y}-\frac{k_y}{k_x}\right)(I_{21}\sin(k_yl)-I_{23}\cos(k_yl))\sin(k_ml)+\frac{k_y^2}{k_x^2}(I_7\sin(k_xl)-I_2\cos(k_xl))\cos(k_ml)\right.\\
&+\frac{k_m}{k_x}(I_{10}\cos(k_yl)+I_{17}\sin(k_yl))\sin(k_ml)+\frac{k_m}{k_y}(I_{16}\sin(k_yl)-I_{23}\cos(k_yl))\sin(k_ml)\\
&\left.-(I_1+I_2)\cos(k_xl)\cos(k_ml)-I_7\sin(k_xl)\cos(k_ml)\right]\\
&+x'p_\tau\frac{1}{\beta_0}\left[\left(\frac{k_x}{k_y}-\frac{k_y}{k_x}\right)(I_{22}\sin(k_yl)-I_{24}\cos(k_yl))\sin(k_ml)+\frac{k_y^2}{k_x^2}(I_8\sin(k_xl)-I_5\cos(k_xl))\cos(k_ml)\right.\\
&+\frac{k_m}{k_x}(I_{13}\cos(k_yl)+I_{20}\sin(k_yl))\sin(k_ml)+\frac{k_m}{k_y}(I_{19}\sin(k_yl)-I_{14}\cos(k_yl))\sin(k_ml)\\
&\left.-(I_4+I_6)\cos(k_xl)-I_8\sin(k_xl)\cos(k_ml)\right]\\
&+y\tau\beta_0\gamma_0^2 k_z\left\{\left(\frac{k_x^2}{k_y}-k_y\right)[(I_{25}-I_1-I_{26})\sin(k_yl)-I_{29}\cos(k_yl)]\sin(k_ml)\right.\\
&+\left(\frac{k_y^2}{k_x}-k_x\right)(I_{21}\cos(k_xl)-I_9\sin(k_xl))\cos(k_ml)+\frac{k_yk_m}{k_x}(I_{11}\sin(k_xl)-I_{17}\cos(k_xl))\cos(k_ml)\\
&\left.-k_m(I_{10}\sin(k_xl)+I_{16}\cos(k_xl))\cos(k_ml)\right\}\\
&+yp_\tau\frac{1}{\beta_{0z}}\left\{\left(\frac{k_x^2}{k_y}-k_y\right)[(I_{27}-I_4-I_{28})\sin(k_yl)-I_{30}\cos(k_yl)]\sin(k_ml)\right.\\
&+\left(\frac{k_y^2}{k_x}-k_x\right)(I_{22}\cos(k_xl)-I_{12}\sin(k_xl))\cos(k_ml)+\frac{k_yk_m}{k_x}(I_{14}\sin(k_xl)-I_{20}\cos(k_xl))\cos(k_ml)\\
&\left.-k_m(I_{13}\sin(k_xl)+I_{19}\cos(k_xl))\cos(k_ml)\right\}\\
&+y'\tau\beta_0\gamma_0^2 k_z\left[\left(\frac{k_x}{k_y}-\frac{k_y}{k_x}\right)(I_{15}\sin(k_xl)-I_{23}\cos(k_xl))\cos(k_ml)+\frac{k_x^2}{k_y^2}(I_{29}\sin(k_yl)-I_{26}\cos(k_yl))\sin(k_ml)\right.\\
&+\frac{k_m}{k_x}(I_{10}\cos(k_xl)-I_{16}\sin(k_xl))\cos(k_ml)-\frac{k_m}{k_y}(I_{11}\cos(k_xl)+I_{17}\sin(k_xl))\cos(k_ml)\\
&\left.-((I_1+I_{25})\cos(k_yl)+I_{29}\sin(k_yl))\sin(k_ml)\right]\\
&+y'p_\tau\frac{1}{\beta_0}\left[\left(\frac{k_x}{k_y}-\frac{k_y}{k_x}\right)(I_{18}\sin(k_xl)-I_{24}\cos(k_xl))\cos(k_ml)+\frac{k_x^2}{k_y^2}(I_{30}\sin(k_yl)-I_{28}\cos(k_yl))\sin(k_ml)\right.\\
&+\frac{k_m}{k_x}(I_{13}\cos(k_xl)-I_{19}\sin(k_xl))\cos(k_ml)-\frac{k_m}{k_y}(I_{14}\cos(k_xl)+I_{20}\sin(k_xl))\cos(k_ml)\\
&\left.-((I_4+I_{27})\cos(k_yl)+I_{30}\sin(k_yl))\sin(k_ml)\right] \qquad (17)
\end{aligned}
$$

In Eq.(17), I_1-I_{30} are some integrations, which can be evaluated numerically, as listed below:

$$I_1 = -\int_0^l \cos^2(k_m l)\sinh(k_z l)\mathrm{d}z, \qquad I_2 = -\int_0^l \sin^2(k_x l)\sin^2(k_m l)\sinh(k_z l)\mathrm{d}z,$$

$$I_3 = -\int_0^l \cos^2(k_x l)\sin^2(k_m l)\sinh(k_z l)\mathrm{d}z, \qquad I_4 = -\int_0^l \cos^2(k_m l)\cos(k_z l)\mathrm{d}z,$$

$$I_5 = -\int_0^l \sin^2(k_x l)\sin^2(k_m l)\cosh(k_z l)\mathrm{d}z, \qquad I_6 = -\int_0^l \cos^2(k_x l)\sin^2(k_m l)\cosh(k_z l)\mathrm{d}z,$$

$$I_7 = -\int_0^l \sin(k_x l)\cos(k_x l)\sin^2(k_m l)\sinh(k_z l)\mathrm{d}z, \qquad I_8 = -\int_0^l \sin(k_x l)\cos(k_x l)\sin^2(k_m l)\cosh(k_z l)\mathrm{d}z,$$

$$I_9 = -\int_0^l \cos(k_x l)\cos(k_y l)\sin(k_m l)\cos(k_m l)\sinh(k_z l)\mathrm{d}z, \qquad I_{10} = -\int_0^l \sin(k_x l)\cos(k_y l)\sinh(k_z l)\mathrm{d}z,$$

$$I_{11} = -\int_0^l \cos(k_x l)\sin(k_y l)\sinh(k_z l)\mathrm{d}z,$$

$$I_{12} = -\int_0^l \cos(k_x l)\cos(k_y l)\sin(k_m l)\cos(k_m l)\cosh(k_z l)\mathrm{d}z,$$

$$I_{13} = -\int_0^l \sin(k_x l)\cos(k_y l)\cosh(k_z l)\mathrm{d}z, \qquad I_{14} = -\int_0^l \cos(k_x l)\sin(k_y l)\cosh(k_z l)\mathrm{d}z,$$

$$I_{15} = -\int_0^l \cos(k_x l)\sin(k_y l)\sin(k_m l)\cos(k_m l)\sinh(k_z l)\mathrm{d}z, \qquad I_{16} = -\int_0^l \cos(k_x l)\cos(k_y l)\sinh(k_z l)\mathrm{d}z,$$

$$I_{17} = -\int_0^l \sin(k_x l)\sin(k_y l)\sinh(k_z l)\mathrm{d}z,$$

$$I_{18} = -\int_0^l \cos(k_x l)\sin(k_y l)\sin(k_m l)\cos(k_m l)\cosh(k_z l)\mathrm{d}z,$$

$$I_{19} = -\int_0^l \cos(k_x l)\cos(k_y l)\cosh(k_z l)\mathrm{d}z, \qquad I_{20} = -\int_0^l \sin(k_x l)\sin(k_y l)\cosh(k_z l)\mathrm{d}z,$$

$$I_{21} = -\int_0^l \sin(k_x l)\cos(k_y l)\sin(k_m l)\cos(k_m l)\sinh(k_z l)\mathrm{d}z,$$

$$I_{22} = -\int_0^l \sin(k_x l)\cos(k_y l)\sin(k_m l)\cos(k_m l)\cosh(k_z l)\mathrm{d}z,$$

$$I_{23} = -\int_0^l \sin(k_x l)\sin(k_y l)\sin(k_m l)\cos(k_m l)\sinh(k_z l)\mathrm{d}z,$$

$$I_{24} = -\int_0^l \sin(k_x l)\sin(k_y l)\sin(k_m l)\cos(k_m l)\cosh(k_z l)\mathrm{d}z,$$

$$I_{25} = -\int_0^l \cos^2(k_y l)\sin^2(k_m l)\sinh(k_z l)\mathrm{d}z, \qquad I_{26} = -\int_0^l \sin^2(k_y l)\sin^2(k_m l)\sinh(k_z l)\mathrm{d}z,$$

$$I_{27} = -\int_0^l \cos^2(k_y l)\sin^2(k_m l)\cosh(k_z l)\mathrm{d}z, \qquad I_{28} = -\int_0^l \sin^2(k_y l)\sin^2(k_m l)\cosh(k_z l)\mathrm{d}z,$$

$$I_{29} = -\int_0^l \sin(k_y l)\cos(k_y l)\sin^2(k_m l)\sinh(k_z l)\mathrm{d}z, \qquad I_{30} = -\int_0^l \sin(k_y l)\cos(k_y l)\sin^2(k_m l)\cosh(k_z l)\mathrm{d}z. \tag{18}$$

Due to the page limit, the second terms of x', y, y', τ and p_τ are not listed here.

5. Discussions

It is a very complicated procedure to calculate the nonlinear transport of intense pulsed beams. Because the electrical potential of the beams depends on the beam dimensions, and the beam dimensions are re-

lated to the electric potential also, we can only solve the problem by iterations. Usually, we should provide the initial beam dimensions first, and the first step is: calculate the electric potential; next step: calculate the particle trajectories; then go back to the first step .... After several iterations we can obtain accurate solutions. The analytical results will be incorporated into a computer program LEADS.[3]

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