Vol 13 No 5, May 2004
1009-1963/2004/13(05)/0618-07

Chinese Physics

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A new type of global bifurcation in Hénon map*

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(Received 27 August 2003; revised manuscript received 20 October 2003)

A simple branch of solution on a bifurcation diagram, which begins at static bifurcation and ends at boundary crisis (or interior crisis in a periodic window), is generally a period-doubling cascade. A domain of solution in parameter space, enclosed by curves of static bifurcation and that of boundary crisis (or the interior of a periodic window), is the trace of branches of solution. A P-n branch of solution refers to the one starting from a period-n ($n \geq 1$) solution, and the corresponding domain in parameter space is named the P-n domain of solution. Because of the co-existence of attractors, there may be several branches within one interval on a bifurcation diagram, and different domains of solution may overlap each other in some areas of the parameter space. A complex phenomenon, concerned both with the co-existence of attractors and the crises of chaotic attractors, was observed in the course of constructing domains of steady state solutions of the Hénon map in parameter space by numerical methods. A narrow domain of period-m solutions firstly co-exists with (lies on) a big period-n ($m \geq 3n$) domain. Then it enters the chaotic area of the big domain and becomes period-m windows. The co-existence of attractors disappears and is called the landing phenomenon. There is an interaction between the two domains in the course of landing: the chaotic area in the big domain is enlarged, and there is a crisis step near the landing area.

Keywords: global bifurcation, crisis, Hénon map
PACC: 0545

1. Introduction

It is a common phenomenon that multiple attractors co-exist at one point in the parameter space of nonlinear dynamic systems. The number of attractors (or steady state solutions) changes suddenly at static bifurcation points when a parameter of the system is varied. A new branch of steady-state solution may emerge or an old one may disappear in a bifurcation diagram. Several branches may occupy the same interval of the system parameter, i.e. they co-exist within the same interval. The numerical global bifurcation analysis (NGBA) method,[1] which combines the local bifurcation analysis with the global one, is a synthetic numerical method. It can obtain both the periodic and chaotic solutions, trace both lower and higher period branches of solution, as well as main and subordinate ones, and show the coexisting phenomena of attractors on its bifurcation diagrams. The method is good for bifurcation and chaos analysis of the lower dimensional nonlinear dynamic systems.

There are crises of chaotic attractors in nonlinear systems universally:[2] the number or size of chaotic attractors changes suddenly with the variation of a parameter of the system. According to Grebogi *et al*,[3,4] there are three types of crises: the boundary crisis, the interior crisis and the merging crisis, each of which is accompanied by a characteristic temporal behaviour of orbit after the crisis. In all cases a time scale τ can be defined which quantifies the observed post-crisis behaviour. Hong and Xu[5−9] have examined the crises in Duffing and Mathieu systems by means of the generalized cell mapping digraph (GCMD) method.

A complex and synthetic phenomenon was observed in the course of constructing domains of steady-state solution of Hénon map in the parameter space by using the numerical global bifurcation analysis method (NGBA). The phenomenon, concerned with both the co-existence of attractors and crises of chaotic attractors, occurs in a relatively wide range of the parameter space.

*Project supported by the National Natural Science Foundation of China (Grant No 50275024) and the Research Project of Science of the Education Department of Anhui Province (Grant No 2003kj013zd).

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2. Domains of solution of Hénon map in parameter space

The Hénon map

$$\begin{cases} x_{n+1} = 1 + by_n - ax_n^2 \\ y_{n+1} = x_n \end{cases} \tag{1}$$

has a two-dimensional parameter space, i.e. the parameter plane (b, a). Taking b (or a) as the control parameter, as is shown in Fig.1, a series of bifurcation diagrams can be drawn at different values of a (or b). The latter is called 'fixed parameter'. It can be seen from Figs.1(a) and (b) that P-3 and P-6 branches co-exist with P-1 branch within some intervals of the control parameter. The evolving traces of branch on the parameter plane, i.e. the domain of solutions as shown in Fig.2, can be plotted by collecting the beginning and ending points of the branches on bifurcation diagrams.

The area $ABCFEDGA$ in Fig.2 is the P-1 domain Ω_0 collected from P-1 branches. AB is the curve of static bifurcation. The points on it comply with the equation of $a = -(1-b)^2/4$. The map has no fixed point below this curve. a_{I_1} is the curve of period-doubling bifurcation from I-2 to I-1 chaotic solutions, and $E_{68}FEE_{568}$ is the curve of boundary crises of I-1 chaotic solutions. They outline the chaotic area of I-1 solutions in Ω_0. BG (i.e. curve a_1, $a = 3(1-b)^2/4$) is the curve of period-doubling bifurcation from P-1 to P-2 periodic solutions.

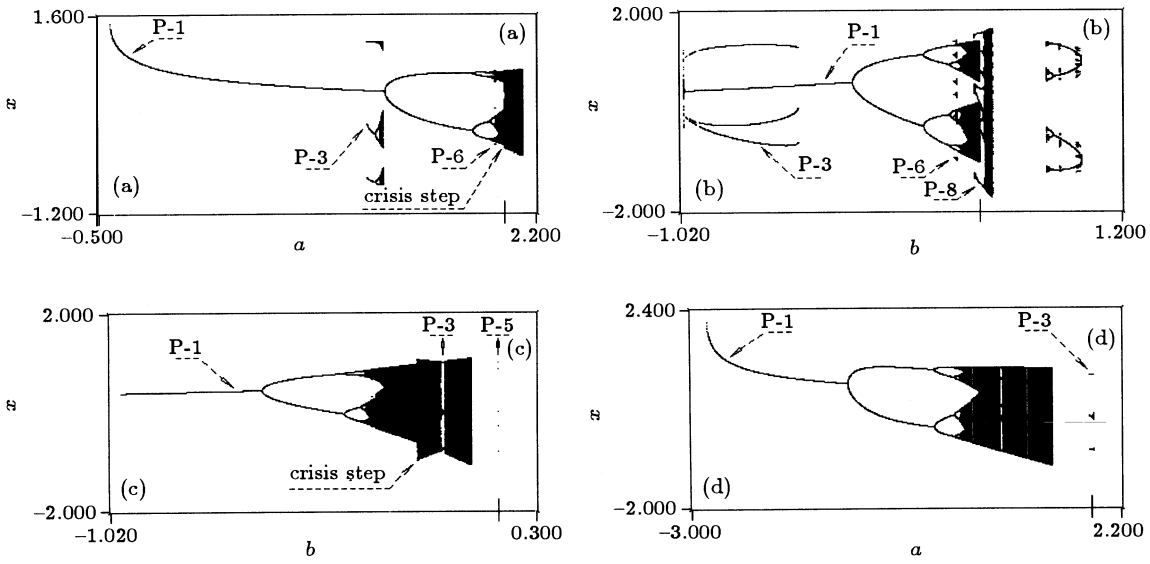


Fig.1. The bifurcation diagrams (a) b=0.3, (b) a=1.0, (c) a=1.8, (d) b=0.1.

The horn-like area L in Fig.2 is the P-3 domain of solution. The area near the points E and F in Fig.2 is shown enlarged in Fig.3. Its lower edge is the curve of static bifurcation. L is narrower than Ω_0 and can be divided into three stages in terms of the relationship between the two domains of solutions.

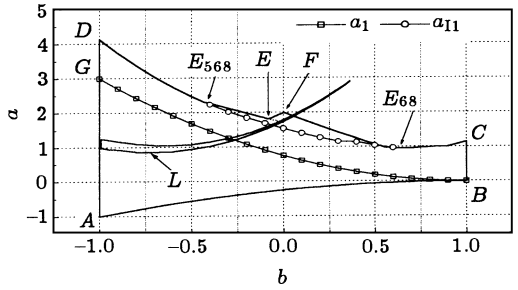


Fig.2. The P-1 domain Ω_0 and P-3 domain L.

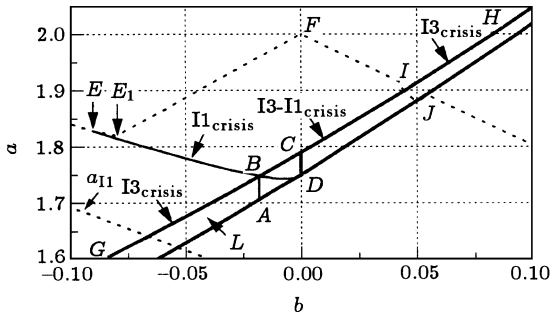


Fig.3. Zoomed area near points E and F in Fig.2.

The *co-existing stage* is the area below curve BD in Fig.3. In this stage, L co-exists with Ω_0 and its upper edge is the curve of boundary crises of I-3 chaotic solutions. P-3 branches of solution fly above or below the P-1 branch on the bifurcation diagrams as shown

in Figs.1(a) and (b). In the interval of existence of the P-3 branch, the map has at least two attractors at each value of the parameter.

The *window stage* is the area between curves BD and IJ. The P-3 branch becomes a period-3 window in the I-1 chaotic area of domain Ω_0, as shown in Fig.1 c). The upper edge (BCI) of L is the curve of interior crises from I-3 to I-1 chaotic attractors. The co-existing phenomenon of the attractors disappears once the curve BD is stepped over from the co-existing stage to the window stage.

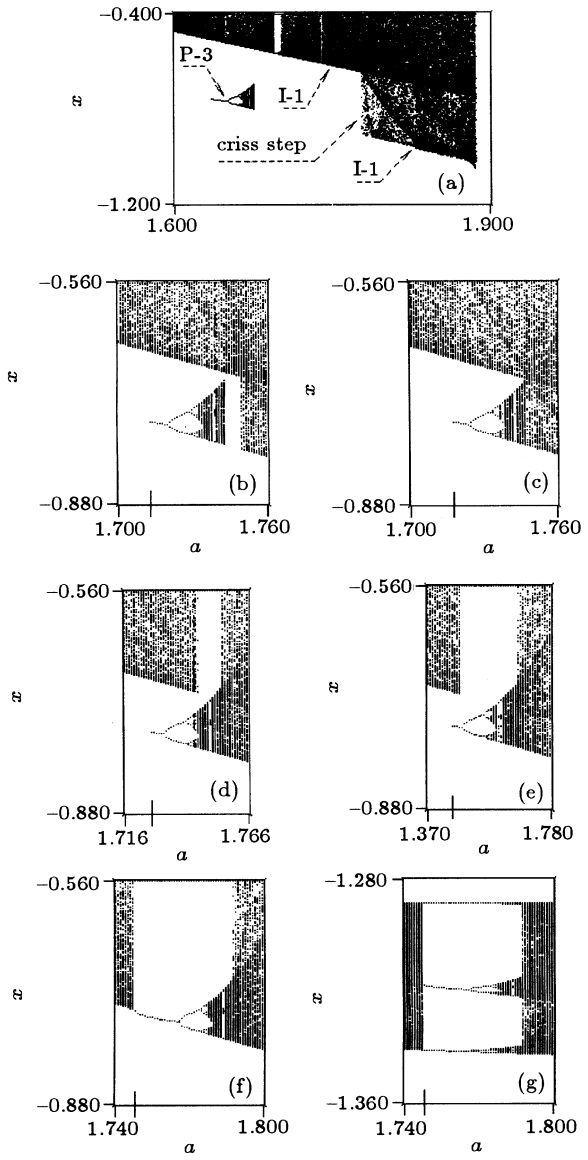


Fig.4. The landing phenomenon of the P-3 domain (a is the control parameter). (a) a=–0.050; (b) b=–0.020; (c) b=–0.0182; (d) b=–0.015; (e) b=–0.010; (f) b=–0.001; (g) b=–0.001.

The area of L above the curve IJ is the *cantilever stage*. L is separated from Ω_0 in this stage. The upper edge IH is the curve of boundary crises of I-3 chaotic attractors (seen in Fig.1(d)).

Now, a question arises: how does the domain L step over the curve BD from the co-existing stage to the window stage?

3. The evolution from the co-existing stage to the window stage, a 'landing' course

It can be seen from Figs.2 and 3 that the curve of boundary crises of I-1 attractors changes its direction at point E_1 when L approaches the chaotic area of I-1 solutions in Ω_0 (in other words, when the system approaches the one-dimensional quadric map, i.e. b=0 in Hénon map). Meanwhile, it can be seen from Figs.1(c), 4(a) and 4(b) that the size of the chaotic attractors of I-1 solutions expands suddenly as the control parameter is varied, and a step is formed on the bifurcation diagrams. This is an interior crisis according to Grebogi *et al*,[2–4] but a kind of crisis from a small I-1 chaotic attractor to a big one. We call it a 'step-like crisis' temporarily and the step is called the 'crisis step'. Curve EB in Fig.3 is a collection of critical values of parameters at such crises on bifurcation diagrams.

It can be seen from the bifurcation diagrams on which a is taken as the control parameter and b the fixed parameter (see Figs.4(a) and (b)) that the crisis step of I-1 chaotic attractors is in front of the end of P-3 branch where the boundary crises of I-3 chaotic attractors occur. The end of P-3 branch approaches the crisis step of I-1 attractors on the P-1 branch when the fixed parameter $b(< -0.0182)$ is increased. They collide with each other at $b \approx -0.0182$. The boundary crisis of I-3 chaotic attractors and the step-like crisis of I-1 chaotic attractors disappear at the same time, shown in Fig.4(c).

The P-1 branch breaks into two parts when b is larger than -0.0182 (see Figs.4(d) and (e)). A gap spanning the area BCD (Fig.3) is formed. The boundary crises of I-1 chaotic attractors occur at the points on curve BD. While the P-3 branch attaches to the chaotic solutions on the right-hand side of the gap (side of curve BC in Fig.3), the interior crises from I-3 to I-1 chaotic attractors occur along curve BC. The I-1 chaotic solutions of the P-1 branch co-exist with the steady state solutions of the P-3 branch in area ABD in Fig.3. But there are only steady state solutions of the P-3 branch in area BCD.

With the increase of fixed parameter b, the gap of

the I-1 solution widens and the head of the P-3 branch approaches the I-1 solutions on the left-hand side of the gap gradually (see Figs.4(d) and (e)). When b is -0.001 (near to line DC), the width of the gap equals the span of the P-3 branch exactly, the head of the P-3 branch collides with the solutions on the left-hand side of the gap and the boundary crises of I-1 solutions along curve BD disappear as shown in Figs.4(f) and (g).

Up to now, the P-3 branch has altered its state from co-existing with the P-1 branch to existing alone in the form of a periodic window in the chaotic area of I-1 solutions on the P-1 branch. The P-3 domain L changes its role from the co-existing stage to the window stage and merges into the P-1 domain Ω_0. They no longer overlap and the co-existing phenomenon of attractors disappears.

The procedure of transition of L domain above can be thought of as 'the landing course' of the P-3 branch, i.e. the 'flying' P-3 branch 'lands at' the chaotic area of I-1 solutions on P-1 branch and changes into P-3 window. We call area $ABCD$ the 'landing area' and point B the 'landing point' in Fig.3. The 'landing curve' is the curve BD that is the curve of boundary crises of I-1 solutions and the great divide between the co-existing stage and the window stage.

The kind of crisis changes dramatically near the landing point B. The step-like crises along curve EB change into boundary crises on curve BD. The boundary crises of I-3 chaotic attractors on the end of the P-3 branch change into interior crises from I-3 to I-1 on curve BCI and are interior crises in the general sense.[2,4]

The landing course is reversible. If the bifurcation diagrams are examined in reverse sequence, we can see the 'inverse landing' phenomenon (i.e. the 'taking off' phenomenon). The flying branch of solution changes into a periodic window in the course of landing and the co-existing phenomena of attractors disappear. But in the inverse course of landing, the periodic window changes into a flying branch and the co-existing phenomena of attractors come into being.

4. The geometric construction in the state-parameter space near the landing area of Ω_0

With the increase of system parameter values, the general trend of alteration of I-1 chaotic attractors is expanding, as shown in Fig.1. Let these bifurcation diagrams stand on the parameter plane as shown in Fig.2. Then we get a three-dimensional bifurcation diagram. The P-1 domain of solutions is focused on and marked with Ω in the space (b, a, $-x$). It has an upper surface and a lower one. The thickness is different in the chaotic domain of I-1 solutions. It is thin at the curve a_{I1}, but thick at the curve $E_{68}FEE_{568}$. If one were to stand on the upper surface at the centre of curve a_{I1} and looks at the edge of Ω above curve $E_{68}FEE_{568}$, there would be a big slope in front. The chaotic attractors expand gradually from the points E_{568} to E, F to E_{68} on the big slope.

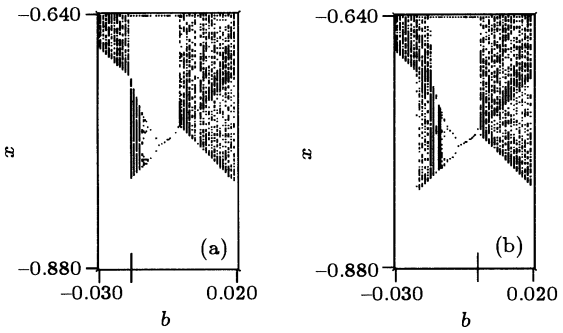


Fig.5. The landing phenomenon of the P-3 domain (b is the control parameter. (a) a=1.748; (b) a=1.751.

But the slope is not smooth. Ω has a special structure in the area near $b = 0$. There is a small slope above area $BCFE$ (in Fig.3) on the big slope. The two slopes meet at the curves BC and CF and have the same side of the curve EF. The height goes down slightly from points E to F. But the sides CF and BC go up gradually from the points C to F, B to C with the uptrend of the big slope.

One edge of the small slope is the curve EB, the curve of step-like crises of I-1 solutions. A crisis step lies transversely on the upper surface of the big slope along the curve EB. One must jump to the step to go through the curve EB from the big slope to the small one. So bifurcation diagrams like Figs.1(c), 4(a), 4(b) and 5(b) are produced. The small slope with crisis step is called 'small crisis slope'. The height of the crisis step gradually decreases from point E to B. At the landing point B, it is just as high as the height of the end of the P-3 branch (see Fig. 4(c)). The fall from the small slope to the big one is equalized, and the step-like crises of I-1 attractors and the boundary crisis of I-3 attractors at the end of the P-3 branch disappear when the P-3 branch collides with the crises step.

5. The interaction between the domains, the macro-cause of step-like crises

How does the special conformation of Ω near $b = 0$ form? It is obvious that there are some interactions between domain L and Ω_0.

It can be seen from Fig.3 that the end of the P-3 branch goes ahead along the curve $GBCH$ before landing. But the situation is different after the landing point B has been reached. In Fig.5, with the increase of fixed parameter $a(> 1.748)$, it seems that the end of the P-3 branch does not go along the curve BCH, but expands along curve EB with inertia and co-exists with P-1 branch. As a matter of fact, there are no I-3 solutions co-existing with I-1 solutions in the area between the curves EB and BCH. The steady state solutions are still I-1 solutions. But the evolution of I-1 solutions in this area is of the evolving trend of I-3 solutions on the P-3 branch because of the effect of the P-3 domain. This trend is opposite to the one of I-1 solutions on the P-1 branch before the crisis step. So a small crisis slope is formed that has a downward trend just the same as for the trend of evolution of I-3 solutions on the tail of P-3 branch.

So the 'inertia effect' of the window stage of L results in the emergence of the small slope and the crisis step on domain Ω, bringing on the special structure in the area near $b = 0$.

On the other hand, the cause of the crisis step can be illustrated by the 'repulsion effect' between Ω_0 and the co-existing stage of L. The repulsion effect mainly exists between the I-3 and I-1 chaotic solutions. It can be seen from Figs.2 and 3 that, without the effect of L, the curve of boundary crises on the P-1 domain Ω_0 would follow the trend within the range of $-1 < b < -0.09$ and stretch from point E to B. When the co-existing stage of L approaches the area of I-1 solutions of Ω_0, the strong repulsion effect makes the area of I-1 solutions on Ω_0 produce the crisis step along curve EB, changes the direction of boundary crises curve at point E_1, enlarges the chaotic area of I-1 solution, and creates the small crisis slope (note that the edge E_1F is almost parallel to the $BCIH$ side of P-3 domain). So the emergence of the peak at point F $(a = 2, b = 0)$ on the boundary crises curve has to do with the effect of the P-3 domain.

In any event, it is true that there is interaction between L and Ω_0. This kind of interaction is called 'landing effect'. The step of the small crisis slope indicates the existence of the effect obviously. So, it is the landing effect that causes the step-like crises of I-1 chaotic solutions.

The landing effect on Ω_0 is limited. It only changes the topological structure of I-1 attractors but the number and the character of attractors remain the same at the step. The range of the effect is locally confined to the area $BCFE$ covered by the small slope in Fig.3. If the unstable steady state solutions are calculated,[10−13] the landing effect may be interpreted satisfactorily. So the illustration presented above is only a 'macro' one.

6. The landing phenomenon in other domains of solution

The P-5 domain JK, P-6 domain MN and P-8 domain RS collected from the corresponding branches on the bifurcation diagrams are shown in Fig.6. There are landing and inverse landing phenomena in the course of their evolution. The P-6 and P-8 domains land in the I-2 chaotic area of the P-1 domain Ω_0, not the I-1 chaotic area (see Figs.7 and 8). The thick solid lines indicate the landing areas. The landing points are marked as B_5, B_6 and B_8 respectively. The step-like crises begin near point E_{568} on the curve of boundary crises of Ω_0. Here we consider that they meet at this point. So the curves of step-like crises are $E_{568}B_5$, $E_{568}B_6$ and $E_{568}B_8$ respectively. The flying branches before landing can be seen in Figs.1(a) and 9. The crises steps of I-1 or I-2 chaotic attractors are in front of them.

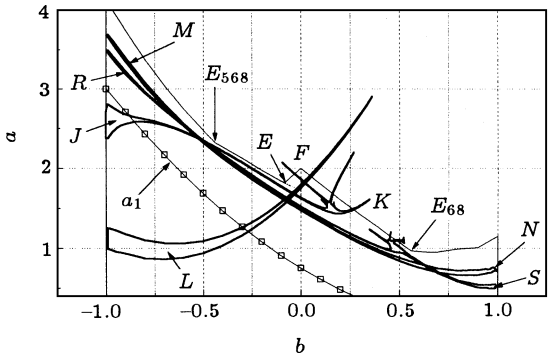


Fig.6. The P-3, P-5, P-6 and P-8 domains L: P-3; JK: P-5; MN: P-6; RS: P-8).

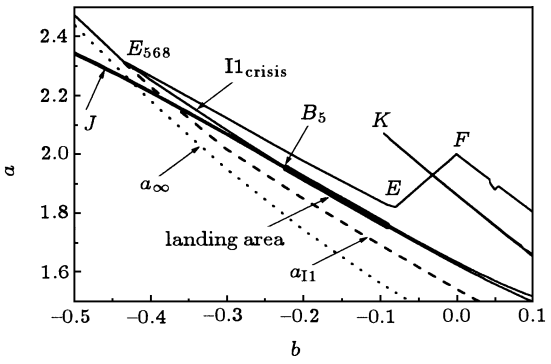


Fig.7. The landing course of P-5 domain J.

The evolution of P-5 domain JK undergoes the co-existing stage, the window stage and the cantilever stage, similar to the P-3 domain L. Landing occurs between the co-existing stage and the window stage. Crises curve $E_{568}B_5$ and landing area are above curve a_{I1}. So the P-5 domain JK lands in the I-1 chaotic area of the P-1 domain Ω_0 and become P-5 windows in the I-1 chaotic area. The landing effect forces the curve of boundary crises of Ω_0 to change its direction at point E_{568} and the I-1 chaotic area is enlarged. As a matter of fact, the I-1 chaotic area begins at just about this point. The P-5 domain J is weaker than the P-3 domain L and the direction of its landing effect is basically consistent with the original trend of the boundary crises curve of Ω_0. So the curve changes its trend at point E_{568} slightly and there is no peak point like point F arising on it.

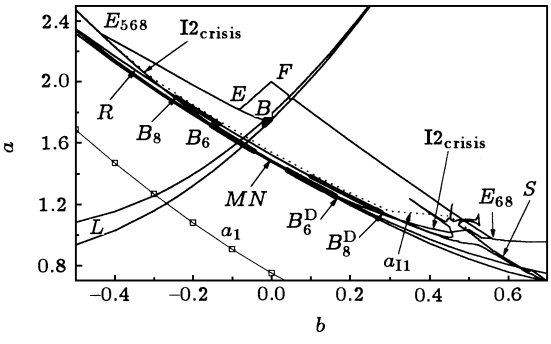


Fig.8. The landing course of P-6 and P-8 domains (L: P-3; MN: P-6; RS: P-8).

The course of evolution of P-6 domain MN and P-8 domain RS is different from that of the P-3 and P-5 domains. They firstly land and become periodic windows in the I-2 chaotic area of P-1 domain Ω_0 in the left half of the parameter plane ($b < 0$). Then they undergo an inverse landing course and co-exist with the P-1 domain again in the right half of the parameter plane ($b > 0$). The step-like crisis curves $E_{568}B_8$ and $E_{568}B_6$ overlap each other and separate near the landing point B_8. At the beginning ($-0.44 \le b \le -0.4$) near point E_{568}, the crisis curve is the flip curve (a_{I1}) of I-2 to I-1 chaotic motion at the same time. These separate gradually when $b > -0.4$. The crises curves and the landing areas are under curve a_{I1}. So the P-6 and P-8 domains land at the I-2 chaotic area of P-1 domain Ω_0 and become P-6 and P-8 windows in the I-2 chaotic area. The crises caused by the landing effect are interior crises from I-2 to I-2 chaotic solutions.

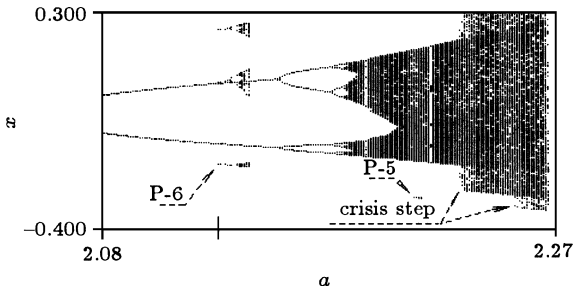


Fig.9. The bifurcation diagram at $b = -0.40$.

The bottom right part of Fig.8 shows the inverse landing course of P-6 and P-8 domains. B_6^D and B_8^D are the inverse landing points of the two domains. Point E_{68} is the jumping-off point of step-like crisis curves $E_{68}B_6^D$ and $E_{68}B_8^D$. If the evolution of P-6 and P-8 domains in area $b > 0$ is observed in the direction from bottom-right to top-left, we find that the two domains undergo evolution courses very similar to the landing courses in area $b < 0$.

We call the small crisis slopes caused by landing of domains MN, RS and J the MN slope, RS slope and J slope, respectively. They overlap each other (see Figs.7–9): the MN slope lies on the big slope of Ω_0, the RS slope lies on the MN slope, and the J slope perches on the RS slope. Because of the superposition near point E_{568} of the step-like crisis curves $E_{568}B_8$ and $E_{568}B_6$, the RS and MN slopes combine together in most parts of them. The three slopes all hunch at the same point E_{568} on the boundary crisis curve of Ω_0.

The height of crisis steps on the edge EB of small crisis slopes MN, RS and J shrinks gradually from point E to point B, similar to the situation of emergence of the L slope caused by the landing effect of the P-3 domain. The height of steps at point B equals the one of chaotic solutions on the ends of P-6, P-8 and P-5 branches. The branches collide with their crisis steps respectively and the falls from the lower slope to the upper one are equalized, and the step-like crises of I-2

or I-1 attractors and the boundary crisis of I-6, I-8 and I-5 attractors at the end of their branches disappear.

7. Conclusions

A complex global bifurcation phenomenon, the so-called 'landing phenomenon', is introduced in this paper with the concept of a solution domain in parameter space and a solution branch in the bifurcation diagram. There is an interaction between domains in the course of their evolution.

(1) A narrow period-m domain firstly co-exists with a wide period-$n(m \geq 3n)$ domain. Then it goes into the chaotic area of the wide domain and becomes period-m windows. The co-existence of attractors disappears. This is the landing phenomenon.

(2) There is an interaction, called the 'landing effect', between the domains in the course of landing. It produces step-like crises of chaotic attractors and a small crisis slope in the wide domain in the state-parameter space. The chaotic area of the wide domain is also enlarged.

(3) The character of crises changes suddenly and dramatically at the landing point. The step-like crises change to boundary crises in the wide domain while the boundary crises in the small domain change to interior ones in the big domain.

(4) The small domain may land in the I-n as well as the I-$2n$ chaotic area of the big domain.

(5) The landing course is invertible. In the course of landing, the co-existing phenomena of attractors disappear. But in the inverse course of landing, the co-existing phenomena of attractors come into being.

(6) All the domains, which traverse line $b = 0$ (i.e. one-dimensional quadric map) on the parameter plane, must undergo a course of landing or an inverse course of landing.

References

[1] Xu P M, Cao Z J and Wen B C 2001 In *Proc. of the Asia-Pacific Vibration Conference 2001*, ed. Wen B C (Changchun: Jilin Science and Technology Press) p77–84

[2] Grebogi C, Ott E and Yorke J A 1982 *Phys. Rev. Lett.* **48** 1507

[3] Grebogi C, Ott E and Yorke J A 1986 *Phys. Rev. Lett.* **57** 1284

[4] Grebogi C, Ott E, Romeiras F and Yorke J A 1987 *Phys. Rev.* A **36**(11) 5365

[5] Hong L and Xu J X 1999 *Phys. Lett.* A **262** 361

[6] Hong L and Xu J X 2001 *Acta Mech. Sin.* **33** 423 (in Chinese)

[7] Hong L and Xu J X 2000 *Acta Phys. Sin.* **49** 1228 (in Chinese)

[8] Hong L and Xu J X 2001 *Acta Phys. Sin.* **50** 612 (in Chinese)

[9] Hong L and Xu J X 2002 *Acta Phys. Sin.* **51** 2694 (in Chinese)

[10] Ling F H 1985 *Phys. Lett.* A **110** 116

[11] Huang Y N 1985 *Chin. Phys. Lett.* **2** 525

[12] Gallas J A C, Grebogi C and Yorke J A 1993 *Phys. Rev. Lett.* **71** 1359

[13] Stewart H B, Ueda Y, Grebogi C and Yorke J A 1995 *Phys. Rev. Lett.* **75** 2478