Vol 13 No 5, May 2004
1009-1963/2004/13(05)/0602-10

Chinese Physics

© 2004 Chin. Phys. Soc.
and IOP Publishing Ltd

New infinite-dimensional hidden symmetries for the stationary axisymmetric Einstein–Maxwell equations with multiple Abelian gauge fields*

Gao Ya-Jun(高亚军)†

Department of Physics, Bohai University, Jinzhou 121003, China

(Received 13 June 2003; revised manuscript received 24 September 2003)

By proposing a so-called extended hyperbolic complex (EHC) function method, an Ernst-like $(p+2)\times(p+2)$ matrix EHC potential is introduced for the stationary axisymmetric (SAS) Einstein–Maxwell theory with p Abelian gauge fields (EM-p theory, for short), then the field equations of the SAS EM-p theory are written as a so-called Hauser–Ernst-like self-dual relation for the EHC matrix potential. Two Hauser–Ernst-type EHC linear systems are established, based on which some new parametrized symmetry transformations for the SAS EM-p theory are explicitly constructed. These hidden symmetries are found to constitute an infinite-dimensional Lie algebra, which is the semidirect product of the Kac–Moody algebra $su(p+1,1)\otimes R(t,t^{-1})$ and Virasoro algebra (without centre charges). All of the SAS EM-p theories for $p=0,1,2,\ldots$ are treated in a unified formulation, $p=0$ and $p=1$ correspond, respectively, to the vacuum gravity and the Einstein–Maxwell cases.

Keywords: stationary axisymmetric Einstein-p-Maxwell theory, extended hyperbolic complex function, infinite-dim hidden symmetries

PACC: 0420, 0290, 0220

1. Introduction

The stationary axisymmetric (SAS) Einstein–Maxwell (EM) field equations, owing to their great important in physics and mathematics, attracted much attention in the past years (e.g. Ref.[1–16]). It has been pointed out [1,2,3,5,11] that the SAS EM theory possesses larger symmetry group than that of the SAS vacuum gravity. However, the complications arising from the existence of Maxwell fields make the study more difficult. For example, when we express the field equations in a σ-model form, the fundamental matrix of the field functions is Hermitian [10,16] rather than real-symmetric as in the vacuum gravity case and its dimension is higher. Thus some study methods need to be extended.

In this paper, we further investigate the SAS EM equations with p Abelian gauge fields, which we call SAS EM-p theory for short. We find that if we use the so-called extended hyperbolic complex (EHC) function method presented in the following, all of the SAS EM-p theories with $p=0,1,2,\ldots$ can be studied in a unified form. (As special cases, $p=0$ corresponds to the vacuum gravity; $p=1$, the EM theory.) We shall show that, from a suitable form of the field equations, an $(p+2)\times(p+2)$ matrix Ernst-like EHC potential can be introduced, then the field equations can be written as an EHC self-dual relation and, further, two Hauser–Ernst (HE)-type EHC linear systems can be established. Based on the solutions of these EHC linear systems, some explicit formulations of parametrized symmetry transformations for the considered theory are constructed. Moreover, the infinite-dimensional Lie algebra structure of these hidden symmetries is calculated out and is found to constitute the semidirect product of the Kac–Moody algebra $su(p+1,1)\otimes R(t,t^{-1})$ and the Virasoro algebra (without centre charges), where $R(t,t^{-1})$ is the algebra of Laurent polynomials in t and t^{-1} with real coefficients. These results show that the EM-p theory under consideration possesses very rich symmetry structures.

In the Preliminaries below, we present the related

*Project supported by the Science Foundation from the Education Department of Liaoning Province of China (Grant No 202142036).
†E-mail: yajgao@jznu.edu.cn

concepts and notation of the so-called EHC function method. In Sec.2, a $(p+2)\times(p+2)$ matrix Ernst-like EHC potential for the SAS EM-p theory is introduced and two HE-type EHC linear systems are established. In Sec.3, some explicit expressions of parametrized transformations for the studied theory are constructed and then it is verified that they are all hidden symmetries, leaving the field equations invariant. The infinite-dimensional Lie algebra structure of these hidden symmetries is calculated in Sec.4. Finally, Sec.5 contains a summary and discussion.

Preliminaries: In a previous paper,[17] we developed a so-called extended double complex function method and used it to study the SAS vacuum gravitation theory. The EHC function method presented here is essentially a restriction of the extended double one in Ref.[17] to the case $J=\varepsilon$. Explicitly, let ε be the extended hyperbolic imaginary unit, i.e. $\varepsilon^2=1$, $\varepsilon\neq\pm1$. When a and b both are ordinary complex numbers, we call

$$c(\varepsilon)=a+\varepsilon b \tag{1}$$

an extended hyperbolic complex (EHC) number and denote $\mathrm{Re}_{\mathrm{EHC}}(c(\varepsilon)):=a$, $\mathrm{Im}_{\mathrm{EHC}}(c(\varepsilon)):=b$. All of the EHC numbers with usual addition and multiplication constitute a commutative ring which we call the EHC ring.

There are two imaginary units in the EHC ring: ε and the ordinary imaginary unit i ($i^2=-1$). Correspondingly, we have two complex conjugations: EHC conjugation '$\star$' and the ordinary complex conjugation '$-$':

$$c(\varepsilon)^\star:=a-\varepsilon b,\qquad \overline{c(\varepsilon)}:=\overline{a}+\varepsilon\overline{b}, \tag{2}$$

where $\overline{a}$ and $\overline{b}$ are ordinary complex conjugations of a and b. The definitions (2) imply that $\varepsilon^\star=-\varepsilon$, $i^\star=i$, $\overline{i}=-i$, $\overline{\varepsilon}=\varepsilon$.

If in (1), a and b are ordinary complex functions of some ordinary complex (or real) variables $z_1,\ldots,z_n$, then

$$c(z_1,\ldots,z_n;\varepsilon)=a(z_1,\ldots,z_n)+\varepsilon b(z_1,\ldots,z_n) \tag{3}$$

is called an EHC function, we say $c(z_1,\ldots,z_n;\varepsilon)$ is continuous, analytical, etc., if $a(z_1,\ldots,z_n)$ and $b(z_1,\ldots,z_n)$ both, as ordinary complex functions, have the same properties. In this paper, we also need EHC (function) matrices. We use the dagger '$\dagger$' and double dagger '$\ddagger$' to stand for, respectively, the ordinary and EHC Hermitian conjugations. That is, for an EHC matrix A, we have

$$A^\dagger:=\overline{A}^{\top},\qquad A^\ddagger:=(A^\dagger)^\star, \tag{4}$$

here '$\top$' denotes the transposition.

2. EHC matrix potential and HE-type EHC linear systems

If there exist p Abelian gauge fields (p Maxwell fields) in space–time, the Einstein-p-Maxwell (EM-p) field equations are

$$\begin{aligned}&R_{\mu\nu}=\delta_{ab}\left(F^a_{\mu\nu}F^{a\ \alpha}_{\ \nu}-\frac{1}{4}g_{\mu\nu}F^a_{\alpha\beta}F^{b\alpha\beta}\right),\\&F^{a\mu\nu}{}_{;\nu}=0,\qquad F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu,\\&\mu,\nu=1,2,3,4,\qquad a,b=1,2,\ldots,p\\&(p\text{ is a nonnegative integer}),\end{aligned} \tag{5}$$

where $R_{\mu\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$ (signature $+---$), and A^a_μ ($a=1,2,\ldots,p$) are multiple $U(1)$ gauge potentials.

Now we consider the stationary axisymmetric (SAS) case, in which the four-dimensional space–time line element can be chosen as [2]

$$\mathrm{d}s^2=f_{AB}\mathrm{d}x^A\mathrm{d}x^B-h_{LN}\mathrm{d}x^L\mathrm{d}x^N,$$

$$(A,B=1,2;\quad L,N=3,4), \tag{6}$$

where f_{AB} and h_{LN} depend only on x^3, x^4. For simplicity, we denote x^3, x^4 by x, y in the following. After reducing to the SAS case and defining $\rho^2:=-\det(f_{AB})$ ($\rho>0$), the field equations (5) admit the introduction of real functions B^a_A and ψ_{AB} such that [2,10,15]

$$\begin{aligned}\mathrm{d}B^a_A=&-\rho^{-1}f_A{}^B\,{}^*\mathrm{d}A^a_B,\\\mathrm{d}\psi_{AB}=&-\rho^{-1}(f_A{}^B\,{}^*\mathrm{d}f_{BC}-2f_A{}^B\delta_{ab}A^a_C\,{}^*\mathrm{d}A^b_B\\&-2f_C{}^B\delta_{ab}A^a_A\,{}^*\mathrm{d}A^b_B),\end{aligned} \tag{7}$$

and the function ρ satisfies

$$\mathrm{d}^*\mathrm{d}\rho=0. \tag{8}$$

Here we have adopted the notation of differential form for convenience and denoted the two-dimensional dual operation '$*$' as

$${}^*\mathrm{d}x=\mathrm{d}y,\qquad {}^*\mathrm{d}y=-\mathrm{d}x. \tag{9}$$

Since the metric coefficients h_{LN} can be determined simply by f_{AB}, A^a_A and ρ,[2,10] then the essentially complete set of field equations can be expressed in

terms of fields f_{11}, ψ_{11}, A^a_1, B^a_1 (which will be denoted by f, ψ, A^a, B^a, respectively) and ρ.[1,2,10] Further, defining the ordinary complex fields Φ^a ($a = 1,2,\ldots,p$) and $\mathcal{E}$ as

$$\Phi^a = A^a - \mathrm{i}B^a, \qquad \mathcal{E} = f - \delta_{ab}\Phi^a\overline{\Phi}^b + \mathrm{i}\psi, \tag{10}$$

then from the discussions given in Refs.[10,16] and noting Eq.(8), the field equations of the SAS EM-p theory can be written as

$$\mathrm{d}(\rho M^{-1}{}^*\mathrm{d}M) = 0, \tag{11}$$

where the $(p+2)\times(p+2)$ ordinary complex matrix function M is defined by

$$M = \frac{\rho}{f}\begin{pmatrix} 1 & \sqrt{2}\,\overline{\Phi}^1 & \sqrt{2}\,\overline{\Phi}^2 & \cdots & \sqrt{2}\,\overline{\Phi}^p & \mathrm{i}(f-\mathcal{E}) \\ \sqrt{2}\Phi^1 & 2\Phi^1\overline{\Phi}^1 - f & 2\Phi^1\overline{\Phi}^2 & \cdots & 2\Phi^1\overline{\Phi}^p & -\mathrm{i}\sqrt{2}\Phi^1\overline{\mathcal{E}} \\ \sqrt{2}\Phi^2 & 2\Phi^2\overline{\Phi}^1 & 2\Phi^2\overline{\Phi}^2 - f & \cdots & 2\Phi^2\overline{\Phi}^p & -\mathrm{i}\sqrt{2}\Phi^2\overline{\mathcal{E}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{2}\Phi^p & 2\Phi^p\overline{\Phi}^1 & 2\Phi^p\overline{\Phi}^2 & \cdots & 2\Phi^p\overline{\Phi}^p - f & -\mathrm{i}\sqrt{2}\Phi^p\overline{\mathcal{E}} \\ \mathrm{i}(\overline{\mathcal{E}} - f) & \mathrm{i}\sqrt{2}\mathcal{E}\overline{\Phi}^1 & \mathrm{i}\sqrt{2}\mathcal{E}\overline{\Phi}^2 & \cdots & \sqrt{2}\mathcal{E}\overline{\Phi}^p & \mathcal{E}\overline{\mathcal{E}} \end{pmatrix}. \tag{12}$$

It can be directly verified that the matrix M in (12) satisfies the conditions

$$M^\dagger = M, \qquad M\gamma M = \rho^2\gamma, \tag{13}$$

where

$$\gamma := \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{i}I_p & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \gamma^\dagger = -\gamma, \quad \gamma^2 = -I, \tag{14}$$

I_p and I are, respectively, the p-dim and $(p+2)$-dim unit matrices. Obviously, Eqs.(11) and (13) are invariant under the global transformations of $SU(p+1,1)$ group:[1)] $M \to U^\dagger M U$, $U \in SU(p+1,1)$. Of course, as will be seen in the following, the symmetry structures of the considered EM-p theory are very much richer than this.

Considering (13), Eq.(11) can be rewritten as

$$\mathrm{d}(\rho^{-1}M\gamma\,{}^*\mathrm{d}M) = 0. \tag{15}$$

This implies that we may introduce a $(p+2)\times(p+2)$ complex matrix twist potential $Q = Q(x,y)$ such that

$$\mathrm{d}Q = -\rho^{-1}M\gamma\,{}^*\mathrm{d}M. \tag{16}$$

Using (13) again and noting (9) and (14), we obtain

$$\mathrm{d}M = -\rho^{-1}M\gamma\,{}^*\mathrm{d}Q. \tag{17}$$

Now introducing a $(p+2)\times(p+2)$ matrix EHC potential,

$$H := M + \varepsilon Q, \tag{18}$$

then Eqs.(16) and (17) can be equivalently written into a single EHC matrix equation,

$$\mathrm{d}H = -\varepsilon\rho^{-1}M\gamma\,{}^*\mathrm{d}H, \tag{19}$$

which is also equivalent to the SAS EM-p field equation (11). When taking $p = 0$ (vacuum gravity), the above matrix M becomes real-symmetric and the EHC potential H is hyperbolic complex, which corresponds to the case $J = \varepsilon$ in Ref.[17]. We call H in (18) the matrix Ernst-like EHC potential and give a unified treatment for all cases of $p = 0,1,2,\ldots$.

From (13) and (16) we have

$$\mathrm{d}(Q - Q^\dagger) = -2\,{}^*\mathrm{d}\rho\gamma. \tag{20}$$

Moreover, (2.4) implies that we can introduce a real field $z = z(x,y)$ such that

$${}^*\mathrm{d}\rho = \mathrm{d}z. \tag{21}$$

Substituting (21) into (20) and choosing appropriate integral constants, we obtain

$$Q - Q^\dagger = -2z\gamma. \tag{22}$$

With the aid of (13), (18) and (22), we can equivalently express Eq.(19) as

$$2(z+\rho^*)\mathrm{d}H = (H + H^\ddagger)(\varepsilon\gamma)\mathrm{d}H, \tag{23}$$

where '‡' denotes the EHC Hermitian conjugate operation defined by (4). Equation (23) is a generalization of the so-called self-duality relation in vacuum gravity.[18,19]

Moreover, from (23) we can obtain the following useful relations:

$$\mathrm{d}H^\ddagger(\varepsilon\gamma)\mathrm{d}H = \mathrm{d}H^\ddagger(\varepsilon\gamma)\,{}^*\mathrm{d}H = 0. \tag{24}$$

1) Similar to [10], here we take the complex metric γ as in (14), and equivalently $SU(p+1,1) = \{U \,|\, U^\dagger\gamma U = \gamma,\ \det U = 1\}$.

Here and in the following, we omit the wedge symbol '$\wedge$' in the exterior products of forms for simplicity.

In order to obtain the linear system associated with the SAS EM-p field equations, motivated by Refs.[18,19], we introduce a (ordinary complex) parameter t and rewrite Eq.(23) as

$$t\mathrm{d}H = A(t)\Gamma(t), \tag{25}$$

where

$$A(t) := I - t(H + H^{\ddagger})(\varepsilon\gamma), \tag{26}$$
$$\Gamma(t) := t\Lambda(t)^{-1}\mathrm{d}H, \tag{27}$$
$$\Lambda(t) := 1 - 2t(z + \rho^*),$$
$$\Lambda(t)^{-1} = \lambda(t)^{-2}[1 - 2t(z - \rho^*)], \tag{28}$$
$$\lambda(t) := [(1 - 2zt)^2 + (2\rho t)^2]^{1/2}. \tag{29}$$

Then from the above definitions and (25) and (24), we can obtain

$$\mathrm{d}\Gamma(t) = \Gamma(t)(\varepsilon\gamma)\Gamma(t), \tag{30}$$

which is just the complete integrability condition of the following EHC linear differential system:

$$\mathrm{d}F(t) = \Gamma(t)(\varepsilon\gamma)F(t), \tag{31}$$

where $F(t) = F(x, y, t; \varepsilon)$ is an EHC $(p + 2) \times (p + 2)$ matrix function of x, y and t. We call (31) an HE-type EHC linear system of the SAS EM-p theory.

The EHC linear system (31) does not define $F(t)$ uniquely, thus we need to suppress some subsidiary conditions consistent with the above equations and the requirement that the EHC matrix function $F(t)$ be holomorphic in a neighbourhood of $t = 0$. To this end, we first note that we can verify the following equations:

$$\mathrm{d}F(0) = \mathrm{d}[\dot{F}(0) - H(\varepsilon\gamma)F(0)] = 0, \tag{32}$$
$$\mathrm{d}[\lambda(t)F(t)^{+}(\varepsilon\gamma)F(t)] = 0, \tag{33}$$
$$\mathrm{d}[F(t)^{\times}(\varepsilon\gamma)A(t)F(t)] = 0, \tag{34}$$

where $\dot{F}(t) := \partial F(t)/\partial t$, $F(t)^{+} := F(\bar{t})^{\dagger}$, $F(t)^{\times} := F(\bar{t})^{\ddagger}$, and $\bar{t}$ is the complex conjugate of t. Also, from these we see that for any EHC matrix function $K(x, y; \varepsilon)$ (independent of t), we have $K(x, y; \varepsilon)^{+} = K(x, y; \varepsilon)^{\dagger}$, $K(x, y; \varepsilon)^{\times} = K(x, y; \varepsilon)^{\ddagger}$.

Equations (32) follow from Eq.(31) and the definitions (27), (28). Equation (33) is deduced from Eq.(31) and the relations

$$H - H^{\dagger} = -2z(\varepsilon\gamma),$$
$$2t\Lambda^{-1}\mathrm{d}z = -\lambda(t)^{-1}\mathrm{d}\lambda(t). \tag{35}$$

Equation (34) is derived from Eqs.(25) and (31) using the facts that $(\varepsilon\gamma)^{\ddagger} = \varepsilon\gamma$ and $[(\varepsilon\gamma)A(t)]^{\times} = (\varepsilon\gamma)A(t)$.

Noticing that the linear system (31) and Eqs.(32)–(34) determine $F(t)$ only up to the right-multiplication by an arbitrary non-degenerate $(p+2)\times(p + 2)$ matrix function of t, we can use this freedom and choose the integral constants of (32)–(34) consistently so that

$$F(0) = I, \quad \dot{F}(0) = H(\varepsilon\gamma), \tag{36}$$
$$\lambda(t)F(t)^{+}(\varepsilon\gamma)F(t) = \varepsilon\gamma, \tag{37}$$
$$F(t)^{\times}(\varepsilon\gamma)A(t)F(t) = \varepsilon\gamma. \tag{38}$$

Moreover, following the spirit of Wu [20] and Hauser,[21] besides (31), we can establish another EHC linear system associated with SAS EM-p field equation (23). Now, for another ordinary complex parameter τ, we define

$$\tilde{A}(\tau) := \tau - (H + H^{\ddagger})(\varepsilon\gamma), \tag{39}$$
$$\tilde{\Gamma}(\tau) := \tilde{\Lambda}(\tau)^{-1}\mathrm{d}H, \tag{40}$$
$$\tilde{\Lambda}(\tau) := \tau - 2(z + \rho^*),$$
$$\tilde{\Lambda}(\tau)^{-1} = \tilde{\lambda}(\tau)^{-2}[\tau - 2(z - \rho^*)], \tag{41}$$
$$\tilde{\lambda}(\tau) := [(\tau - 2z)^2 + (2\rho)^2]^{1/2}. \tag{42}$$

Then the field equation (23) can be rewritten as

$$\mathrm{d}H = \tilde{A}(\tau)\tilde{\Gamma}(\tau), \tag{43}$$

and by derivations similar to the above, we obtain the desired linear system,

$$\mathrm{d}\tilde{F}(\tau) = \tilde{\Gamma}(\tau)(\varepsilon\gamma)\tilde{F}(\tau), \tag{44}$$

where $\tilde{F}(\tau) = \tilde{F}(x, y, \tau; \varepsilon)$ is another EHC $(p + 2) \times (p + 2)$ matrix function of x, y and τ. It is obvious that the EHC equation (23) is also the compatibility condition of Eq.(44). In addition, similar to (33) and (34), from Eqs.(43), (44) and (39)–(42) we have

$$\mathrm{d}[\tilde{\lambda}(\tau)\tilde{F}(\tau)^{+}(\varepsilon\gamma)\tilde{F}(\tau)] = 0, \tag{45}$$
$$\mathrm{d}[\tilde{F}(\tau)^{\times}(\varepsilon\gamma)\tilde{A}(\tau)\tilde{F}(\tau)] = 0. \tag{46}$$

Thus we may require consistently that $\tilde{F}(\tau)$ be analytic around $\tau = 0$ and satisfy

$$\tilde{\lambda}(\tau)\tilde{F}(\tau)^{+}(\varepsilon\gamma)\tilde{F}(\tau) = \varepsilon\gamma, \tag{47}$$
$$\tilde{F}(\tau)^{\times}(\varepsilon\gamma)\tilde{A}(\tau)\tilde{F}(\tau) = \varepsilon\gamma. \tag{48}$$

Later discussion will show that the existence of the additional EHC linear system (44) implies that the SAS

EM-p theory under consideration possesses more and richer symmetry structures.

3. Parametrized hidden symmetry transformations

In this section, by virtue of solutions $F(t)$ and $\tilde{F}(\tau)$ of the EHC linear systems (31) and (44), we explicitly construct some parametrized transformations and then verify that these transformations are all hidden symmetries for the considered EM-p theory. The Lie algebra structures of these symmetries will be calculated in the next section.

First, motivated by the discussions of Wu and Ge for the pure gravity case,[19] now we consider the following parametrized infinitesimal transformation of the EHC Ernst-like matrix potential H:

$$\delta H = \frac{1}{l}[F(l)TF(l)^{-1} - T](\varepsilon\gamma), \tag{49}$$

where l is a (finite) real parameter, $F(l)$ is a solution of (31) and (36)–(38) with t being replaced by l, $T = T_a\alpha^a \in su(p+1,1)$ (the Lie algebra of $SU(p+1,1)$). T_a are generators of $su(p+1,1)$, α^a are infinitesimal real constants. Thus we have the relations

$$T^\dagger\gamma + \gamma T = 0, \qquad \mathrm{tr}T = 0, \tag{50}$$

and from the definitions of the operations '+', '×' below (33) and (34), we also have $T^\times = T^+ = T^\dagger$.

Now we prove that (49) is a hidden symmetry transformation of the SAS EM-p field equation (23). First, from (49), (50) and (37) we have

$$\begin{aligned}\delta H - \delta H^\dagger =& \frac{1}{l}[F(l)TF(l)^{-1} - T](\varepsilon\gamma)\\ &+ \frac{1}{l}(\varepsilon\gamma)[(F(l)^{-1})^+T^\dagger F(l)^+ - T^\dagger]\\ =& \frac{1}{l}F(l)[TF(l)^{-1}(\varepsilon\gamma)(F(l)^+)^{-1}\\ &+ F(l)^{-1}(\varepsilon\gamma)(F(l)^{-1})^+T^\dagger]F(l)^+\\ =& \frac{\lambda(t)}{l}F(l)[T(\varepsilon\gamma) + (\varepsilon\gamma)T^\dagger]F(l)^+ = 0.\end{aligned} \tag{51}$$

Hence δH is ordinary Hermitian, and $\delta M = \mathrm{Re}_{\mathrm{EHC}}(\delta H)$ is Hermitian as it should be. Moreover, from (35) and (51) it follows that

$$(H+\delta H)-(H+\delta H)^\dagger = -2z(\varepsilon\gamma), \quad \text{i.e. } \delta z = 0. \tag{52}$$

The transformation of the function ρ can be calculated as follows. Equations (13) and (14) give $\rho^2 = -\frac{1}{p+2}\mathrm{tr}(M\gamma M\gamma)$, while from relation $H + H^\ddagger = 2M - 2z(\varepsilon\gamma)$ and (52) we have

$$\begin{aligned}&\delta[\mathrm{tr}(M\gamma M\gamma)] = 2\mathrm{tr}(\delta M\gamma M\gamma)\\ &= -(1/2)\mathrm{tr}[(\delta H + \delta H^\ddagger)(\varepsilon\gamma)(H + H^\ddagger - 2z(\varepsilon\gamma))(\varepsilon\gamma)].\end{aligned} \tag{53}$$

On the other hand, Eqs.(49) and (50) lead to

$$\begin{aligned}\delta H + \delta H^\ddagger =& \frac{1}{l}[F(l)TF(l)^{-1}(\varepsilon\gamma)\\ &+ (\varepsilon\gamma)F(l)^{\times-1}T^\dagger F(l)^\times].\end{aligned} \tag{54}$$

Substituting (54) into (53) and using the definition (26) and relation (38), we obtain

$$\begin{aligned}&\delta[\mathrm{tr}(M\gamma M\gamma)]\\ &= -\frac{1}{2l^2}\mathrm{tr}\{[F(l)TF(l)^{-1}(\varepsilon\gamma)\\ &\quad + (\varepsilon\gamma)F(l)^{\times-1}T^\dagger F(l)^\times](\varepsilon\gamma)[1 - 2lz - A(l)]\}\\ &= \frac{1}{2l^2}\mathrm{tr}\{(\varepsilon\gamma)F(l)^{\times-1}[(\varepsilon\gamma)T + T^\dagger(\varepsilon\gamma)]F(l)^{-1}\} = 0\end{aligned}$$

by (50). Thus, in addition to (52), we also have $\delta\rho = 0$ under the transformation (49).

Now we investigate the equation satisfied by δH. From (49) and (31), it follows that

$$\mathrm{d}(\delta H) = \frac{1}{l}[\Gamma(l)(\varepsilon\gamma),\ F(l)TF(l)^{-1}](\varepsilon\gamma). \tag{55}$$

This and (27) and (23) are further followed by

$$\begin{aligned}&2(z+\rho^*)\mathrm{d}(\delta H)\\ &= (H + H^\ddagger)(\varepsilon\gamma)\mathrm{d}(\delta H) - \frac{1}{l}[(H + H^\ddagger)(\varepsilon\gamma),\\ &\quad F(l)TF(l)^{-1}]\Gamma(l).\end{aligned} \tag{56}$$

On the other hand, from Eqs.(54) and (25) we have

$$\begin{aligned}&(\delta H + \delta H^\ddagger)(\varepsilon\gamma)\mathrm{d}H\\ &= \frac{1}{l^2}[F(l)TF(l)^{-1}(\varepsilon\gamma)\\ &\quad + (\varepsilon\gamma)F(l)^{\times-1}T^\dagger F(l)^\times](\varepsilon\gamma)A(l)\Gamma(l).\end{aligned}$$

Then, using (38), (50) and (26), we arrive at

$$\begin{aligned}&(\delta H + \delta H^\ddagger)(\varepsilon\gamma)\mathrm{d}H\\ &= -\frac{1}{l}[(H + H^\ddagger)(\varepsilon\gamma),\ F(l)TF(l)^{-1}]\Gamma(l).\end{aligned}$$

Substituting this into Eq.(56), we finally obtain

$$\begin{aligned}&2(z+\rho^*)\mathrm{d}(\delta H)\\ &= (H + H^\ddagger)(\varepsilon\gamma)\mathrm{d}(\delta H) + (\delta H + \delta H^\ddagger)(\varepsilon\gamma)\mathrm{d}H.\end{aligned} \tag{57}$$

Noticing the invariance of ρ and z under the transformation (49), it is clear that Eq.(57) shows that $H+\delta H$

with δH given by (49) satisfies the same EHC equation (23) as H does, i.e. (49) is indeed a symmetry transformation for the SAS EM-p field equations.

Similarly, by using solution $\tilde{F}(s)$ of the EHC linear system (44) with (47), (48), we can construct another infinitesimal transformation of the EHC potential H as

$$\tilde{\delta}H = -s[\tilde{F}(s)T\tilde{F}(s)^{-1} - T](\varepsilon\gamma), \tag{58}$$

where the parameter s is finite and real, and the infinitesimal operator T is the same as in (49). Similar to the above calculations, it can be verified that $\tilde{\delta}H$ satisfies

$$\tilde{\delta}H^{\dagger} = \tilde{\delta}H, \qquad \tilde{\delta}[\mathrm{tr}(M\gamma M\gamma)] = 0, \tag{59}$$

$$\begin{aligned} &2(z+\rho^*)\mathrm{d}(\tilde{\delta}H) \\ =&(H+H^{\ddagger})(\varepsilon\gamma)\mathrm{d}(\tilde{\delta}H) + (\tilde{\delta}H + \tilde{\delta}H^{\ddagger})(\varepsilon\gamma)\mathrm{d}H. \end{aligned} \tag{60}$$

These imply that

$$(\tilde{\delta}M)^{\dagger} = \tilde{\delta}M, \qquad \tilde{\delta}\rho = \tilde{\delta}z = 0, \tag{61}$$

and (58) is also a symmetry transformation of the EM-p field equation (23).

Not only that, but the symmetry group of the SAS EM-p theory can be further enlarged. Inspired by Hou and Li's discussions of the pure gravity,[22,23] for the present EM-p case, in addition to (49) and (58), we propose two other infinitesimal transformations,

$$\Delta H = -\sigma\dot{F}(l)F(l)^{-1}(\varepsilon\gamma), \tag{62}$$

$$\tilde{\Delta}H = -\epsilon s\left[s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1} + \frac{1}{2}\right](\varepsilon\gamma), \tag{63}$$

where l, s both are finite real parameters, and σ, ϵ are infinitesimal real constants.

From (18), (22) and (37), the transformation (62) gives

$$\begin{aligned} &2\Delta z(\varepsilon\gamma) = -(\Delta H - \Delta H^{\dagger}) \\ =&\sigma[\dot{F}(l)F(l)^{-1}(\varepsilon\gamma) + (\varepsilon\gamma)F(l)^{+-1}\dot{F}(l)^{+}] \\ =&-\sigma\lambda(l)^{-1}\frac{\partial}{\partial l}\lambda(l)(\varepsilon\gamma), \end{aligned}$$

thus we have

$$\Delta z = \frac{\sigma}{\lambda(l)^2}[z(1-2lz) - 2l\rho^2], \tag{64}$$

and this also implies that

$$(\Delta M)^{\dagger} = \Delta M \tag{65}$$

as it should be. Moreover, from (65), (13), (22), (62), (37) and (38), we obtain

$$\begin{aligned} &2(p+2)\rho\Delta\rho = -\Delta\mathrm{tr}(M\gamma M\gamma) = -2\mathrm{tr}(\Delta M\gamma M\gamma) \\ =&-\mathrm{tr}(\Delta M\gamma M\gamma) - \mathrm{tr}(\Delta M\gamma M\gamma)^{\dagger} = -\mathrm{tr}[\Delta(H+H^{\ddagger})\gamma M\gamma] - \mathrm{tr}[\Delta(H+H^{\ddagger})\gamma M\gamma]^{\dagger} \\ =&-\frac{\sigma}{4l}\mathrm{tr}\left[\left(F(l)\frac{\partial}{\partial l}F(l)^{-1}(\varepsilon\gamma) + (\varepsilon\gamma)\frac{\partial}{\partial l}F(l)^{\times -1}\cdot F(l)^{\times}\right)(\varepsilon\gamma)(2lz-1+A(l))\right] \\ &-\frac{\sigma}{4l}\mathrm{tr}\left[\left(F(l)\frac{\partial}{\partial l}F(l)^{-1}(\varepsilon\gamma) + (\varepsilon\gamma)\frac{\partial}{\partial l}F(l)^{\times -1}\cdot F(l)^{\times}\right)(\varepsilon\gamma)(2lz-1+A(l))\right]^{+} \\ =&-\frac{(p+2)\sigma(2lz-1)}{2l\lambda(l)}\frac{\partial}{\partial l}\lambda(l) - \frac{\sigma}{4l}\frac{\partial}{\partial l}[\mathrm{tr}(A(l)+A(l)^{+})] = \frac{2(p+2)\sigma\rho^2}{\lambda(l)^2}. \end{aligned}$$

This gives

$$\Delta\rho = \frac{\sigma}{\lambda(l)^2}\rho. \tag{66}$$

We would like to point out that the result (66) may also be obtained by using Eqs.(21) and (64).

Similarly, from (13), (22), (47) and (48), the transformation (63) gives

$$\tilde{\Delta}z = \frac{\epsilon s}{\tilde{\lambda}(s)^2}[z(2z-s) + 2\rho^2], \quad \tilde{\Delta}\rho = -\frac{\epsilon s^2}{\tilde{\lambda}(s)^2}\rho. \tag{67}$$

Now we prove that (62) and (63) are also symmetry transformations of the SAS EM-p field equations. For (62) we need to prove the following equation:

$$\begin{aligned} &2(\Delta z + \Delta\rho^*)\mathrm{d}H + 2(z+\rho^*)\mathrm{d}\Delta H \\ =&(\Delta H + \Delta H^{\ddagger})(\varepsilon\gamma)\mathrm{d}H + (H+H^{\ddagger})(\varepsilon\gamma)\mathrm{d}\Delta H. \end{aligned} \tag{68}$$

From (64), (66), (27) and (23), we have

$$\begin{aligned} 2(\Delta z + \Delta\rho^*)\mathrm{d}H =&2\sigma(z+\rho^*)[1-2l(z+\rho^*)]^{-1}\mathrm{d}H \\ =&\frac{\sigma}{l}(H+H^{\ddagger})(\varepsilon\gamma)\Gamma(l). \end{aligned} \tag{69}$$

Moreover, from (62), (27), (28) and (31) we obtain

$$\begin{aligned} \mathrm{d}\Delta H =&-\sigma\mathrm{d}(\dot{F}(l)F(l)^{-1})(\varepsilon\gamma) \\ =&\sigma\dot{\Gamma}(l) - \sigma l\Lambda(l)^{-1}[\mathrm{d}H(\varepsilon\gamma),\ \dot{F}(l)F(l)^{-1}](\varepsilon\gamma). \end{aligned} \tag{70}$$

Multiplying Eq.(70) from the left by $2(z+\rho^*)$ and using (23) and (70) again, it follows that

$$\begin{aligned}&2(z+\rho^*)\mathrm{d}\Delta H\\ =&2\sigma(z+\rho^*)\dot{\Gamma}(l)-\sigma l\Lambda(l)^{-1}\\ &\times[(H+H^{\ddagger})(\varepsilon\gamma)\mathrm{d}H(\varepsilon\gamma),\dot{F}(l)F(l)^{-1}](\varepsilon\gamma)\\ =&\sigma[(H+H^{\ddagger})(\varepsilon\gamma),\ \dot{F}(l)F(l)^{-1}]\Gamma(l)\\ &+(H+H^{\ddagger})(\varepsilon\gamma)\mathrm{d}\Delta H.\end{aligned}\tag{71}$$

On the other hand, from (62), (25), (26) and (38) we have

$$\begin{aligned}&(\Delta H+\Delta H^{\ddagger})(\varepsilon\gamma)\mathrm{d}H\\ =&-\sigma[\dot{F}(l)F(l)^{-1}(\varepsilon\gamma)+(\varepsilon\gamma)F(l)^{\times-1}\dot{F}(l)^{\times}](\varepsilon\gamma)\mathrm{d}H\\ =&\sigma[(H+H^{\ddagger})(\varepsilon\gamma),\ \dot{F}(l)F(l)^{-1}]\Gamma(l)\\ &+\frac{\sigma}{l}(H+H^{\ddagger})(\varepsilon\gamma)\Gamma(l).\end{aligned}\tag{72}$$

Then (69), (71), (72) give (68) which shows that (62) is indeed a symmetry transformation of the field equation (23).

Similarly, we can prove

$$\begin{aligned}&2(\tilde{\Delta}z+\tilde{\Delta}\rho^*)\mathrm{d}H+2(z+\rho^*)\mathrm{d}\tilde{\Delta}H\\ =&(\tilde{\Delta}H+\tilde{\Delta}H^{\ddagger})(\varepsilon\gamma)\mathrm{d}H+(H+H^{\ddagger})(\varepsilon\gamma)\mathrm{d}\tilde{\Delta}H,\end{aligned}\tag{73}$$

i.e. (63) is also a symmetry transformation of the field equation (23).

4. Infinite-dimensional algebra structures of the symmetries

Noticing the structures of the transformations (49), (58) and the analytic property of $F(l)$, $\tilde{F}(s)$ around $l=0$, $s=0$, we can expand the right-hand sides of (49) and (58) in powers of l and s, respectively, as

$$\delta H=\sum_{n=0}^{\infty}l^n\delta^{(n)}H,\qquad \tilde{\delta}H=\sum_{m=1}^{\infty}s^m\tilde{\delta}^{(m)}H.\tag{74}$$

It can be easily seen that each of $\delta^{(n)}$ and $\tilde{\delta}^{(m)}$ satisfies the same equations and conditions as δ and $\tilde{\delta}$ do. Thus we have, in fact, obtained infinitely many infinitesimal hidden symmetry transformations of the SAS EM-p theory.

Next we calculate the commutators between the members of this infinite set of symmetry transformations. Firstly, noticing the dependence of (49) and (58) on the parameters l, s and the infinitesimal constants α^a in T, we denote the transformations (49) and (58) by $\delta_\alpha(l)$, $\tilde{\delta}_\alpha(s)$, respectively. Then we have

$$\begin{aligned}&[\delta_\alpha(l),\ \delta_\beta(l')]H\\ =&\delta_\alpha(l)(H+\delta_\beta(l')H)-\delta_\alpha(l)H\\ &-\delta_\beta(l')(H+\delta_\alpha(l)H)+\delta_\beta(l')H\\ =&\frac{1}{l}[\delta_\beta(l')F(l)\cdot F(l)^{-1},\ F(l)T_\alpha F(l)^{-1}](\varepsilon\gamma)\\ &-\frac{1}{l'}[\delta_\alpha(l)F(l')\cdot F(l')^{-1},\ F(l')T_\beta F(l')^{-1}](\varepsilon\gamma),\end{aligned}\tag{75}$$

and similarly

$$\begin{aligned}&[\delta_\alpha(l),\ \tilde{\delta}_\beta(s)]H\\ =&\frac{1}{l}[\tilde{\delta}_\beta(s)F(l)\cdot F(l)^{-1},\ F(l)T_\alpha F(l)^{-1}](\varepsilon\gamma)\\ &+s[\delta_\alpha(l)\tilde{F}(s)\cdot\tilde{F}(s)^{-1},\ \tilde{F}(s)T_\beta\tilde{F}(s)^{-1}](\varepsilon\gamma),\end{aligned}\tag{76}$$

$$\begin{aligned}&[\tilde{\delta}_\alpha(s),\ \tilde{\delta}_\beta(s')]H\\ =&-s[\tilde{\delta}_\beta(s')\tilde{F}(s)\cdot\tilde{F}(s)^{-1},\ \tilde{F}(s)T_\alpha\tilde{F}(s)^{-1}](\varepsilon\gamma)\\ &+s'[\tilde{\delta}_\alpha(s)\tilde{F}(s')\cdot\tilde{F}(s')^{-1},\ \tilde{F}(s')T_\beta\tilde{F}(s')^{-1}](\varepsilon\gamma),\end{aligned}\tag{77}$$

where $T_\alpha=\alpha^aT_a$, $\delta(l')F(l)=F(l,H+\delta(l')H)-F(l,H)$, etc.

From (75)–(77) we see that, to obtain the commutators explicitly, we need to know the variations of $F(l)$ and $\tilde{F}(s)$ induced by $\delta(l')H$, $\tilde{\delta}(s')H$. Motivated by Refs.[19,20], it may be verified by tedious but straightforward calculations that we can take

$$\begin{aligned}\delta_\alpha(l')F(l)=&\frac{l}{l-l'}[F(l')T_\alpha F(l')^{-1}\\ &-F(l)T_\alpha F(l)^{-1}]F(l),\end{aligned}\tag{78}$$

$$\begin{aligned}\tilde{\delta}_\alpha(s)F(l)=&\frac{ls}{1-ls}[\tilde{F}(s)T_\alpha\tilde{F}(s)^{-1}\\ &-F(l)T_\alpha F(l)^{-1}]F(l),\end{aligned}\tag{79}$$

$$\begin{aligned}\delta_\alpha(l)\tilde{F}(s)=&\frac{1}{1-ls}[F(l)T_\alpha F(l)^{-1}\\ &-\tilde{F}(s)T_\alpha\tilde{F}(s)^{-1}]\tilde{F}(s),\end{aligned}\tag{80}$$

$$\begin{aligned}\tilde{\delta}_\alpha(s')\tilde{F}(s)=&\frac{s'}{s-s'}[\tilde{F}(s')T_\alpha\tilde{F}(s')^{-1}\\ &-\tilde{F}(s)T_\alpha\tilde{F}(s)^{-1}]\tilde{F}(s)\end{aligned}\tag{81}$$

such that $F(l)+\delta_\alpha(l')F(l)$, $F(l)+\tilde{\delta}_\alpha(s)F(l)$ satisfy the same equation (31) and conditions (36)–(38) as $F(l)$ does; while $\tilde{F}(s)+\delta_\alpha(l)\tilde{F}(s)$ and $\tilde{F}(s)+\tilde{\delta}_\alpha(s')\tilde{F}(s)$ satisfy the same equation (44) and conditions (47) and

(48) as $\tilde{F}(s)$ does. For example, from (40), (81), (50), (58), (39) and the relation $\tilde{A}(s) = \tilde{A}(s') + s - s'$, it can be verified that

$$\begin{aligned}&\tilde{\delta}(s')\tilde{F}(s)^{\times}(\varepsilon\gamma)\tilde{A}(s)\tilde{F}(s)\\&+\tilde{F}(s)^{\times}(\varepsilon\gamma)\tilde{\delta}(s')\tilde{A}(s)\tilde{F}(s)\\&+\tilde{F}(s)^{\times}(\varepsilon\gamma)\tilde{A}(s)\tilde{\delta}(s')\tilde{F}(s)=0.\end{aligned}\tag{82}$$

This implies that $\tilde{F}(s)+\tilde{\delta}_\alpha(s')\tilde{F}(s)$ satisfies the same condition (48) as $\tilde{F}(s)$ does.

Substituting (78)–(81) into (75)–(77), using (49) and (58) again and writing $\delta_\alpha(l)H = \alpha^a\delta_a(l)H$, etc., we obtain

$$[\delta_\alpha(l),\ \delta_\beta(s)]H = \frac{\alpha^a\beta^b}{l-l'}C^c_{ab}\Big(l\delta_c(l)H - l'\delta_c(l')H\Big),\tag{83}$$

$$[\delta_\alpha(l),\ \tilde{\delta}_\beta(s)]H = \frac{\alpha^a\beta^b}{1-ls}C^c_{ab}\Big(ls\delta_c(l)H + \tilde{\delta}_c(s)H\Big),\tag{84}$$

$$[\tilde{\delta}_\alpha(s),\ \tilde{\delta}_\beta(s')]H = \frac{\alpha^a\beta^b}{s-s'}C^c_{ab}\Big(s'\tilde{\delta}_c(s)H - s\tilde{\delta}_c(s')H\Big),\tag{85}$$

where C^c_{ab} are structure constants of the Lie algebra $su(p+1,1)$. Writing the transformations in (74) in the explicitly α related form as

$$\begin{aligned}\delta_\alpha(l)H &= \alpha^a\sum_{n=0}^{\infty}l^n\delta^{(n)}_a H,\\ \tilde{\delta}_\alpha(s)H &= \alpha^a\sum_{m=1}^{\infty}s^m\tilde{\delta}^{(m)}_a H,\end{aligned}\tag{86}$$

and then expanding both sides of (83)–(85), we finally obtain

$$[\delta^{(n)}_a,\ \delta^{(m)}_b]H = C^c_{ab}\delta^{n+m)}_c H,\quad n,m = 0,\pm1,\pm2,\ldots,\tag{87}$$

where $\delta^{(-m)}_a H := \tilde{\delta}^{(m)}_a H$ for $m \geq 1$. Thus, the infinite set of symmetry transformations $\{\delta^{(n)}_a,\ n = 0,\pm1,\pm2,\ldots\}$ constitute the Kac–Moody algebra $\widehat{su(p+1,1)}$ (without centre charge).

However, the symmetry group for the studied EM-p theory is larger. Now we consider parametrized symmetry transformations (62) and (63). From their structures and analytic property around $l = 0$ and $s = 0$, they can be expanded as

$$\begin{cases}\Delta H = \sigma\sum_{n=0}^{\infty}l^n\Delta^{(n)}H,\\ \tilde{\Delta}H = \epsilon\sum_{m=1}^{\infty}s^m\tilde{\Delta}^{(m)}H.\end{cases}\tag{88}$$

Thus we obtain another infinite set of symmetry transformations $\{\Delta^{(n)},\ \tilde{\Delta}^{(m)},\ n = 0,1,2,\ldots;\ m = 1,2,\ldots\}$ of the EM-p theory. In order to calculate their commutators, we first note that from (62) and (63) and denoting them by $\Delta_\sigma(l)H$, $\tilde{\Delta}_\epsilon(s)H$, respectively, we have

$$\begin{aligned}&[\Delta_\sigma(l),\ \Delta_{\sigma'}(l')]H\\&= -\sigma\frac{\partial}{\partial l}\Big(\Delta_{\sigma'}(l')F(l)\cdot F(l)^{-1}\Big)(\varepsilon\gamma)\\&\quad+\sigma'\frac{\partial}{\partial l'}\Big(\Delta_\sigma(l)F(l')\cdot F(l')^{-1}\Big)(\varepsilon\gamma)\\&\quad-\sigma[\Delta_{\sigma'}(l')F(l)\cdot F(l)^{-1},\ \dot{F}(l)F(l)^{-1}](\varepsilon\gamma)\\&\quad+\sigma'[\Delta_\sigma(l)F(l')\cdot F(l')^{-1},\ \dot{F}(l')F(l')^{-1}](\varepsilon\gamma),\end{aligned}\tag{89}$$

$$\begin{aligned}&[\Delta_\sigma(l),\ \tilde{\Delta}_\epsilon(s)]H\\&= -\sigma\frac{\partial}{\partial l}\Big(\tilde{\Delta}_\epsilon(s)F(l)\cdot F(l)^{-1}\Big)(\varepsilon\gamma)\\&\quad+\epsilon s^2\frac{\partial}{\partial s}\Big(\Delta_\sigma(l)\tilde{F}(s)\cdot\tilde{F}(s)^{-1}\Big)(\varepsilon\gamma)\\&\quad-\sigma[\tilde{\Delta}_\epsilon(s)F(l)\cdot F(l)^{-1},\ \dot{F}(l)F(l)^{-1}](\varepsilon\gamma)\\&\quad+\epsilon s^2[\Delta_\sigma(l)\tilde{F}(s)\cdot\tilde{F}(s)^{-1},\ \dot{\tilde{F}}(s)\tilde{F}(s)^{-1}](\varepsilon\gamma),\end{aligned}\tag{90}$$

$$\begin{aligned}&[\tilde{\Delta}_\epsilon(s),\ \tilde{\Delta}_{\epsilon'}(s')]H\\&= -\epsilon s^2\frac{\partial}{\partial s}\Big(\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s)\cdot\tilde{F}(s)^{-1}\Big)(\varepsilon\gamma)\\&\quad+\epsilon' s'^2\frac{\partial}{\partial s'}\Big(\tilde{\Delta}_\epsilon(s)\tilde{F}(s')\cdot\tilde{F}(s')^{-1}\Big)(\varepsilon\gamma)\\&\quad-\epsilon s^2[\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s)\cdot\tilde{F}(s)^{-1},\ \dot{\tilde{F}}(s)\tilde{F}(s)^{-1}](\varepsilon\gamma)\\&\quad+\epsilon' s'^2[\tilde{\Delta}_\epsilon(s)\tilde{F}(s')\cdot\tilde{F}(s')^{-1},\ \dot{\tilde{F}}(s')\tilde{F}(s')^{-1}](\varepsilon\gamma).\end{aligned}\tag{91}$$

As for $\Delta_\sigma(l')F(l)$, $\Delta_\sigma(l)\tilde{F}(s)$, etc., we propose

$$\begin{aligned}\Delta_\sigma(l')F(l) =&\sigma\frac{l}{l-l'}[l\dot{F}(l)F(l)^{-1}\\&-l'\dot{F}(l')F(l')^{-1}]F(l),\end{aligned}\tag{92}$$

$$\begin{aligned}\tilde{\Delta}_\epsilon(s)F(l) =&\epsilon\frac{ls}{1-ls}[l\dot{F}(l)F(l)^{-1}\\&-s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1}+\tfrac{1}{2}]F(l),\end{aligned}\tag{93}$$

$$\begin{aligned}\Delta_\sigma(l)\tilde{F}(s) =&\sigma\frac{1}{ls-1}[s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1}\\&+l\dot{F}(l)F(l)^{-1}+\tfrac{1}{2}]\tilde{F}(s),\end{aligned}\tag{94}$$

$$\begin{aligned}\tilde{\Delta}_\epsilon(s')\tilde{F}(s) =&\epsilon\frac{s'}{s-s'}[s\dot{\tilde{F}}(s)\tilde{F}(s)^{-1}\\&-s'\dot{\tilde{F}}(s')\tilde{F}(s')^{-1}]\tilde{F}(s).\end{aligned}\tag{95}$$

By some lengthy but straightforward calculations, it can be verified that (92) and (93) are symmetry transformations of Eq.(31) with conditions (36)–(38); and (94) and (95) are symmetry transformations of Eq.(44) with conditions (47), (48). For example, we can prove that $F(l)+\Delta_\sigma(l')F(l)$ satisfies Eq.(31) by checking that

$$\begin{aligned}&\Delta_\sigma(l')\Lambda(l)\mathrm{d}F(l)+\Lambda(l)\mathrm{d}(\Delta_\sigma(l')F(l))\\=&l\mathrm{d}(\Delta_\sigma(l')H)(\varepsilon\gamma)F(l)+l\mathrm{d}H(\varepsilon\gamma)\Delta_\sigma(l')F(l).\end{aligned}\quad(96)$$

This may be done by using (31), (27), (28), (62) and the relations $\Delta_\sigma(l')\Lambda(l)=-2\sigma l(z+\rho^*)\Lambda(l')^{-1}$ and $\dfrac{1}{l'-l}[l'\Lambda(l')^{-1}-l\Lambda(l)^{-1}]=\Lambda(l')^{-1}\Lambda(l)^{-1}$.

Substituting (92)–(95) into (89)–(91) and using (62), (63) again, it follows that

$$\begin{aligned}&[\Delta_\sigma(l),\ \Delta_{\sigma'}(l')]H\\=&\sigma\frac{\partial}{\partial l}\left[\frac{l}{l-l'}\Big(l\Delta_{\sigma'}(l)H-l'\Delta_{\sigma'}(l')H\Big)\right]\\&-\sigma'\frac{\partial}{\partial l'}\left[\frac{l'}{l'-l}\Big(l'\Delta_\sigma(l')H-l\Delta_\sigma(l)H\Big)\right],\end{aligned}\quad(97)$$

$$\begin{aligned}&[\Delta_\sigma(l),\ \tilde\Delta_\epsilon(s)]H\\=&\sigma\frac{\partial}{\partial l}\left[\frac{ls}{1-ls}\Big(l\Delta_\epsilon(l)H+s^{-1}\tilde\Delta_\epsilon(s)H\Big)\right]\\&-\epsilon s^2\frac{\partial}{\partial s}\left[\frac{1}{ls-1}\Big(s^{-1}\tilde\Delta_\sigma(s)H+l\Delta_\sigma(l)H\Big)\right],\end{aligned}\quad(98)$$

$$\begin{aligned}&[\tilde\Delta_\epsilon(s),\ \tilde\Delta_{\epsilon'}(s')]H\\=&\epsilon s^2\frac{\partial}{\partial s}\left[\frac{s'}{s-s'}\Big(s^{-1}\tilde\Delta_{\epsilon'}(s)H-s'^{-1}\tilde\Delta_{\epsilon'}(s')H\Big)\right]\\&-\epsilon' s'^2\frac{\partial}{\partial s'}\left[\frac{s}{s'-s}\Big(s'^{-1}\tilde\Delta_\epsilon(s')H-s^{-1}\tilde\Delta_\epsilon(s)H\Big)\right].\end{aligned}\quad(99)$$

By using (88) to expand both sides of (97)–(99), we obtain

$$[\Delta^{(m)},\ \Delta^{(n)}]H=(m-n)\Delta^{(m+n)}H,$$

$$m,n=0,\pm1,\pm2,\ldots,\quad(100)$$

where we have written $\Delta^{(-n)}H:=\tilde\Delta^{(n)}H$ for $n\geq1$. This shows that the infinite set of symmetry transformations $\{\Delta^{(n)},\ n=0,\pm1,\pm2,\ldots\}$ constitute the Virasoro algebra (without centre charge).

Now we investigate the commutators between the members of $\{\delta^{(m)}\}$ and $\{\Delta^{(n)}\}$. For example, from (49), (62), (78) and (92) we have, by some calculations,

$$\begin{aligned}&[\Delta_\sigma(l),\ \delta_a(s)]H\\=&\sigma\frac{\partial}{\partial l}\left[\frac{l}{l-s}\Big(l\delta_a(l)H-s\delta_a(s)H\Big)\right]\\&-\sigma\frac{l}{l-s}\frac{\partial}{\partial l}\Big(l\delta_a(l)H\Big)\\&+\sigma\frac{s}{l-s}\frac{\partial}{\partial s}\Big(s\delta_a(s)H\Big).\end{aligned}\quad(101)$$

Similarly, we can calculate out the expressions of $[\Delta_\sigma(l),\ \tilde\delta_a(s)]H$, $[\tilde\Delta_\sigma(l),\ \delta_a(s)]H$ and $[\tilde\Delta_\sigma(l),\ \tilde\delta_a(s)]H$. Then by using (74) and (88) to expand both sides of these results, we finally obtain

$$[\Delta^{(m)},\ \delta_a^{(n)}]H=-n\delta_a^{(m+n)}H,$$

$$m,n=0,\pm1,\pm2,\ldots.\quad(102)$$

The equations (87), (100) and (102) give the structure of semidirect product of the affine Kac–Moody algebra $su(\widehat{p+1,1})$ and the Virasoro algebra. This reveals that the EM-p theory under consideration possesses the symmetry group of semidirect product of the affine Kac–Moody group $SU(\widehat{p+1,1})$ and Virasoro group. The explicit expressions (49), (58), (62) and (63) for parametrized infinitesimal symmetry transformations give a representation of the corresponding infinite-dimensional semidirect product Lie algebra.

5. Summary and discussion

By proposing and using the so-called EHC function method, the SAS EM-p field equations, when being expressed in a suitable σ-model form, enable us to introduce an Ernst-like $(p+2)\times(p+2)$ matrix EHC potential H. Then the field equations can be equivalently written as a so-called HE-like self-dual relation for the potential H. Two HE-type EHC linear systems (31) and (44) associated with these equations are established they essentially can be interrelated with each other by a continuation $t\leftrightarrow\tau=1/t$. Based on solutions $F(t)$ and $\tilde F(\tau)$ of these EHC linear systems, we explicitly construct the parametrized transformations (49), (58), (62) and (63), and then verify that all of them are hidden symmetries of the studied EM-p theory. These symmetries are proven to constitute an infinite-dimensional Lie algebra, which is the semidirect product of the Kac–Moody algebra $su(p+1,1)\otimes R(t,t^{-1})$ and the Virasoro algebra (without centre charges). This paper treats all of the SAS

EM-p theories for $p = 0, 1, 2, \ldots$ in a unified formulation. When taking $p = 0$, Eqs.(12) and (14) give

$$M|_{p=0} = \frac{\rho}{f}\begin{pmatrix} 1 & -\psi \\ -\psi & f^2+\psi^2 \end{pmatrix},$$

$$\gamma|_{p=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (103)$$

thus the above discussions give the results for the SAS vacuum gravity theory but in a 'dualized' formulation. When taking $p = 1$, we obtain the results about the SAS EM theory.

We would like to indicate that the infinite-dimensional symmetry algebras obtained in this paper are entirely different from that in the quantum field theories. The symmetry algebras here are defined as Lie algebras of classical transformations, which operate in the tangent space of the SAS EM-p solution manifold and generate new solutions from the known one for the field equations. However, finite symmetry transformations relating to the above infinitesimal ones need more and further investigations and will be considered in some forthcoming works.

References

[1] Ernst F J 1968 *Phys. Rev.* **168** 1415 Ernst F J 1982 *Lect. Notes Phys.* **189** 128

[2] Kinnersley W 1977 *J. Math. Phys.* **18** 1529 Kinnersley W 1980 *J. Math. Phys.* **21** 2231

[3] Kinnersley W and Chitre D M 1977 *J. Math. Phys.* **18** 1538 Kinnersley W and Chitre D M 1978 *J. Math. Phys.* **19** 1926

[4] Tanabe Y 1978 *J. Math. Phys.* **19** 1808

[5] Hauser I and Ernst F J 1979 *Phys. Rev.* D **20** 1783 Hauser I and Ernst F J 1980 *J. Math. Phys.* **21** 1418

[6] Jones T C 1980 *J. Math. Phys.* **21** 1790

[7] Kaliappan P and Lakshmanan M 1981 *J. Math. Phys.* **22** 2447

[8] Gürses M and Xanthopoulos B C 1982 *Phys. Rev.* D **26** 1912

[9] Eris A, Gürses M and Karasu A 1984 *J. Math. Phys.* **25** 1489

[10] Gürses M 1984 *Phys. Rev.* D **30** 486 Gürses M 1984 *Lect. Notes Phys.* **205** 186

[11] Kramer D and Neugebauer G 1981 *J. Phys.* A **14** L333

[12] Kramer D 1982 *J. Phys.* A **15** 2201

[13] Neugebauer G and Kramer D 1983 *J. Phys.* A **16** 1927

[14] Cosgrove C M 1981 *J. Math. Phys.* **22** 2624

[15] Leaute B and Marcilhacy G 1985 *J. Math. Phys.* **26** 1938

[16] Hashimoto I and Sawae R 1994 *Commun. Math. Phys.* **160** 15

[17] Gao Y J, Zhong Z Z and Gui Y X 1997 *J. Math. Phys.* **38** 3155

[18] Hauser I and Ernst F J 1980 *J. Math. Phys.* **21** 1126

[19] Wu Y S and Ge M L 1983 *J. Math. Phys.* **24** 1187

[20] Wu Y S 1983 *Phys. Lett.* A **96** 179

[21] Hauser I 1984 *Lect. Notes Phys.* **205** 113

[22] Hou B Y and Li W 1987 *Lett. Math. Phys.* **13** 1

[23] Li W 1988 *Phys. Lett.* A **129** 301