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Multi-valued solitary waves in multidimensional soliton systems*

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Considering that folded phenomena are rather universal in nature and some arbitrary functions can be included in the exact excitations of many (2+1)-dimensional soliton systems, we use adequate multivalued functions to construct folded solitary structures in multi-dimensions. Based on some interesting variable separation results in the literature, a common formula with arbitrary functions has been derived for suitable physical quantities of some significant (2+1)-dimensional soliton systems like the generalized Ablowitz–Kaup–Newell–Segur (GAKNS) model, the generalized Nizhnik–Novikov–Veselov (GNNV) system and the new (2+1)-dimensional long dispersive wave (NLDW) system. Then a new special type of two-dimensional solitary wave structure, i.e. the folded solitary wave and foldon, is obtained. The novel structure exhibits interesting features not found in the single valued solitary excitations.

Keywords: multidimensional soliton system, multivalued solitary wave, foldon
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1. Introduction

Over the last few decades, single valued localized excitations such as solitoffs, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic and fractal patterns, etc., have been extensively studied in the context of some high-dimensional nonlinear physical systems.[1] However, in the natural world, there exist very complicated folded phenomena such as the folded protein,[2] folded brain, folded skin surface and many other kinds of folded biology systems.[3] Bubbles on (or under) a fluid surface may be the simplest folded waves. Furthermore, various kinds of ocean waves are folded waves. How to describe these universal folded phenomena is an interesting topic. The loop solitons, which are often thought of as a class of the simplest folded waves in the (1+1)-dimensional case, have been found in many (1+1)-dimensional integrable systems [4] and applied in certain physical fields, such as string interaction with external field,[5] quantum field theory [6] and particle physics.[7] Compared to these lower-dimensional folded waves, we know little on folded solitary waves in higher-dimensional systems. Fortunately, in a recent study we found that some arbitrary functions can be included in exact excitations of many multidimensional soliton systems, which may be used to depict these natural folded phenomena. Our present work is motivated by Tang and Lou's pioneer work,[8] which first pointed out that these special folded localized excitations may occur in some (2+1)-dimensional nonlinear models, such as the (2+1)-dimensional dispersive long wave equation and the (2+1)-dimensional Burgers equation. They defined a new type of soliton, the foldon, for the case that the interaction between the folded solitary waves is completely elastic. In this paper, we extend the discussion to other significant soliton systems and pay attention to their possible multivalued localized solitary structures such as folded solitary waves and foldons.

As a concrete example, the first model we consider is the (2+1)-dimensional generalized Nizhnik–Novikov–Vesselov system (GNNV),[9] which reads as

$$v_t + av_{xxx} + bv_{yyy} + cv_x + dv_y = 3a(uv)_x + 3b(vw)_y,$$

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$$v_x = u_y, \quad v_y = w_x, \tag{1}$$

where a, b, c and d are arbitrary constants. If $c = d = 0$, the (2+1)-dimensional GNNV system will be degenerated to the usual two-dimensional Nizhnik–Novikov–Vesselov (NNV) system,[10] which is an isotropic Lax extension of the classical (1+1)-dimensional shallow water wave Korteweg–de Vries (KdV) model.

The second model we consider is the (2+1)-dimensional generalized Ablowitz–Kaup–Newell–Segur (GAKNS) [11–13]

$$\begin{aligned} &iq_t + aq_{xx} + bq_{yy} + 2(aw + bv)q = 0, \\ &-ir_t + ar_{xx} + br_{yy} + 2(aw + bv)r = 0, \\ &v_x + \lambda(qr)_y = 0, \ w_y + \lambda(qr)_x = 0, \end{aligned} \tag{2}$$

where a, b are arbitrary constants and $\lambda = \pm 1$. This system is derived from the inner parameter symmetry constraint of the generalized Schrödinger system.[14]

The third is a new (2+1)-dimensional long dispersive wave (NLDW) system,

$$\begin{aligned} &\lambda q_t + q_{xx} - 2q\int (qp)_x \mathrm{d}y = 0, \\ &\lambda r_t - r_{xx} + 2r\int (qr)_x \mathrm{d}y = 0, \end{aligned} \tag{3}$$

where $\partial_x = \partial_\xi$, $\partial_y = \partial_\xi - \lambda\partial_\eta$, and λ is a constant. This system obviously differs from the usual or the traditional long dispersive wave system.[15]

Since the detailed physical backgrounds of the GNNV, GAKNS and NLDW systems have been given in Refs.[16], [17], [18] and [19, 20] respectively, we neglect the corresponding description. For the GAKNS system, we only discuss the case of $\lambda = 1$ in the soliton system. If $\lambda = -1$, we can make the simple variable transformation $x \to -x$ (or $y \to -y$), then the situation will be identical with the former case.

This paper is organized as follows. In Sec.2, a common variable separation solution to the GNNV, GAKNS and NLDW systems is obtained. In Sec.3, we discuss the possible multivalued solitary structures for the common solution based on result obtained by the variable separation approach. A brief discussion and summary is given in the last section.

2. A common variable separation solution to the GNNV, GAKNS and NLDW systems

In Ref.[16], applying the variable separation approach,[21] we have proven that the (2+1)-dimensional GNNV system (1) possesses a quite general excitation,

$$v = \frac{2(a_1a_2 - a_0a_3)F_xG_y}{(a_0 + a_1F + a_2G + a_3FG)^2}, \tag{4}$$

$$\begin{aligned} u =& \frac{2(a_1 + a_3G)^2F_x^2}{(a_0 + a_1F + a_2G + a_3FG)^2} \\ &- \frac{2(a_1 + a_3G)F_{xx}}{a_0 + a_1F + a_2G + a_3FG} \\ &+ \frac{F_t + aF_{xxx} + cF_x}{3aF_x}, \end{aligned} \tag{5}$$

$$\begin{aligned} w =& \frac{2(a_2 + a_3F)^2G_y^2}{(a_0 + a_1F + a_2G + a_3FG)^2} \\ &- \frac{2(a_2 + a_3F)G_{yy}}{a_0 + a_1F + a_2G + a_3FG} \\ &+ \frac{G_t + bG_{yyy} + dG_y}{3bG_y}, \end{aligned} \tag{6}$$

where $F \equiv F(x,t)$ and $G \equiv G(y,t)$ are two arbitrary functions of $\{x,t\}$ and $\{y,t\}$, respectively, a_0, a_1, a_2 and a_3 are arbitrary constants.

In Ref.[18], using a similar variable separation approach, we have derived a general exact solution to the (2+1)-dimensional GAKNS system (2),

$$q = \frac{\delta_1\delta_2\sqrt{a_1a_2F_xG_y}\exp(p+s)}{a_1F + a_2G + a_3FG}, \tag{7}$$

$$r = \frac{\delta_1\delta_2\sqrt{a_1a_2F_xG_y}}{(a_1F + a_2G + a_3FG)\exp(p+s)}, \tag{8}$$

$$\begin{aligned} w =& F_0 + \left(\frac{a_1F_{xx} + a_3GF_{xx}}{a_1F + a_2G + a_3FG}\right. \\ &\left. - \frac{(a_1F_x + a_3GF_x)^2}{(a_1F + a_2G + a_3FG)^2}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} v =& G_0 + \left(\frac{a_2G_{yy} + a_3FG_{yy}}{a_1F + a_2G + a_3FG}\right. \\ &\left. - \frac{(a_2G_y + a_3FG_y)^2}{(a_1F + a_2G + a_3FG)^2}\right), \end{aligned} \tag{10}$$

where $F \equiv F(x,t)$ and $G \equiv G(y,t)$ are also arbitrary functions of $\{x,t\}$ and $\{y,t\}$, respectively, a_0, a_1, a_2 and a_3 are arbitrary constants, $p(x,t), s(y,t)$ are related to F and G by

$$\begin{aligned} p_x =& [-F_t + c_1(a_2 + a_3F)^2 \\ &+ c_2(a_2 + a_3F) + a_1a_2c_3]/2iaF_x, \end{aligned} \tag{11}$$

$$\begin{aligned} s_y =& -[G_t + c_3(a_1 + a_3G)^2 \\ &+ c_2(a_1 + a_3G) + a_1a_2c_1]/2ibG_y, \end{aligned} \tag{12}$$

while F_0 and G_0 are also related to F and G by the following equations:

$$F_0 = (8aF_x^2)^{-1}[aF_{xx}^2 - 2aF_xF_{xxx} - 4F_x^2(p_t + ap_x^2 + c_4)], \quad (13)$$

$$G_0 = (8bG_y^2)^{-1}[bG_{yy}^2 - 2bG_yG_{yyy} - 4G_y^2(s_t + bs_y^2 - c_4)], \quad (14)$$

where c_1, c_2, c_3, c_4 are all arbitrary functions of time t.

Here we are interested in the physical quantity qr. From expressions (7) and (8), we have

$$qr = \frac{a_1a_2F_xG_y}{(a_1F + a_2G + a_3FG)^2}. \quad (15)$$

It is of interest to find that the solution (15) is also valid for the new (2+1)-dimensional long dispersive wave (NLDW). In Ref.[19], we take a simple transformation $qr = w_y$ (here w is some arbitrary potential) and making use of a similar approach, a general excitation of the NLDW system is obtained,

$$q = \frac{\delta_1\delta_2\sqrt{a_1a_2F_xG_y}\exp[\lambda(p + g(y) + b(t))]}{a_1F + a_2G + a_3FG}, \quad (16)$$

$$r = \frac{\delta_1\delta_2\sqrt{a_1a_2F_xG_y}}{(a_1F + a_2G + a_3FG)\exp[\lambda(p + g(y) + b(t))]}, \quad (17)$$

$$w = F_0 - \frac{a_1F_x + a_3F_xG}{a_1F + a_2G + a_3FG}, \quad (18)$$

where $F \equiv F(x,t)$, $\delta_1^2 = \delta_2^2 = 1$, $g(y)$, $b(t)$ are three arbitrary functions, a_0, a_1, a_2 and a_3 are arbitrary constants, and $G \equiv G(y,t)$ satisfies the following Riccati equation:

$$a_1^2G_t - (c_1 - c_2a_3 + c_3a_3^2)G^2 - a_1(-c_2 + 2c_3a_3)G - a_1^2c_3 = 0, \quad (19)$$

Its general solution has the form

$$G(y,t) = \frac{A_1(t)}{A_2(t) + U(y)} + A_3(t), \quad (20)$$

where $U(y)$ is an arbitrary function of y while A_1, A_2 and A_3 are arbitrary functions of time t, which are linked with c_1, c_2 and c_3 by

$$c_1 = \frac{-1}{A_1}(2a_1a_3A_3A_{2t} + a_1a_3A_{1t} + a_1^2A_{2t} - a_3^2A_1A_{3t} + a_3^2A_3^2A_{2t} + a_3^2A_3A_{1t}), \quad (21)$$

$$c_2 = \frac{-1}{A_1}(2a_1A_3A_{2t} + a_1A_{1t} - 2a_3A_1A_{3t} + 2a_3A_3^2A_{2t} + 2a_3A_3A_{1t}), \quad (22)$$

$$c_3 = \frac{-1}{A_1}(A_3A_{1t} + A_3^2A_{2t} - A_1A_{3t}). \quad (23)$$

Using the relations (21)–(23), Eq.(19) becomes

$$G_t = \frac{-1}{A_1}[A_{2t}G^2 - (A_{1t} + 2A_3A_{2t})G + A_3^2A_{2t} + A_3A_{1t} - A_1A_{3t}]. \quad (24)$$

One can verify easily that Eq.(20) is a general excitation of Eq.(24), while the functions p and F_0 are only related to $F(x,t)$ and $b(t)$ by

$$p_x = \frac{-1}{2a_1a_2F_x}(c_1F^2 + c_2a_2F + c_3a_2^2 + a_1a_2F_t), \quad (25)$$

$$F_{0x} = \frac{1}{8F_x^2}[2F_xF_{xxx} + 4\lambda^2F_x^2(p_x^2 + p_t + b_t) - F_{xx}^2]. \quad (26)$$

For the potential qr, according to the solutions (16) and (17), we have

$$qr = \frac{a_1a_2F_xG_y}{(a_1F + a_2G + a_3FG)^2}. \quad (27)$$

Comparing the excitation (4) of GNNV system with the solution (15) or (27) to the GAKNS system and the NLDW system, one can easily see they are essentially identical when setting the arbitrary constant $a_0 = 0$ and dividing by 2 for the excitation (4). Therefore, the expression (4) or (27) can be regarded as a common formula for the above soliton systems. Actually, besides the above-mentioned systems, there are a large class of soliton models [1] admitting to the common formula (4).

Given the arbitrariness of the functions F and G, many types of usual localized excitations, such as dromion, lump, ring, oscillated dromion, breather, peakon, compacton, fractal and chaotic soliton structures can be easily constructed by selecting adequate single valued analytic functions or numerical solutions, which have been much discussed in the literature. Now we concentrate on how to find some kinds of folded solitary waves and foldons for the GNNV system (1), the GAKNS system (2) and the NLDW system (3).

3. Folded localized excitations for the multidimensional systems

In order to construct various kinds of interesting folded localized excitations and/or foldons for the common formula (4), we need to introduce some suitable multivalued functions.[8] For example,

$$F_x = \sum_{j=1}^{M} f_j(\zeta + \omega_j t),$$

$$x = \zeta + \sum_{j=1}^{M} X_j(\zeta + \omega_j t), \quad (28)$$

where f_j, X_j are localized excitations with properties $f_j(\pm\infty) = 0$, $X_j(\pm\infty)$=constants. From Eq.(28), one knows that ζ may be a multivalued function in some adequate regions of x by selecting the functions X_j appropriately. Therefore, the function F_x, which is an interaction solution of M localized excitations since the property $\zeta|_{x\to\infty} \to \infty$, may be a multivalued function of x in these areas though it is a single valued function of ζ. In fact, most of the known multiloop solutions are special cases of Eq.(28). We treat the function $G(y,t)$ in a similar way:

$$G_y = \sum_{j=1}^{N} g_j(\eta + \mu_j t),$$
$$y = \eta + \sum_{j=1}^{N} Y_j(\eta + \mu_j t). \tag{29}$$

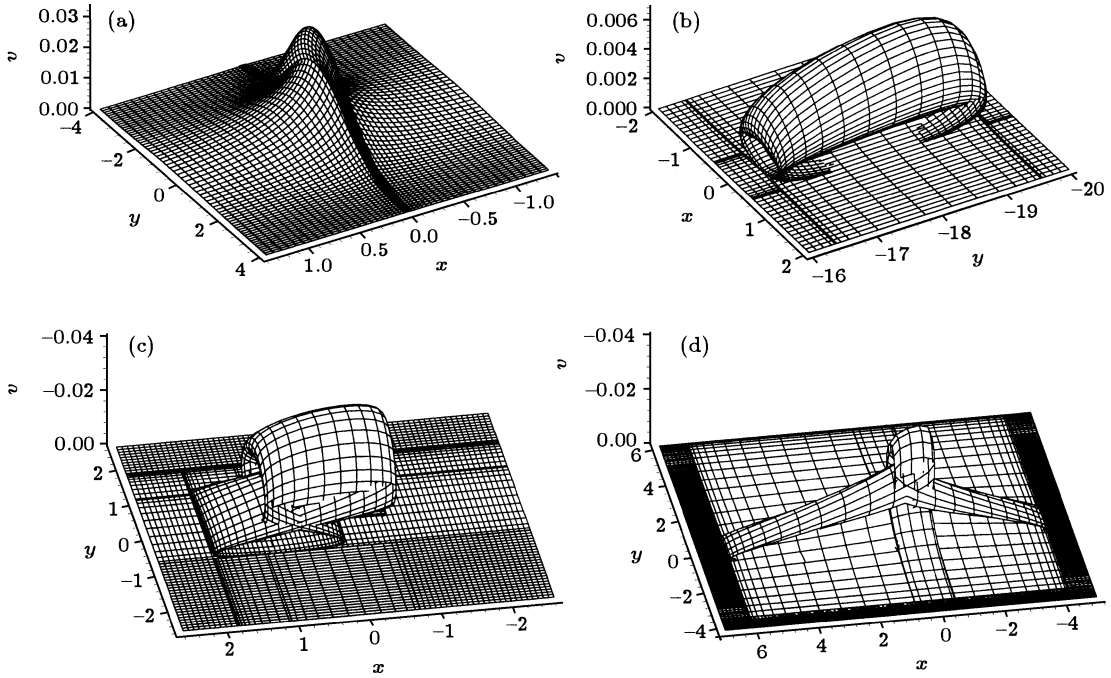


Fig.1. Four types of folded solitary waves are depicted for the common formula (4) at time $t = 0$ with the selections (a) $F_x = \mathrm{sech}^2(\zeta - \omega t), x = \zeta - 1.2\tanh(\zeta - \omega t), G_y = \mathrm{sech}^2(\eta), y = \eta + k\tanh(\eta)$ and $a_1 = a_2 = a_3 = k = C_1/8 = 1, a_0 = C_2 = 0$; (b) F_x and the related parameters are the same as (a), but $C_1 = 20$, $x = \zeta - 2\tanh(\zeta - \omega t)$, $G_y = 1.5\mathrm{sech}^2(\eta), y = \eta - 3\tanh(\eta) - ct$; (c) F_x and the related parameters are the same as (a), however, $C_1 = C_2 = 50$, $x = \zeta + 2\tanh(\zeta - \omega t) + \tanh(\zeta - \omega t)^2 - k\tanh(\zeta - \omega t)^3$, $G_y = \mathrm{sech}^6 2(\eta) + \mathrm{sech}^2(\eta), y = \eta + 2\tanh(\eta) + \tanh(\eta)^2 - k\tanh(\eta)^3, k = 5.5$; (d) As (c), but $k = 10$ and C_2=260.

Consequently, the functions F and G can be expressed by the following integral equations:

$$F = \int^{\xi} F_x x_\xi \mathrm{d}\xi + C_1, \quad G = \int^{\eta} G_y y_\eta \mathrm{d}\eta + C_2, \tag{30}$$

where C_1 and C_2 are integration constants.

In Fig.1, four types of folded solitary waves are presented for the common formula (4). The related function selections of the plots are directly given in the figure captions. From Fig.1, one can see that the folded solitary waves may be folded in rather complicated ways. Now we further discuss the properties of the interaction between the folded solitary waves. If we select F and G to be some appropriate multivalued functions, then we can see that the interactions between the folded solitary waves are completely elastic. For example,

$$F_x = \mathrm{sech}^2(\zeta), x = \zeta - 2\tanh(\zeta), \tag{31}$$

$$G_y = \frac{4}{5}\mathrm{sech}^2(\eta) + \frac{1}{2}\mathrm{sech}^2(\eta - t/4),$$
$$y = \eta - \frac{3}{2}(\tanh(\eta) + \tanh(\eta - t/4)), \tag{32}$$

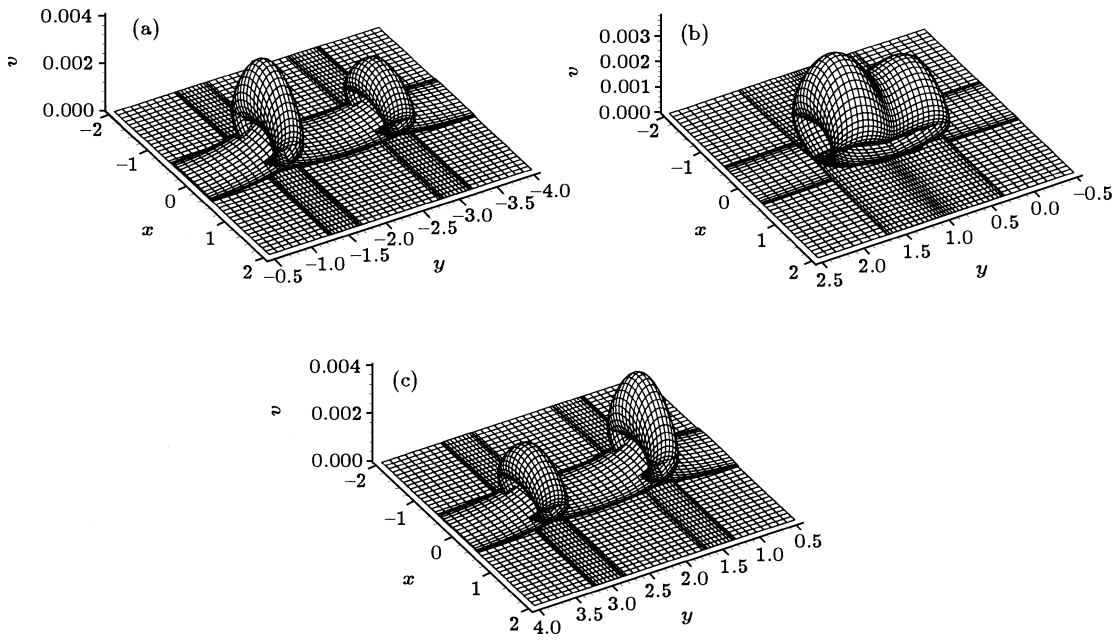


Fig.2. Evolution plots of two foldons for the common formula (4) with conditions (31), (32) and $a_1 = a_2 = a_3 = C_1/20 = 1$, $a_0 = C_2 = 0$ at different times (a) $t = -18$, (b)$t = 8$, (c)$t = 18$, respectively.

and $a_1 = a_2 = a_3 = 1$, then we can derive some interesting foldons as depicted in Figs.2(a), (b) and (c) at different times (a) $t = -18$, (b)$t = 8$, (c)$t = 18$, respectively. From Figs.2(a) and (c), one can find the interaction between the two foldons is completely elastic since the velocity of one of the foldons has been set to zero and there are still phase shifts for the two foldons. To see in more detail, one can easily find that the position of the large static foldon is altered from about $y = -1.5$ to $y = 1.5$ and its shape is completely preserved after interaction.

4. Summary and discussion

In summary, for the (2+1)-dimensional GNNV, GAKNS and NLDW systems, in addition to the usual localized coherent soliton structures like dromion, solitoff, peakon, compacton, breather, instanton, chaotic and fractal structures, etc., there are new types of multivalued localized structures, i.e. folded solitary waves and/or foldons for the multidimensional soliton systems, which may be used to describe some natural folded phenomena. To our knowledge, the folded solitary wave and/or foldon structures for the multidimensional GNNV , GAKNS and NLDW systems has not been reported in the literature. Since the common excitation (4) is a rather 'universal' formula for many (2+1)-dimensional physical models [1] and the GNNV system and GAKNS systems can present several significant physical models, all the discussions in this work are valid for the various physical models mentioned above. This work is only a preliminary attempt, further investigation to find the folded localized excitations like new types of folded solitary waves, foldons and their applications is still necessary.

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