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A new conserved quantity of mechanical systems with differential constraints*

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A new conserved quantity of non-Noether symmetry for the mechanical systems with differential constraints is studied. First, the differential equations of motion of the systems are established. Then, the determining equations and restriction equations of the non-Noether symmetry are obtained and a new conserved quantity is given. Finally, an example is given to illustrate the application of the results.

Keywords: analytical mechanics, differential constraint, symmetry, conserved quantity
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1. Introduction

Researches on the conserved quantities of mechanical systems possess not only important mathematical significance, but also profound physical background. The symmetry is an invariance. The invariance method of mechanical systems is a modern method to study the conserved quantity. There are the Noether symmetry, the Lie symmetry, the adjoint symmetry, the quasi-symmetry and so on.[1−11] The Noether theorem associates one conserved quantity to each Noether symmetry and the Lie symmetry does not imply a conserved quantity. If the generators of the Lie symmetry satisfy the structure equation (i.e. Noether identity), then the Lie symmetry can lead to a conserved quantity.[1,3,5,7] Hojman [12] and Lutzky [13] have given a direct method to find a conserved quantity. Pillay and Leach have demonstrated that the invariant derived by using Hojman's conservation law is trivial for all Noether symmetries.[14] Recently, Zhang Yi extended Hojman's conservation law to the system of generalized classical mechanics.[15] In this paper, we generalize the use of Hojman's method to mechanical systems with differential constraints and obtain nontrivial conserved quantities.

2. Equations of motion of systems

Let the position of a mechanical system be determined by n generalized coordinates q_s $(s = 1, \ldots, n)$ and its motion be subjected to g ideal bilateral differential constraints

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \qquad (\beta = 1, \ldots, g). \tag{1}$$

If the constraints (1) are integrable, then the system is a holonomic system with redundant coordinates; if the constraints (1) are not integrable, then the system is a nonholonomic system of Chetaev type. To avoid the loss of symmetry, an integrable differential system can be considered as a nonholonomic system.[16].

For the system with the constraints (1), whether it is integrable or not, the equations of motion of the system can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \qquad (s = 1, \ldots, n), \tag{2}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian of the system, Q_s are the generalized forces and λ_β the multipliers of constraint. Suppose that the system is not singular, i.e.

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0,$$

then we can obtain all the accelerations as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \qquad (s = 1, \ldots, n). \tag{3}$$

3. Symmetry and conserved quantity

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Introduce the infinitesimal transformations of time and coordinates as

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \triangle q_s \quad (s = 1, \ldots, n), \quad (4)$$

or their expanded form

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (5)$$

where ε is an infinitesimal parameter, ξ_s are the infinitesimal generators.

The invariance of the differential equations (3) under the infinitesimal transformations (5) leads to the satisfaction of the following determining equations:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \quad (s, k = 1, \ldots, n), \quad (6)$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}. \quad (7)$$

The invariance of the differential constraints (1) under the infinitesimal transformations (5) leads to the satisfaction of the following restriction equations:

$$\frac{\partial f_\beta}{\partial q_s}\xi_s + \frac{\partial f_\beta}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = 0. \quad (8)$$

We have

Criterion 1 If the infinitesimal generators ξ_s satisfy the determining equations (6), then the corresponding symmetry is a Lie symmetry of the system (3).

Criterion 2 If the infinitesimal generators ξ_s satisfy the determining equations (6) and the restriction equations (8), then the corresponding symmetry is a Lie symmetry of the system [(1),(2)] with differential constraints.

We can demonstrate the following results.

Proposition 1 Under Criterion 1, if α_s satisfy the following condition:

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} = 0, \quad (9)$$

then the system (3) possesses the following conserved quantity:

$$I = \frac{\partial \xi_s}{\partial q_s} + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const.} \quad (10)$$

If there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\ln \mu) = 0, \quad (11)$$

then the system (3) possesses the following conserved quantity:

$$I = \frac{1}{\mu}\frac{\partial(\mu\xi_s)}{\partial q_s} + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const.} \quad (12)$$

Proof We have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} \\ &+ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right], \quad (13)\end{aligned}$$

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) =& \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ &- \frac{\partial \alpha_k}{\partial \dot{q}_s}\frac{\partial}{\partial \dot{q}_k}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} =& \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \alpha_k}{\partial q_s}\frac{\partial \xi_s}{\partial \dot{q}_k}. \quad (14)\end{aligned}$$

Taking the partial derivative of Eqs.(6) with respect to $\dot{q}_s$ and the sum over s, we have

$$\frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial q_k}\xi_k\right) + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right). \quad (15)$$

Substituting Eqs.(14) and (15) into Eqs.(13), we obtain

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I =& \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ &+ \frac{\partial^2 \alpha_s}{\partial q_k \partial \dot{q}_s}\xi_k + \frac{\partial^2 \alpha_s}{\partial \dot{q}_k \partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \quad (16)\end{aligned}$$

Substituting Eq.(11) into Eq.(16) and eliminating $\frac{\partial \alpha_s}{\partial \dot{q}_s}$, we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I =& \left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\right) - \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\ln \mu)\right\}\xi_s \\ &+ \left\{\frac{1}{\mu}\frac{\partial \mu}{\partial q_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\right) - \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\ln \mu)\right\}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s \\ &+ \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ =& \frac{1}{\mu}\left\{\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial \mu}{\partial \dot{q}_k}\frac{\partial \alpha_k}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s \right. \\ &\left. - \frac{\partial \mu}{\partial \dot{q}_k}\frac{\partial \alpha_k}{\partial q_s}\xi_s\right\} \\ =& 0.\end{aligned}$$

In Ref.[12], Hojman has taken $\mu = \mu(\boldsymbol{q})$.

Proposition 2 Under Criterion 2, if α_s satisfy the condition (9), then the system [(1),(2)] with differential constraints possesses the conserved quantity (10); if there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that the condition (11) holds, then the system [(1),(2)] with differential constraints possesses the conserved quantity (12).

4. Illustrative example

Example The Lagrangian of a system is

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}\omega^2 r^2 \quad (\omega = \text{const.}) \tag{17}$$

and the integrable differential constraint is

$$f = \frac{\dot{r}}{r} - C = 0 \quad (C = \text{const}). \tag{18}$$

Equations (2) give

$$\ddot{r} - r\dot{\theta}^2 + \omega^2 r = \frac{\lambda}{r},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0.$$

Equations (3) become

$$\ddot{r} = \frac{\dot{r}^2}{r} \equiv \alpha_r,$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} \equiv \alpha_\theta.$$

Equations (6) and (8) give

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r\right) = -\frac{\dot{r}^2}{r^2}\xi_r + \frac{2\dot{r}}{r}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r,\\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\theta\right) = \frac{2\dot{r}\dot{\theta}}{r^2}\xi_r - \frac{2\dot{\theta}}{r}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r - \frac{2\dot{r}}{r}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\theta,\\
&-\frac{\dot{r}}{r^2}\xi_r + \frac{1}{r}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_r = 0.
\end{aligned} \tag{19}$$

Equations (19) have a solution

$$\xi_r = r^3\dot{\theta}, \quad \xi_\theta = 0. \tag{20}$$

It is verifiable that the condition (9) holds and the conserved quantity (10) gives

$$I = 4r^2\dot{\theta} = \text{const.} \tag{21}$$

The generators (20) are non-Noether.

5. Conclusion

The results in this paper are generalizations of a theorem of Hojman on the non-Noether conserved quantity to the mechanical systems with integrable or non-integrable differential constraints. The result in Ref.[12] is a special case of this paper. In the above example, the symmetry is non-Noether, but it also possesses the corresponding conserved quantity.

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