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Bound states of the Klein–Gordon equation for ring-shaped Kratzer-type potential

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The exact normalized bound state wavefunctions and energy equations of Klein–Gordon equation with equal scalar and vector ring-shaped Kratzer-type potential have been obtained.

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1. Introduction

It is well known that when a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics. Except for a few examples, such as the hydrogen atom and electrons in a uniform magnetic field, perhaps because of the difficulties in mathematics, in relativistic quantum mechanics the problems that can be exactly solved for Klein–Gordon or Dirac equations are few. In recent years, some researchers, by assuming that the scalar potential is equal to the vector potential, solved the Klein–Gordon and also the Dirac equations, then obtained the s-wave bound-state solution of particles in some typical potential fields.[1−8] We also solved Klein–Gordon and Dirac equations for some equal scalar and vector central potentials and obtained complete solutions instead of only s-wave solutions.[9,10] Shen solved Liouville equation for some typical potential.[11] It is obvious that the studies mentioned above are only aiming at central potential fields. On the other hand, some authors solved the ring-shaped central potentials,[12−16] but for non-relativistic quantum mechanics. We also solved the Klein–Gordon equation for equal scalar and vector potentials in ring-shaped harmonic oscillators.[17] In this paper we shall solve the Klein–Gordon equation for the ring-shaped Kratzer-type potential. The so-called ring-shaped Kratzer-type potential is a combination of a Kratzer-type potential and a ring-shaped potential. A Kratzer-type potential is also a combination of a Coulomb potential with an inverse square potential, but a ring-shaped potential can be in the form of $C\sec^2\theta/r^2$ or $C\csc^2\theta/r^2$.

The Klein–Gordon equation with equal scalar and vector potentials is ($\hbar = c = 1$)

$$[\boldsymbol{P}^2 - (E - V(\boldsymbol{r})/2)^2]\psi = -(\mu + V(\boldsymbol{r})/2)^2\psi, \quad (1)$$

where $\boldsymbol{P}$ is the momentum operator, and E and μ are the energy and rest mass of a particle respectively. For a ring-shaped potential

$$V(\boldsymbol{r}) = V(r) + \frac{V(\theta)}{r^2} + \frac{V(\phi)}{r^2\sin^2\theta}, \quad (2)$$

we separate variables in Eq.(1), and obtain three equations:

$$K''(\phi) - \left[\alpha_2^2\, V(\phi) - m^2\right] K(\phi) = 0, \quad (3)$$

$$H''(\theta) + \cot(\theta)H'(\theta) - [m^2\csc^2(\theta) + \alpha_2^2\, V(\theta) - s(s+1)]H(\theta) = 0, \quad (4)$$

$$(r^2R'(r))' - \left[s(s+1) + \alpha_2^2\,(\alpha_1^2 + V(r))\, r^2\right] R(r) = 0, \quad (5)$$

where m and s are separating-variable constants, and

$$\alpha_1 = \sqrt{\mu - E}, \qquad \alpha_2 = \sqrt{\mu + E}. \quad (6)$$

For $V(\phi) = 0$ Eq.(3) becomes

$$K''(\phi) + m^2K(\phi) = 0, \quad (7)$$

which has the solution satisfying periodic boundary conditions

$$K(\phi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\phi} \quad m = 0, \pm1, \pm2, \ldots \quad (8)$$

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In Secs.2 and 3, we focus our attention on the potentials including a ring-shaped potential $C\sec^2\theta/r^2$ and give the corresponding wavefunction and energy equation. In Sec.4 we briefly give the main result for the potential containing the term $C\csc^2\theta/r^2$. We will discuss the non-relativistic limit of our part result in the last section.

2. The solution of $\theta-$dimensional equation

For the potentials

$$V(r)=\frac{A}{r^2}-\frac{B}{r},V(\phi)=0,\quad V(\theta)=C\sec^2\theta, \tag{9}$$

Eq.(4) becomes

$$\begin{aligned}&H''(\theta)+\cot(\theta)\,H'(\theta)\\&+[s\,(s+1)-m^2\,\csc^2(\theta)\\&-C\,\alpha_2^2\,\sec^2(\theta)]H(\theta)=0.\end{aligned} \tag{10}$$

In order to solve Eq.(10), letting

$$\begin{aligned}&x=\cos^2(\theta),\qquad \nu=\sqrt{1+4\,C\alpha_2^2},\\&H(\theta)=(\cos(\theta))^{\frac{1}{2}(1+\nu)}\sin^{|m|}(\theta)\,f(x),\end{aligned} \tag{11}$$

we obtain

$$\begin{aligned}&(1-x)xf''(x)+\left[1+\frac{\nu}{2}-\left(2+|m|+\frac{\nu}{2}\right)x\right]f'(x)\\&-\frac{1}{4}\left[(|m|+1)^2-s(s+1)\right.\\&\left.-\frac{1-\nu^2}{4}+(|m|+1)\nu\right]f(x)=0,\end{aligned} \tag{12}$$

which is a hypergeometric equation and its solution is [18]

$$f(x)=\,{}_2F_1(\alpha,\beta,\gamma,x), \tag{13}$$

where ${}_2F_1(\alpha,\beta,\gamma,x)$ is a hypergeometric function and

$$\begin{aligned}&\alpha=\frac{1}{4}(1+2\,|m|-2\,s+\nu),\\&\beta=\frac{1}{4}(3+2\,|m|+2\,s+\nu),\\&\gamma=1+\frac{\nu}{2}.\end{aligned} \tag{14}$$

In order that $f(x)$ is finite in the range $[0,1]$, $f(x)$ must terminate as a polynomial. This demands that α or β be equal to zero or negative integers. But if

$$\beta=\frac{1}{4}(3+2\,|m|+2\,s+\nu)=-k,\quad k=0,1,2,\ldots, \tag{15}$$

then $s=-\left(2\,k+|m|+\nu+\frac{3}{2}\right)<0$. It is easy to see that when there is no ring-shaped potential, i.e. $C=0$, the potential turns into a central potential. The corresponding solution of Eq.(10) is an associated Legendre function $P_s^{|m|}(\cos\theta)$. In this situation s must be positive integer. So generally, we can only take

$$\alpha=\frac{1}{4}(1+2\,|m|-2\,s+\nu)=-k,\quad k=0,1,2,3,\ldots, \tag{16}$$

and obtain

$$\beta=1+|m|+k+\frac{\nu}{2},\quad s=2k+|m|+\frac{1+\nu}{2}. \tag{17}$$

Finally, we have the normalized wavefunction of θ-dimension

$$\begin{aligned}H(\theta)=&C_{k,m}(\cos(\theta))^{\frac{1}{2}(1+\nu)}\sin^{|m|}(\theta)\\&\times\,{}_2F_1\left[-k,1+|m|+k+\frac{\nu}{2},1+\frac{\nu}{2},\cos^2(\theta)\right],\end{aligned} \tag{18}$$

where the normalization constant is

$$\begin{aligned}C_{k,m}=&\frac{1}{\sqrt{S_{k,m}}},\\S_{k,m}=&\frac{|m|!\,[1-(-1)^\nu]}{2}\\&\times\sum_{q=0}^{k}\frac{(-k)_q\left(1+|m|+k+\frac{\nu}{2}\right)_q\,\Gamma\left(1+q+\frac{\nu}{2}\right)}{q!\left(1+\frac{\nu}{2}\right)_q\,\Gamma\left(2+|m|+q+\frac{\nu}{2}\right)}\\&\times\,{}_pF_q\left[-k,1+q+\frac{\nu}{2},1+|m|+k+\frac{\nu}{2},\right.\\&\left.1+\frac{\nu}{2},2+|m|+q+\frac{\nu}{2},1\right],\end{aligned} \tag{19}$$

and $(\lambda)_n$ is the Pochhammer symbol; ${}_pF_q$ stands for the general hypergeometric series.[18]

3. The solution of the r-dimensional equation

If we substitute the first formula of Eq.(9) into Eq.(5), we obtain

$$\begin{aligned}&\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR(r))-\left[\frac{s(s+1)}{r^2}+(\alpha_1\,\alpha_2)^2\right.\\&\left.+{\alpha_2}^2(\frac{A}{r^2}-\frac{B}{r})\right]R(r)=0.\end{aligned} \tag{20}$$

This equation is the same as Eq.(3) of Ref.[10] in form and s in Eq.(20) corresponds to l in Eq.(3) of Ref.[10]. So we can directly write down the solution of Eq.(20) and obtain the energy equation as follows:

$$R(r)=C\,r^{\frac{\delta-1}{2}}\,\mathrm{e}^{-\alpha_1\,\alpha_2\,r}\,F[-n,1+\delta,2\,\alpha_1\,\alpha_2\,r], \tag{21}$$

where the normalization constant is

$$C_{n,k}=\frac{(2\,\alpha_1\,\alpha_2)^{\frac{\delta}{2}+1}}{\Gamma(1+\delta)}\sqrt{\frac{\Gamma(1+n+\delta)}{(1+2\,n+\delta)\,\Gamma(1+n)}}, \quad (22)$$

and

$$\begin{aligned}\delta =&\sqrt{(2\,s+1)^2+4\,A\,\alpha_2^2}\\ =&\sqrt{(4\,k+2|m|+2+\nu)^2+4\,A\,\alpha_2^2}. \end{aligned}\quad (23)$$

The energy equation is

$$\begin{aligned}&\left(1+2\,n+\sqrt{(2\,s+1)^2+4\,A\,(\mu+E)}\right)\\ &\times\sqrt{\mu-E}-B\,\sqrt{\mu+E}=0. \end{aligned}\quad (24)$$

Recalling that s is given by the second formula of Eq.(17), we can alternatively write the energy equation as

$$\left(1+2\,n+\sqrt{(2\,l'+\sqrt{1+4\,C(\mu+E)})^2+4\,A\,(\mu+E)}\right)\sqrt{\mu-E}-B\,\sqrt{\mu+E}=0, \quad (25)$$

where

$$l'=2\,k+|m|+1=1,2,3,\ldots \quad (26)$$

4. The solution for ring-shaped potential $C\csc^2(\theta)/r^2$

When $V(r)$ and $V(\phi)$ in Eq.(9) are unchanged, but the ring-shaped potential becomes

$$V(\theta)=\frac{C\csc^2\theta}{r^2}, \quad (27)$$

Eq.(4) turns into

$$\begin{aligned}&H''(\theta)+\cot(\theta)H'(\theta)\\ &+[s(s+1)-(m^2+C\alpha_2^2)\csc^2(\theta)]H(\theta)=0. \end{aligned}\quad (28)$$

Except that here we have used some different notation, this equation is the same as Eq.(9) of Ref.[17]. So we can directly obtain the solution of Eq.(28) from Ref.[17],

$$\begin{aligned}H_k(\theta)=&C_{k,m}\sin^{\nu}(\theta)\\ &\times{}_2F_1[-k,k+\nu+1/2,1+\nu,\sin^2(\theta)], \end{aligned}\quad (29)$$

where

$$\begin{aligned}C_{k,m}=&\frac{1}{2^{\nu+1/2}\Gamma(1+\nu)}\\ &\times\sqrt{\frac{(1+4\,k+2\,\nu)\Gamma(1+2\,k+2\,\nu)}{(2\,k)!}},\\ \nu=&\sqrt{m^2+4\,C\alpha_2^2},\\ s=&2\,k+\nu,\qquad k=0,1,2,\ldots \end{aligned}\quad (30)$$

On the other hand, the radial equation for this ring-shaped potential is unchanged as in Eq.(20) and also is the same as Eq.(3) of Ref.[10] except that l is replaced by s. So, if we replace l by s in Eqs.(15) and (16) of Ref.[10], we can easily obtain the solution of the radial equation for this ring-shaped potential,

$$R(r)=C\,r^{\frac{\sigma-1}{2}}\,\mathrm{e}^{-\alpha_1\,\alpha_2\,r}\,F[-n,1+\sigma,2\,\alpha_1\,\alpha_2\,r], \quad (31)$$

where F represents the confluent hypergeometric function $F[\alpha,\gamma,z]$,[18] and the normalization constant is

$$\begin{aligned}C_{n,k}=&\frac{(2\,\alpha_1\,\alpha_2)^{\frac{\sigma}{2}+1}}{\Gamma(1+\sigma)}\sqrt{\frac{\Gamma(1+n+\sigma)}{n!(1+2\,n+\sigma)}}, \end{aligned}\quad (32)$$

$$\begin{aligned}\sigma=&\sqrt{(2\,s+1)^2+4\,A\,\alpha_2^2}\\ =&\sqrt{(4\,k+2\,\nu+1)^2+4A\alpha_2^2}. \end{aligned}\quad (33)$$

The energy equation is

$$\left(1+2\,n+\sqrt{(4\,k+1+2\sqrt{m^2+4\,C(\mu+E)})^2+4\,A\,(\mu+E)}\right)\sqrt{\mu-E}-B\,\sqrt{\mu+E}=0. \quad (34)$$

When there is no ring-shaped potential, i.e. $C=0$, the potential becomes the one we have researched in Ref.[10], and $\nu=|m|$. Therefore, if we let

$$l=2k+|m|=0,1,2,\ldots, \quad (35)$$

Eq.(34) becomes

$$\begin{aligned}&(1+2\,n+\sqrt{(2\,l+1)^2+4\,A\,(\mu+E)})\\ &\times\sqrt{\mu-E}-B\,\sqrt{\mu+E}=0. \end{aligned}\quad (36)$$

This is just the Eq.(11) of Ref.[10].

5. Discussion

In the above sections, we have exactly solved the Klein–Gordon equation for two kinds of ring-shaped Kratzer-type potential. At this stage it is interesting to analyse the non-relativistic limit of our result.

When $A = 0$, the two potentials discussed above degenerate into the ring-shaped Coulomb potential. Reference [12] had solved Schrödinger's equation for the ring-shaped Coulomb potential

$$V = \left(\frac{\hbar^2}{2\,\mu}\right)\frac{b(b+1)}{r^2}\sec^2\theta - \frac{Ze^2}{r}, \qquad (37)$$

and obtained the energy equation

$$W_n = -\frac{\mu\, e^4\, Z^2}{\hbar^2\,\sigma^2}, \quad \sigma = b+1, b+2, \ldots, b+n, \ldots, \qquad (38)$$

where W_n represents the kinetic energy of a particle. In order to be in accord with our notation, we use μ, instead of M, to express the rest mass. We have resolved this problem and find that there is a little mistake. The energy we found is

$$W_n = -\frac{\mu\, e^4\, Z^2}{2\hbar^2\,\sigma^2}, \qquad \sigma = b+2, b+3, \ldots. \qquad (39)$$

Now we will show that if one takes some proper approximation, one would obtain Eq.(39) from Eq.(25). For this purpose, we let $A = 0$ in Eq.(25) and use α_1 to express the relativistic energy equation for the ring-shaped Coulomb potential as

$$(1 + 2\,n + 2\,l' + \sqrt{1 + 4\,C(2\,\mu - \alpha_1^2)})\alpha_1 - B\sqrt{2\,\mu - \alpha_1^2} = 0, \qquad (40)$$

where $2\,l' + \sqrt{1 + 4\,C(\mu + E)} > 0$ has been used. We expand the left-hand side of Eq.(40) as a series of α_1:

$$-B\sqrt{2\,\mu} + (2\,n + 2\,l' + 1 + \sqrt{1 + 8\,C\,\mu})\alpha_1 + \frac{B}{2\sqrt{2}\,\mu}\alpha_1^2 + O(\alpha_1)^3 = 0. \qquad (41)$$

If we only take the term of α_1, which is not higher than one, we obtain

$$\alpha_1 = \frac{\sqrt{2\,\mu}\,B}{2\,n + 2\,l' + 1 + \sqrt{1 + 8\,C\,\mu}}. \qquad (42)$$

Now, if we make the substitution $C \to b\,(1 + b)/(2\,\mu)$ and $B \to e^2\,Z\,(\hbar = 1)$, then we finally obtain

$$W_n = E - \mu = -\alpha_1^2 = -\frac{\mu\, e^4\, Z^2}{2\,\sigma^2}, \quad \sigma = n + l' + 1 + b = b + 2, b + 3, \ldots, \qquad (43)$$

which is consistent with Eq.(39).

Finally, we point out that we have also solved Schrödinger's equation for the ring-shaped Coulomb potential

$$V = \left(\frac{\hbar^2}{2\,\mu}\right)\frac{b(b+1)}{r^2}\csc^2\theta - \frac{Ze^2}{r}, \qquad (44)$$

and find the energy expression to be

$$W_n = -\frac{\mu\, e^4\, Z^2}{2\hbar^2\,(\sqrt{b(b+1) + |m|^2} + l)^2}, \qquad l = 1, 2, \ldots \qquad (45)$$

Similarly, if one takes some proper approximation, one would obtain Eq.(45) from Eq.(34).

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