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Bound states of the Klein–Gordon and Dirac equations for potential $V_0 \tanh^2(r/d)$

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The exact bound state wavefunctions and energy equations of Klein–Gordon and Dirac equations are given with equal scalar and vector potential $s(r) = v(r) = V(r)/2 = V_0 \tanh^2(r/d)$. The relation between the energy equation and that of relativistic harmonic is discussed.

Keywords: $V_0 \tanh^2(r/d)$ potential, Klein–Gordon equation, Dirac equation, bound state
PACC: 0300, 0365

1. Introduction

It is well known that central force field is a class of solvable problems in quantum mechanics. Typical examples of this kind of problem are the hydrogen atom and the three-dimensional harmonic oscillator, both of which have been solved completely in the non-relativistic quantum mechanics.[1] In the relativistic quantum mechanics the hydrogen atom has also been solved successfully and its result has explained the fine structure of the hydrogen atom.[2] Yet the problems that can be exactly solved are rare in the relativistic quantum mechanics. In the relativistic quantum mechanics in order to solve problems which have been solved in the non-relativistic quantum mechanics, some authors have assumed that the scalar potential is equal to the vector potential and solved the Klein–Gordon and Dirac equations for some typical central potential. For example, we have solved the Klein–Gordon and Dirac equations and obtained the normalized wavefunction and energy equation for the relativistic harmonic oscillator.[3] We have also obtained the bound state solution for a combined potential of a Coulomb potential and a inverse square potential.[4] On the other hand, there are a number of potentials, such as the Hulthen potential,[5,6] Morse potential,[7] Wood–Saxon potential,[8] $\tan^2(\pi\eta r)$ potential,[9] reflectionless type potential, etc.,[10,11] which have also been solved in relativistic quantum mechanics. But unfortunately, for these kinds of potentials, one can only obtain the s wave ($l = 0$ for Klein–Gordon and $K = 0$ for Dirac equations) bound state solutions. In this paper we shall solve the Klein–Gordon and Dirac equations for $V_0 \tanh^2(r/d)$ potential[12] and discuss the relation between this potential and the three-dimensional harmonic oscillator in the following sections.

2. Bound-state solution of the Klein–Gordon equation with scalar and vector potentials equal to each other

According to Ref.[2], the Klein–Gordon equation for equal scalar and vector potential is ($\hbar$=c=1)

$$\left[\boldsymbol{P}^2 - \left(E - \frac{V(r)}{2}\right)^2\right]\psi = -\left(\mu + \frac{V(r)}{2}\right)^2\psi, \quad (1)$$

where $\boldsymbol{P}$ is the momentum operator, and E and μ are the energy and rest mass of the particle, respectively. In spherical coordinates, the radial part of Eq.(1) is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR(r)) - \left[\frac{l(l+1)}{r^2} + \alpha_2(\alpha_1 + V(r))\right]R(r) = 0, \quad (2)$$

where

$$\alpha_1 = \mu - E, \qquad \alpha_2 = \mu + E. \quad (3)$$

The radial wavefunction $R(r)$ must satisfy the boundary condition that $R(r)$ becomes zero when $r \to \infty$,

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and $R(r)$ is finite at $r=0$. Now letting

$$x = r/d, \qquad R(r) = \frac{F(x)}{r}, \tag{4}$$

for the potential

$$V(r) = V_0 \tanh^2(r/d) = V_0 \tanh^2(x) \tag{5}$$

and s wave ($l=0$), we obtain the equation of $F(x)$,

$$F''(x) - d^2\,\alpha_2\,(\alpha_1 + V_0 \tanh^2(x))\,F(x) = 0, \tag{6}$$

where F' stands for $\partial F/\partial x$. Obviously, only when $x \to \infty, F(x)$ is finite and $F(x) = 0$ at point $x = 0(r=0)$, can $R(r)$ satisfy the above boundary conditions. Now if we introduce

$$\begin{aligned} z &= \frac{1-\tanh(x)}{2}, \\ \gamma &= \sqrt{\alpha_2\,(\alpha_1+V_0)}, \\ F(x) &= [\cosh(x)]^{-d\gamma} f(z), \end{aligned} \tag{7}$$

then we obtain the equation of $f(z)$,

$$\begin{aligned} &(1-z)\,z\,f''(z) + (1-2\,z)\,(1+d\,\gamma)\,f'(z) \\ &- d\,(d\,\alpha_1\,\alpha_2+\gamma)\,f(z) = 0. \end{aligned} \tag{8}$$

This is a hypergeometric equation,[13] and its solution, which can satisfy the boundary conditions, is

$$f(z) = {}_2F_1(a,b,c,z), \tag{9}$$

where ${}_2F_1$ represents the hypergeometric function, and

$$\begin{aligned} a &= \frac{1}{2} + d\,\gamma - \frac{1}{2}\sqrt{(1+2\,d\,\gamma)^2 - 4\,(d\,\alpha_1\,\alpha_2+\gamma)}, \\ b &= \frac{1}{2} + d\,\gamma + \frac{1}{2}\sqrt{(1+2\,d\,\gamma)^2 - 4\,(d\,\alpha_1\,\alpha_2+\gamma)}, \\ c &= 1 + d\,\gamma. \end{aligned} \tag{10}$$

In order to use the boundary condition of $f(z)$ at $x=0$, we can use the relation between hypergeometric function and Gegenbauer function [13],

$$\begin{aligned} C_\alpha^\nu(z) =& \Gamma(\alpha+2\nu)\Gamma(\alpha+1)\Gamma(2\nu)^{-1} \\ &\times {}_2F_1\left(\alpha+2\nu, -\alpha, \nu+\frac{1}{2}, \frac{1}{2}-\frac{1}{2}z\right) \end{aligned} \tag{11}$$

to express the solution of Eq.(8) as

$$f(x) = \frac{\Gamma(1+2\,d\,\gamma)}{\Gamma[-a+2\,d\gamma)]\,\Gamma[-a]} C_{-a}^{\frac{1}{2}+d\gamma}(\tanh(x)). \tag{12}$$

Because only when α is an integer can $C_\alpha^\nu(z)$ become a polynomial, Gegenbauer or ultraspherical polynomial, and [13]

$$C_n^\nu(0) = \begin{cases} 0, & n \text{ odd} \\ (-1)^m \dfrac{\Gamma(\nu+m)}{\Gamma(\nu)\Gamma(m+1)} & n \text{ even and } n=2m, \end{cases} \tag{13}$$

so $-a$ must be odd integers, i.e.

$$\begin{aligned} a =& \frac{1}{2} + d\gamma - \frac{1}{2}\sqrt{(1+2\,d\,\gamma)^2 - 4\,(d\,\alpha_1\,\alpha_2+\gamma)} \\ =& -(2\,n+1), \qquad (n=0,1,2,\cdots). \end{aligned} \tag{14}$$

Now we can completely write down the radial wavefunction $R(r)$ as (unnormalized)

$$\begin{aligned} R(r) =& \frac{1}{r}\,[\cosh(r/d)]^{-d\gamma} \\ &\times {}_2F_1\Big(-(2\,n+1), 2\,(n+1+d\,\gamma), \\ &1+d\,\gamma, \frac{1-\tanh(r/d)}{2}\Big) \\ =& \frac{\Gamma(1+2\,d\,\gamma)}{(2\,n+1)!\,\Gamma[2(n+1+d\gamma)]}\frac{1}{r}[\cosh(r/d)]^{-d\gamma} \\ &\times C_{2\,n+1}^{\frac{1}{2}+d\gamma}(\tanh(r/d)). \end{aligned} \tag{15}$$

3. Bound-state solution of the Dirac equation with scalar and vector potentials equal to each other

The Dirac equation with scalar potential $s(r)$ and vector potential $v(r)$ is ($\hbar$=c=1)

$$\{\alpha \cdot \boldsymbol{P} + \beta[\mu + s(r)]\}\psi = [E - v(r)]\psi. \tag{16}$$

In relativistic quantum mechanics, the complete set of the conservative quantities of a particle in a central field can be taken as (H, K, J^2, J_z), the eigenfunctions of which are [14]

$$\psi = \frac{1}{r}\begin{bmatrix} F(r)\phi^A_{jm_j} \\ \mathrm{i}G(r)\phi^B_{jm_j} \end{bmatrix} \qquad (K = j+1/2), \tag{17}$$

$$\psi = \frac{1}{r}\begin{bmatrix} F(r)\phi^B_{jm_j} \\ \mathrm{i}G(r)\phi^A_{jm_j} \end{bmatrix} \qquad (K = -(j+1/2)), \tag{18}$$

where

$$\phi^A_{jm_j} = \frac{1}{\sqrt{2\,l+1}}\begin{bmatrix} \sqrt{l+m+1}\,Y_{l,m} \\ \sqrt{l-m}\,Y_{l,m+1} \end{bmatrix},$$

$$\phi^B_{jm_j} = \frac{1}{\sqrt{2\,l+3}}\begin{bmatrix} -\sqrt{l-m+1}\,Y_{l+1,m} \\ \sqrt{l+m+1}\,Y_{l+1,m+1} \end{bmatrix}. \tag{19}$$

Substituting Eq.(17) or (18) into Eq.(16) we obtain the radial part of the Dirac equation as

$$\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{K}{r}F = [\mu + E + s(r) - v(r)]\,G,$$
$$\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{K}{r}G = [\mu - E + s(r) + v(r)]\,F. \qquad (20)$$

Taking $s(r) = v(r) = \frac{1}{2}V_0\ \tanh^2(r/d)$ in Eq.(20) leads to the following equations:

$$\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{K}{r}F = (\mu + E)\,G,$$
$$\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{K}{r}G = \left(\mu - E + V_0\ \tanh^2(r/d)\right)F. \qquad (21)$$

Eliminating G from Eq.(21), we obtain

$$F''(r) - \frac{K\,(K-1)}{r^2}F(r) - \alpha_2\left(\alpha_1 + V_0\ \tanh^2(r/d)\right)F(r) = 0, \qquad (22)$$

where α_1 and α_2 are the same as in Eq.(3). For the s wave, i.e. $K = 1$, Eq.(22) becomes

$$F''(r) - \alpha_2\left(\alpha_1 + V_0\ \tanh^2(r/d)\right)F(r) = 0. \qquad (23)$$

If we change variable r to $x = r/d$, we obtain

$$F''(x) - d^2\,\alpha_2\,(\alpha_1 + V_0\tanh^2(x))\,F(x) = 0. \qquad (24)$$

This equation is identical with Eq.(6), and of course the solution is the same as that of Eq.(6). So the solution of Eq.(23) is

$$\begin{aligned} F(r) =& [\cosh(r/d)]^{-d\gamma} \\ & \times {}_2F_1\Big(-(2n+1), 2\,(n+1+d\gamma), \\ & 1 + d\gamma, \frac{1-\tanh(r/d)}{2}\Big) \\ =& \frac{\Gamma(1+2\,d\gamma)}{(2n+1)!\,\Gamma[2(n+1+d\gamma)]}\,[\cosh(r/d)]^{-d\gamma} \\ & \times C^{\frac{1}{2}+d\gamma}_{2n+1}(\tanh(r/d)). \end{aligned} \qquad (25)$$

Now, we can substitute $F(r)$ into the first equation of Eq.(21) and keeping in mind that $K = 1$, we obtain

$$\begin{aligned} G(r) =& -\frac{1}{2\,\alpha_2}[\cosh(r/d)]^{-d\gamma}\Big\{\frac{2}{r} \times {}_2F_1\Big[-(2n+1), 2\,(n+1+d\gamma), 1+d\gamma, \frac{1-\tanh(r/d)}{2}\Big] \\ & - \frac{2(1+2n)(1+n+d\gamma)}{d(1+d\gamma)\cosh^2(r/d)} \times {}_2F_1\Big[-2n, 2n+3+2\,d\gamma,\ 2+d\gamma, \frac{1-\tanh(r/d)}{2}\Big] \\ & + 2\,\gamma\,\tanh(r/d) \times {}_2F_1\Big[-(2n+1), 2\,(n+1+d\gamma), 1+d\gamma, \frac{1-\tanh(r/d)}{2}\Big]\Big\} \\ =& \frac{\Gamma(1+2\,d\gamma)}{d\,\alpha_2(2n+1)!\,\Gamma[2(n+1+d\gamma)]}[\cosh(r/d)]^{-(2+d\gamma)} \times \Big\{(1+2\,d\gamma)C^{3/2+d\gamma}_{2n}(\tanh(r/d)) \\ & - d\,\cosh(r/d)\left[\frac{1}{r}\,\cosh(r/d) + \gamma\sinh(r/d)\right]C^{1/2+d\gamma}_{2n+1}(\tanh(r/d))\Big\}. \end{aligned} \qquad (26)$$

Substituting $F(r)$ and $G(r)$ into Eqs.(17) and (18) yields the spinor wavefunction of the Dirac equation.

4. Energy equation and discussion

Because Eqs.(6) and (24) are the same, for the s wave, the Klein–Gordon and Dirac equations give the same energy equation. From Eqs.(14), (3) and (7), we obtain the energy equation

$$\begin{aligned} & 4n + 3 + 2\,d\sqrt{(\mu+E)(\mu-E+V_0)} \\ =& \sqrt{-4\,[d\,(\mu^2 - E^2) + \sqrt{(\mu+E)(\mu-E+V_0)}] + [1 + 2d\sqrt{(\mu+E)(\mu-E+V_0)}]^2}. \end{aligned} \qquad (27)$$

If we square the two sides of above equation, we can the express energy equation in a simpler form:

$$2\,(n+1)\,(2\,n+1) + d\,(\mu^2 - E^2) + [1 + 2\,d(2\,n+1)]\,\sqrt{(V_0 + \mu - E)\,(\mu+E)} = 0. \qquad (28)$$

In Eqs.(27) and (28), n are non-negative integers and their values must ensure $0 \le E - \mu \le V_0$. Although we can exactly solve Eq.(27) or (28) for E, the expression of E is too complicated to see its meaning. So we discuss the energy equation as follows.

If we expand the potential $V_0\ \tanh^2(r/d)$ as the power series of r, we have

$$V(r) = \frac{V_0\,r^2}{d^2} - \frac{2\,V_0\,r^4}{3\,d^4} + O(r)^6. \qquad (29)$$

When a particle moves in the bottom of the potential well, its radial coordinate r is far smaller than the characteristic parameter d. So in this case, d is a large quantity. If we only take the first term in the above

expansion, and identify it with a harmonic oscillator potential, then we obtain the relations of the proper frequency or elastic coefficient of a harmonic oscillator and the maximum V_0 of the potential as follows:

$$V_0 = \frac{1}{2}\mu\, d^2\, \omega^2 = \frac{1}{2}\, d^2\, k. \tag{30}$$

On the other hand, for a particle moving in the bottom of the potential well, its kinetic energy $E_k = E - \mu = -\alpha_1$ is also far smaller than the maximum V_0 of the potential. So compared with $|\alpha_1|$, V_0 is also a large quantity. The above analysis tells us that it is essential for our following discussion that V_0 and d are large. In order to further investigate the relation between potential $V_0 \tanh^2(r/d)$ and the harmonic oscillator potential, let us review the main result of Ref.[3]. The energy equations of a relativistic harmonic oscillator are

$$\begin{array}{ll} (4\,n + 2\,l + 3)\sqrt{k} - (E-\mu)\,\sqrt{2\,(\mu+E)} = 0 & (\text{for Klein}--\text{Gordon}) \\ (4\,n + 2\,K + 1)\sqrt{k} - (E-\mu)\,\sqrt{2\,(\mu+E)} = 0 & (\text{for Dirac}) \end{array} \quad (n = 0, 1, 2, \cdots). \tag{31}$$

For the s wave, i.e. $l = 0$ or $K = 1$, the above two equations have the same form:

$$(4\,n+3)\,\sqrt{k} - (\mu - E)\,\sqrt{2\,(\mu+E)} = 0$$

$$(n = 0, 1, 2, \cdots), \tag{32}$$

where we have expressed the rest mass of the particle by μ instead of m.

In the following, we will show that if we keep in mind that V_0 and d are large and make some approximation, we shall obtain Eq.(32) from Eq.(27). For simplicity, we use α_1 and α_2 to express Eq.(27) as

$$4n + 3 + 2\,d\sqrt{\alpha_2(\alpha_1+V_0)} - \sqrt{[1+2d\sqrt{\alpha_2(\alpha_1+V_0)}]^2 - 4\,[d\,\alpha_1\,\alpha_2 + \sqrt{\alpha_2(\alpha_1+V_0)}]} = 0. \tag{33}$$

To further discuss, we rewrite Eq.(33) as

$$4\,n + 3 + d\sqrt{2}\,\sqrt{2\,\alpha_2}\,f(\alpha_1) = 0, \tag{34}$$

where

$$f(\alpha_1) = \sqrt{V_0+\alpha_1} - \frac{1}{2\,d\sqrt{\alpha_2}} \times \sqrt{(1+2\,d\,\gamma)^2 - 4(d\,\alpha_1\,\alpha_2 + \gamma)}. \tag{35}$$

In Eq.(35) γ is given by the second formula in Eq.(7) and α_2 stands for $2\,\mu - \alpha_1$. If we expand $f(\alpha_1)$ as a power series of α_1 and only take the first two terms, however, we obtain a complicated expression. But recalling that V_0 and d are large and taking a proper approximation, we obtain a very concise formula,

$$f(\alpha_1) \approx \frac{1}{2\sqrt{V_0}}\,\alpha_1. \tag{36}$$

Substituting Eq.(36) and α_1 as well as α_2 into Eq.(34), then using Eq.(30) to simplify, we obtain Eq.(32) approximately.

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