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Frequency sensitivity of sub-excitable systems coupled with different topology*

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We have investigated numerically the behaviours of identical FitzHugh–Nagumo systems, which are coupled into topologies of regular one-dimensional lattice, small-world network and scale-free network, and driven by white noise and an external signal. We found that when a number of uniform systems are coupled into a network, the system's signal-to-noise ratio remains at a high level in a wider frequency range than in the case of a single oscillator. Different architectures manifest different impact, with the scale-free network being the most remarkable. Results presented here suggest a possible approach to improve the sensitivity of a system to external signals and are helpful for designing communication equipments.

Keywords: frequency sensitivity, coupled systems, stochastic resonance, scale-free network, small-world network, FitzHugh–Nagumo

PACC: 0540, 0547, 8820W

1. Introduction

The nonlinear phenomenon of stochastic resonance (SR)[1] has attracted considerable interest of researches from the fields of physics, chemistry and biology in the past decade.[2–12] It was originally put forward by Benzi *et al*[13] to address the problem of periodically recurrent ice ages. In a dynamical system driven by a signal and noise, SR is typically characterized by an optimized signal-to-noise ratio (SNR) at an appropriate intensity of noise. The noise plays a constructive role that enhances the response of the system to signals. The mechanism of this nonlinear phenomenon is simple to explain. Consider an overdamping particle moving in a bistable potential driven by a Gaussian white noise and a periodic signal.[14,15] Without the periodic signal, its behaviour can be represented by the well known Kramers rate r_k which describes the frequency of the particle hopping from one minimum to another. When a signal is incorporated with the double-well potential, the barrier will be made periodically lower and higher. Then we can expect that the system becomes synchronized when the two time scales, $\frac{1}{r_k}$ and Ω, are matchable; and so the SNR of the output of the system reaches its maximum.

There are many experiments on neural systems that have been reported to show the SR, including single mechanoreceptive sensory neurons from crayfish,[2,7] rat skin[16] and neural networks from mammalian brain.[17] Experimental results showed that neurons are more sensitive and have a high SNR for signals with a limited range of frequency.[18] This effective frequency range has been argued to arise from the resonance between the intrinsic oscillation and the driven signal. While the frequency bandwidth is important for detection and transduction of information and is favoured in real applications, it is, however, quite limited in quantitative study.

In this paper, we are interested in the approaches that can achieve broader effective frequency ranges than usual situations. In previous studies regarding array-enhanced stochastic resonance,[19] it was revealed that the resonance behaviour of a single oscillator can be highly enhanced by coupling it locally to an array of identical oscillators. Under an optimal coupling strength, the arrays of oscillators are almost synchronized and the SNR of the oscillators reaches its maximum. The output of an array of oscillators can be much more enhanced when they are coupled to

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a two-dimensional (2D) array than in 1D.[19,20] Moreover, the topology of coupling plays an important role, and the studies of SR showed that the output and the synchronization of the system are improved to different degrees when the oscillators are coupled into regular, random and small-world networks.[20] Inspired by these previous results, we here investigate the property of frequency-response of the system which is constructed by coupling excitable systems into topologies of regular lattice, small-world network [21] and particularly the scale-free network.[22] We show that the architecture of coupled FitzHugh–Nagumo (FHN) systems can significantly expand the effective frequency range of external signals, and thus improve the sensitivity and efficiency of signal detection and transduction for the system.

2. Model and numerical simulations

The system we consider here is constructed by coupling N excitable systems into a network. The excitable system is described by the FHN model.[23,24] It has been employed frequently to investigate the phenomenon of stochastic resonance, because it provides a useful and compact mechanism to study the firing of neural systems driven by external signals. When it is disturbed by a sufficiently strong perturbation, the system will fire from its stable fixed point, and then takes a long journey and finally returns to that point after a period of time. While a sub-threshold signal cannot activate the system to fire, with the help of noise, the system has, however, a probability to be excited from the fixed point; constant firings can be viewed as an oscillation. In our model, the excitable systems are driven by a periodically input signal and white Gaussian noises. N such oscillators are coupled into a network with a certain topology. The dynamics of the oscillator of node i in the network is therefore described by the following equations:

$$\varepsilon_1 \frac{\mathrm{d}v_i}{\mathrm{d}t} = v_i(v_i - a)(1 - v_i) - w_i + I_0 + \sum_{j=1}^{N} \epsilon \Theta_{ij}(v_j - v_i) + A\sin(\varpi t) + \Gamma_i(t), \quad (1a)$$

$$\varepsilon_2 \frac{\mathrm{d}w_i}{\mathrm{d}t} = v_i - w_i - b, \quad (1b)$$

where ϖ and A are respectively the frequency and amplitude of the driving signal; I_0 is a constant, which is smaller than the threshold voltage of the oscillator; Θ is the coupling matrix representing the connectivity between oscillators. We set Θ_{ij} to 1 when the node i and node j are coupled together, otherwise Θ_{ij} is equal to 0. ϵ is the coupling strength. The Gaussian white noise $\Gamma_i(t)$ is time uncorrelated and satisfies

$$\langle \Gamma_i(t) \rangle = 0, \quad (2a)$$

$$\left\langle \Gamma_i(t)\Gamma_j(t') \right\rangle = 2D\delta(t, t')\delta_{ij}, \quad (2b)$$

where $\langle\ \rangle$ is the average over time and D is the intensity of noise. We set the parameters a=0.5, b=0.0, ε_1=1.0×10^{-5}, ε_2=1.0 ×10^{-3} in the rest of our discussions. The values of I_0 and A are chosen in such a way that the sub-systems in the network cannot be activated without the help of noise. We investigate the response of the system to signals of various frequencies at different intensities of noise, particularly the effect of topology of the network. For analysis of the system, we use the average of variable v over the N nodes in the network as the output of the whole system, i.e.

$$I_{\mathrm{output}}(t) = \frac{1}{N}\sum v_i(t). \quad (3)$$

The power spectrum density (PSD) can be calculated from the time series $I_{\mathrm{output}}(t)$ by means of Fourier transform. Before the calculation of the PSD, $I_{\mathrm{output}}(t)$ is pre-processed by a filtration: if $I_{\mathrm{output}}(t) \geq 0.8$, it is set to 1.0, otherwise to 0.0. The PSD thus calculated is a characterization of the system driven by noise and signal. In the PSD, a major spike, together with several sub-spikes among the noisy background, can be recognized to locate at the frequency $f = w/2\pi$ of the input signal. Denoting the intensity of the PSD at f with S, and that of the background with BN, we define the SNR as SNR=10log$_{10}\dfrac{S}{BN}$.

Before checking the dynamics of coupled oscillators, we first examine the properties of a single oscillator. Without input signal and noise, the system is just the original FHN model that has a globally stable fixed point with parameters a=0.5, b=0.0, ε_1=1.0 ×10^{-5}, ε_2=1.0 ×10^{-3}. Under the action of a bias I_0 added as that in Eq.(1), the dynamics is still a damping oscillation when I_0 is below a critical value ($I_c = 0.27$). Figure 1(a) depicts the damping behaviour when I_0=0.26. At $I_0 = I_c$, the system undergoes a transition from a steady state to a limit cycle. Near transition point, a signal with a proper frequency and intensity can make the system fire constantly. The critical intensity A_c that makes the system oscillate depends on the frequency of the input signal $f = \varpi/2\pi$. Figure

1(b) shows the dependence of the critical intensity A_c on the frequency when $I_0 = 0.2$. The curve shows a minimum at a frequency f_d ($\approx$ 1350) where the system needs the least critical intensity in order to activate the oscillation. This result proposes that the signal-free system has an intrinsic frequency. When the frequency of the input signal is close to the intrinsic frequency, they resonate and the signal is most efficient to drive the system to oscillate. We check the frequency of the damping oscillation at $I_0 = 0.2$ when the signal is absent. It is about 1300, which agrees approximately with f_d.

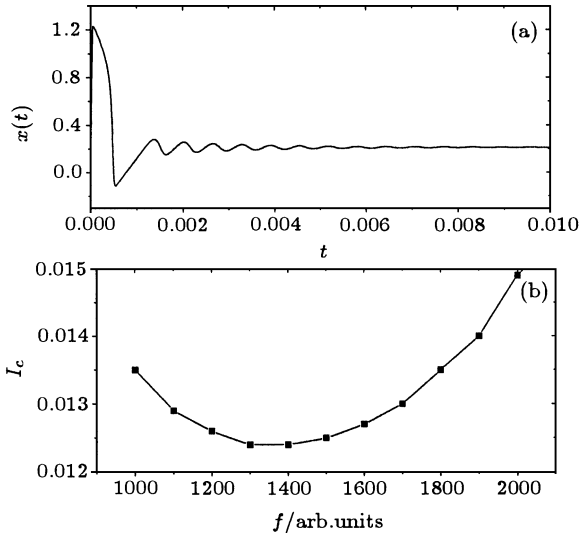


Fig.1. (a) Example of damping oscillations when a single FHN element is subjected with a bias.(b) Relation between the critical signal amplitude I_c and the driving frequency f. The damping frequency in (a) is calculated to be about 1300. Parameters: A=0.0, I_0=0.25, D=0.0 for (a), and I_0=0.2, D=0.0 for (b).

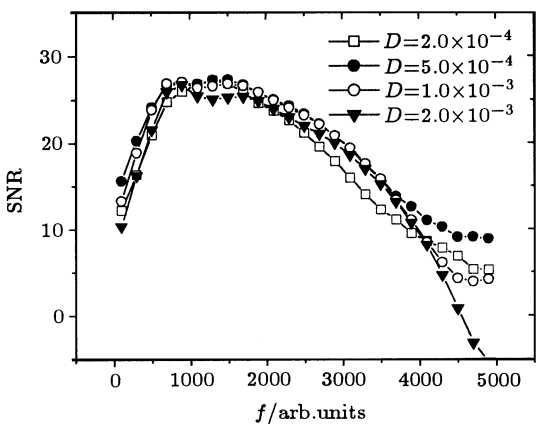


Fig.2. SNR versus the driving signal frequency f for different noise intensities in the case of a single oscillator. Parameters: I_0=0.2, A=0.01.

In the sub-excitable regime, when a white noise is added together with a small signal, the system can be excited irregularly and the dynamics becomes stochastic. We have checked the case when the bias I_0 and the amplitude of the input signal was so small that the system could not be fired without the help of noise. A and I_0 were fixed, and the frequency of the signal was scanned to examine the response of the system. From the time series $v(t)$, the SNR can be calculated for each signal. Figure 2 shows the dependence of SNR on the frequency f when $A = 0.01$ and I_0=0.2. The curve of SNR versus f is tent-shaped, approximately centred at the intrinsic frequency f_d (about 1350) (see Fig.1(b)). Signals with a frequency range around this maximum are more effective in information transmission than signals outside this range. The effective range can be estimated in the frequency regime in which the SNR is higher than $1/\sqrt{2}$ of the maximum. Thus the favoured frequency range in Fig.2 is estimated to be about [340, 3100] with D=1.0 $\times 10^{-3}$, which is the noise density. It is limited, confining the effective functions of the system if it is applied to transmit information that has a wide frequency width. The dependence of SNR of the system on the signal frequency was also examined at different intensities of noise. From Fig.2, one can observe that the noise intensity does not show significant influence on the effective frequency width in the range that has been checked.

In the following, we consider N identical elements that are coupled into a network, and check the influence of its topology on the effective frequency width we discussed above. There are many possibilities to organize the sub-systems. Here we consider three kinds of network topologies: regular 1D lattice, small-world network [21] and scale-free network.[22] They are schematically depicted in Fig.3. In the first case, we organize 200 sub-systems into a 1D lattice in periodic boundary condition. For each node in the lattice, it is coupled with its two nearest neighbours and two next-nearest neighbours. The small-world network is obtained from the 1D lattice by rewiring randomly each link in the network with a small probability. The numbers of vertices and connections in the network are unchanged. In the third case, the scale-free network grows from several isolated initial sites. At each step of growth, a new node is added, which connects with m old nodes. The choice of old nodes is determined by the fraction of connectivities that this old nodes contribute to the total connectivity of the current network. Specifically, node i has the probability $\Pi(k_i) = \dfrac{k_i}{\sum_j k_j}$ to be coupled by the new-coming

node. In the expression, k_i is the number of nodes with which node i is connected. We substitute each node in the networks by an identical oscillator that is driven by a bias, a signal and white noise. The number of nodes in the networks we considered is always kept at 200.

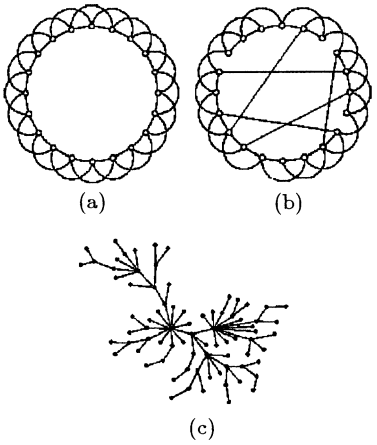


Fig.3. Schematic map for networks of 1D regular lattice (a), small-world network (b), and scale-free network (c).

After coupling the elements into a network, we monitor the behaviour of the whole system by analysing the time series I_{output} defined in Eq.(3). From I_{output}, the SNR can be calculated using fast Fourier transform (FFT). Figure 4(a) demonstrates the value of SNR versus the frequency f of the input signal in the case of a regular lattice. The curve demonstrates a similar behaviour to that in Fig.2. The frequency at which SNR reaches its maximum is still approximately f_d; however, the effective frequency bandwidth is expanded about 14%, from 2760 to 3150 (compare Fig.2 with Fig.4(a)). This expansion is due to the coupling of oscillators; it makes the system more sensitive and improves its ability to transmit information. This result is in accordance with previous findings reported in Ref.[19]. Different noise intensities are also tried, the results are shown in Fig.4(a).

With the same parameters A, I_0 and noise used for the regular lattice, we organize the 200 oscillators into a small-world network and calculate its response to various frequencies of the signal. Figure 4(b) shows the results. The effective frequency bandwidth changes as a function of rewiring probability (p): its range is estimated to be 3440 when p=0.06, and 4230 when p=0.1. Figure 5 shows the effective frequency width as a function of p. The maximum bandwidth is located at about p=0.1.

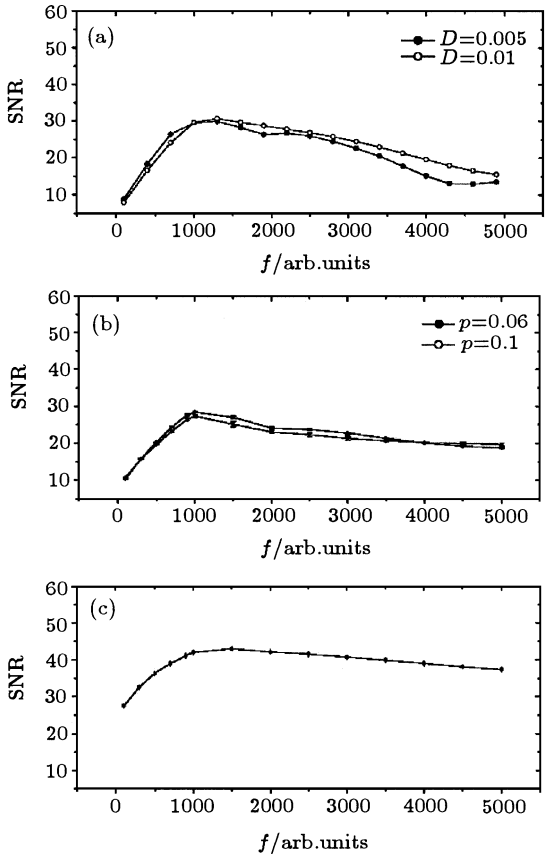


Fig.4. SNR versus the input signal frequency when the excitable elements are coupled into 1D regular lattice (a), small-world network (b) and scale-free network (c). The SNR shown here is an average of SNRs calculated from 50 time series I_{signal}, which are obtained with different seeds for the random number generator. Parameters: ϵ=0.15 for (a), ϵ=0.15, D=0.01 for (b), and ϵ=0.03, D=0.01, m=6 for (c).Other parameters: N=200, I_0=0.2, A=0.01.

When the small-world network is created from the 1D lattice, the rewiring of a small number of couplings between oscillators causes the nodes of the network much closer to each other. The characteristic path length in a small-world network, defined as the averaged number of links of the shortest path between two nodes, decreases rapidly with the increase of rewiring probability p.[21] Our results indicate that with the decrease of the average distance in the network, the oscillators communicate with each other more easily, thus improving the sensitivity of the system with a wider effective frequency range. When the rewiring probability p further increases, the disorder in the network increases, and thus hampers the system to response efficiently to external signals. When p = 1.0, the network becomes completely random, then the system behaves approximately as a single oscillator.

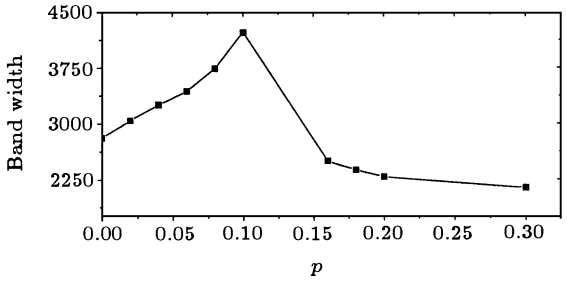


Fig.5. Effective frequency width versus the rewiring probability p of small world networks. The maximal bandwidth is located at p=0.1. Parameters are the same as in Fig.4(b).

When the oscillators are coupled into a topology of scale-free network, the effective frequency range is expanded even wider. Figure 4(c) shows the SNR versus the frequency f of the input signal. The curve shows a wide plateau and the SNR decreases fairly slowly. The favoured frequency range is estimated to be [430,12000], having an extremely large width of 11570. This behaviour of the system must be related to the topology of the network. As described above, nodes in the network with more links have higher probabilities to be linked by newly-added nodes when the network grows. So a few of nodes in the network are most frequently connected with other nodes, and the network is highly clustered. Due to this character, the system manages to sustain high SNRs for a very wide frequency range of signals.

3. Conclusion

In conclusion, we have investigated numerically the responding behaviours to external signals for the coupled FHN excitable systems driven by white noise. Networks of regular 1D lattice, small-world network and scale-free network have been investigated. We found that when a number of identical elements of FHN model are coupled into the networks, generally the system maintains a wider frequency range in which SNR remains relatively high; however, the range is narrower in the case of single oscillator. The topology of the network was found to have an important impact on the effective frequency range of the system. For the parameter space we have checked, the coupled system with a topology of small-world network has a wider effective range than that with a topology of regular lattice; moreover, the case of scale-free network has an even wider range than the case of small-world network. Our results demonstrate a general rule: in a network constructed with identical excitable elements, the system tends to be able to response efficiently to signals within a wider frequency range when the network has long-range connectivities, or is highly clustered, so that the nodes in the system can interact and communicate efficiently.

From the viewpoint of practical applications, the width of the frequency range in which the system is sensitive and has high SNRs is an important factor. For organisms or neural systems in biology, the ability to detect and process external signals within a broad frequency range, such as sounds and electronic signals, is vital for their fitness and survival in the environment. The same is true in the information detection and transmission processes in fields of communication and information technology. The results reported here present a possible approach to improve the sensitivity of a system to external signals by means of coupling oscillators into networks with various topologies, and are helpful to reveal the mechanism of information detection processes in biological organisms and to design communication equipments.

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