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The supersymmetric WKB approximation of ring-shaped oscillator potential in r and θ dimensions with spherical polar coordinates

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According to calculation of the energy spectrum of ring-shape oscillator potential by using the supersymmetric WKB approximation, it is shown that the energy spectrum of some noncentral separable potentials can be exactly obtained in r and θ dimensions by above method.

Keywords: ring-shaped oscillator potential, energy spectrum, supersymmetric WKB approximation
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1. Introduction

With the idea of the factorization method by Infeld and Hull,[1] supersymmetry was first discussed in the simplest case of supersymmetric quantum mechanics by Witten,[2] and Cooper and Freeman.[3] In 1983, the concept of shape invariance potential was introduced by Gendenstein,[4] and the relationship between solvable potentials and shape invariance has been investigated extensively.[5−10] Gendenstein [4] pointed out that essentially all the known solvable potentials are shape invariant, and the energy eigenvalues can be obtained algebraically. In addition, the supersymmetric quantum mechanics and shape invariance can solve relativistic quantum mechanics.[11] Based on the supersymmetric quantum mechanics, the supersymmetry WKB approximation was introduced by Comtet *et al*[12] in 1985. With the ideas of the supersymmetric WKB approximation, a number of potentials could also be solved exactly.[13−15] Although most of these potentials are either one-dimensional or central potential, there are some three-dimensional noncentral potentials, which can be separated in spherical polar coordinates. These potentials should have supersymmetry and shape invariance, and can be exactly solved by means of supersymmetric quantum mechanics and shape invariance.[16] In this paper, we suggest that the energy spectrum of the potentials mentioned above should also be obtained by the supersymmetric WKB quantization condition as in Ref.[17], and the results for these potentials are consistent with the results obtained using the supersymmetric quantum mechanics and shape invariance. As an illustration we discuss the ring-shaped oscillator potential.

The ring-shaped oscillator potential was introduced in quantum chemistry, and is given by the following expression:[18,19]

$$V(r,\theta)=\alpha r^2+\frac{\beta}{(r\sin\theta)^2}. \tag{1}$$

The limiting case $\beta=0$ corresponds to the well-known isotropic harmonic oscillator, which has been investigated in many papers and textbooks.

2. Separable potentials in spherical polar coordinates

In spherical polar coordinates the most general potentials, for which the Schrödinger equation is separable, are given by [7]

$$V(r,\theta,\phi)=V(r)+\frac{V(\theta)}{r^2}+\frac{V(\phi)}{r^2\sin^2\theta}, \tag{2}$$

where $V(r)$, $V(\theta)$, $V(\phi)$ are arbitrary functions of their arguments. If the wavefunction $\Psi(r,\theta,\phi)$ is written as

$$\Psi(r,\theta,\phi)=\frac{R(r)}{r}\frac{H(\theta)}{(\sin\theta)^{1/2}}K(\phi), \tag{3}$$

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then one obtains the following equations (in units of $2\mu = \hbar = 1$):

$$-\frac{\mathrm{d}^2K(\phi)}{\mathrm{d}\phi^2} + V(\phi)K(\phi) = m^2K(\phi), \tag{4}$$

$$-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2} + \left[V(\theta) + \left(m^2 - \frac{1}{4}\right)\csc^2\theta\right]H(\theta) = l^2H(\theta), \tag{5}$$

$$-\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \left[V(r) + \frac{l^2 - 1/4}{r^2}\right]R(r) = ER(r), \tag{6}$$

where m^2 and l^2 are separation constants.

For the ring-shaped oscillator potential we have

$$V(r) = \alpha r^2, \quad V(\theta) = \beta\csc^2\theta, \quad V(\phi) = 0, \tag{7}$$

hence we can obtain the following three Schrödinger equations:

$$-\frac{\mathrm{d}^2K(\phi)}{\mathrm{d}\phi^2} = m^2K(\phi), \tag{8}$$

$$-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2} + \left[\left(m^2 + \beta - \frac{1}{4}\right)\csc^2\theta\right]H(\theta) = l^2H(\theta), \tag{9}$$

$$-\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \left[\alpha r^2 + \frac{l^2 - 1/4}{r^2}\right]R(r) = ER(r). \tag{10}$$

Though $K(\phi)$ solution cannot be given by using supersymmetric quantum mechanics, we can obtain it easily. Its solutions are

$$K(\phi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\phi}, \quad m = 0, \pm1, \pm2, \ldots \tag{11}$$

In the following sections we will use the supersymmetric WKB quantization condition to gain the energy spectrum of the ring-shaped oscillator potential in r and θ dimensions.

3. The supersymmetric WKB approximation in θ dimension

If we put $B(B-1) = \beta + m^2 - \frac{1}{4}$ or $B = \sqrt{\beta + m^2} + \frac{1}{2}$, Eq.(9) turns into

$$-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2} + [B(B-1)\csc^2\theta]H(\theta) = l^2H(\theta). \tag{12}$$

In order to use the supersymmetric WKB quantization condition, its superpotential $W(\theta)$ and ground state E_0 have to be known. From Refs.[20–22], we can easily obtain them. They are respectively given by

$$W(\theta) = -B\cot\theta, \quad E_0 = B^2. \tag{13}$$

The supersymmetric WKB quantization condition is [12]

$$\int_{x_\mathrm{L}}^{x_\mathrm{R}} \sqrt{[E_n^{(-)} - W^2(x)]} = n\pi, \quad n = 0, 1, 2, \ldots, \tag{14}$$

where the two turning points x_L and x_R are given by $[E_n^{(-)} - W^2(x)] = 0$; the energy $E_n^{(-)}$ is the energy level of the supersymmetric partner potentials $V_-(x)$. Substituting the superpotential given in Eq.(13) into the supersymmetric WKB quantization condition, we have

$$\int_{x_\mathrm{L}}^{x_\mathrm{R}} \sqrt{[E_s^{(-)} - B^2\cot^2\theta]} = s\pi, \quad s = 0, 1, 2, \ldots \tag{15}$$

With a change of variable $y = -\cot\theta$, Eq.(15) becomes

$$B\int_{x_\mathrm{L}}^{x_\mathrm{R}} \frac{1}{1+y^2}\sqrt{(y_\mathrm{R} - y)(y - y_\mathrm{L})}\mathrm{d}y = s\pi, \tag{16}$$

where the two turning points are given by $y_\mathrm{L} = -\sqrt{E_s^{(-)}}/B$ and $y_\mathrm{R} = \sqrt{E_s^{(-)}}/B$. Using the integral expression [14]

$$\int_a^b \frac{\mathrm{d}z}{1+z^2}\sqrt{(z-a)(b-z)} = \frac{\pi}{\sqrt{2}}[\sqrt{1+a^2}\sqrt{1+b^2} - ab + 1]^{\frac{1}{2}} - \pi, \tag{17}$$

we can obtain

$$E_s^{(-)} = (B+s)^2 - B^2. \tag{18}$$

For the energy spectrum $E_s = E_s^{(-)} + E_0$, we can obtain

$$l = (B+s). \tag{19}$$

4. The supersymmetric WKB approximation in r dimension

According to Refs.[20–22], we can easily obtain the superpotential $W(r)$ ground state E_0 in Eq.(10) too. They are respectively given by

$$W(r) = \sqrt{\alpha}r - \frac{\left(l + \frac{1}{2}\right)}{r}, \quad E_0 = (2l+2)\sqrt{\alpha}. \tag{20}$$

Substituting $W(r)$ in Eq.(20) into the supersymmetry WKB quantization condition in Eq.(14), we can obtain

$$\int_{x_\mathrm{L}}^{x_\mathrm{R}} \sqrt{E_n^{(-)} - \left[\sqrt{\alpha}r - \left(l + \frac{1}{2}\right)\Big/ r\right]^2}\,\mathrm{d}r = n\pi, \quad n = 0, 1, 2, \ldots \tag{21}$$

With a change of variable $\rho = r^2$, Eq.(21) becomes

$$\frac{\sqrt{\alpha}}{2}\int_{\rho_{\rm L}}^{\rho_{\rm R}}\frac{1}{\rho}\sqrt{(\rho_{\rm R}-\rho)(\rho-\rho_{\rm L})}{\rm d}y = n\pi, \qquad (22)$$

where the two turning points are given by

$$\rho_{\rm R} = \left[\frac{E_n^{(-)}}{2\alpha}+\frac{(2l+1)}{2\sqrt{\alpha}}\right] + \frac{1}{2}\sqrt{\left[\frac{E_n^{(-)}}{2\alpha}+\frac{(2l+1)}{2\sqrt{\alpha}}\right]^2 - \frac{4\left(l+\frac{1}{2}\right)^2}{\alpha}} \qquad (23)$$

and

$$\rho_{\rm L} = \left[\frac{E_n^{(-)}}{2\alpha}+\frac{(2l+1)}{2\sqrt{\alpha}}\right] - \frac{1}{2}\sqrt{\left[\frac{E_n^{(-)}}{2\alpha}+\frac{(2l+1)}{2\sqrt{\alpha}}\right]^2 - \frac{4\left(l+\frac{1}{2}\right)^2}{\alpha}}. \qquad (24)$$

Using the integral expression [14]

$$\int_a^b \frac{{\rm d}z}{z}\sqrt{(z-a)(b-z)} = \frac{\pi}{2}(a+b) - \pi\sqrt{ab}, \qquad (25)$$

we can obtain

$$E_n^{(-)} = 4n\sqrt{\alpha}. \qquad (26)$$

For the energy spectrum $E_n = E_n^{(-)} + E_0$, we can obtain

$$E_n = 2(2n+l+1)\sqrt{\alpha}. \qquad (27)$$

Finally, comparing Eq.(27) with Eq.(19), we can obtain the exact energy spectrum of the ring-shaped oscillator potential

$$E_{n,s} = 2\left(2n+\sqrt{\beta+m^2}+s+\frac{3}{2}\right)\sqrt{\alpha}. \qquad (28)$$

It can be seen that the result Eq.(28) is consistent with the result in Ref.[15].

5. Conclusion

In this paper, we have presented the full energy spectrum of the ring-shaped oscillator potential with the help of the supersymmetric WKB approximation. The energy spectrum is exactly solved and is consistent with the result obtained using the supersymmetric quantum mechanics and shape invariance. By using the supersymmetric WKB approximation we believe that the noncentral separable potentials have an exact energy spectrum, and the method in this paper may be extended to any noncentral separable potential.

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