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A certain critical two-soliton solution of the nonlinear Schrödinger equation*

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An exact two-soliton solution of the nonlinear Schrödinger equation is derived by using the Hirota direct approach. This solution describes such a critical process that two still solitons separated infinitely approach and then pass through each other and keep straight on infinitely.

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It is well known that the nonlinear Schrödinger equation (NLSE)

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0 \tag{1}$$

is one of the most important nonlinear evolution equations whose N-soliton solution has been obtained by the inverse scattering method long ago.[1,2] These solutions can also be derived, in principle, by making use of the other methods such as the Hirota direct method,[3−5] and Bäcklund transformation,[6,7] etc. However, it seems that there are some soliton solutions yet to be obtained. In this short note, as an example, we shall find a special two-soliton solution by using the Hirota method. To do this, according to the usual practice, we put

$$u(x,t) = G(x,t)/F(x,t)\ , \tag{2}$$

where $G(x,t)$ and $F(x,t)$ are two functions of time t and space x to be determined, among which $F(x,t)$ is supposed to be real. It follows from Eq.(2) that

$$u_t = \frac{1}{F^2} D_t(G\cdot F)\ , \tag{3}$$

$$u_{xx} = \frac{1}{F^2} D_x^2(G\cdot F) - \frac{2}{F^3} G D_x^2(F\cdot F)\ , \tag{4}$$

where D_t^n and D_x^n $(n = 1, 2, \ldots)$ are bilinear operators defined as [3−5]

$$\begin{aligned} D_t^n(f\cdot g) &= (\partial_t - \partial_{t'})^n f(x,t)g(x',t')|_{x'=x,t'=t}\ , \\ D_x^n(f\cdot g) &= (\partial_x - \partial_{x'})^n f(x,t)g(x',t')|_{x'=x,t'=t}\ . \end{aligned} \tag{5}$$

Substituting Eqs.(3) and (4) into Eq.(2), and equating the coefficients of F^{-2} and F^{-3}, we obtain the following two equations:

$$(\mathrm{i}D_t + D_x^2)(G\cdot F) = 0\ , \tag{6}$$

$$D_x^2(F\cdot F) - 2|G|^2 = 0\ . \tag{7}$$

Now we expand F and G into power series of a small parameter ϵ as follows:

$$\begin{aligned} F &= 1 + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + \epsilon^3 F^{(3)} + \cdots\ , \\ G &= \epsilon G^{(1)} + \epsilon^2 G^{(2)} + \epsilon^3 G^{(3)} + \cdots \end{aligned} \tag{8}$$

Inserting Eq.(8) into Eqs.(6) and (7), and equating the coefficients of each power of ϵ, we have the following recurrence formulae:

$$\begin{aligned} &(\mathrm{i}\partial_t + \partial_x^2)G^{(1)} = 0\ , \\ &2\partial_x^2 F^{(1)} = 0\ , \end{aligned} \tag{9}$$

$$\begin{aligned} &(\mathrm{i}\partial_t + \partial_x^2)G^{(2)} = -(\mathrm{i}D_t + D_x^2)(G^{(1)}\cdot F^{(1)}), \\ &2\partial_x^2 F^{(2)} = -D_x^2(F^{(1)}\cdot F^{(1)}) + 2|G^{(1)}|^2\ , \end{aligned} \tag{10}$$

$$\begin{aligned} (\mathrm{i}\partial_t + \partial_x^2)G^{(3)} &= -(\mathrm{i}D_t + D_x^2)(G^{(1)} \\ &\quad \times F^{(2)} + G^{(2)}\cdot F^{(1)})\ , \cdots\ , \\ 2\partial_x^2 F^{(3)} &= -D_x^2(F^{(1)}\cdot F^{(2)} + F^{(2)}\cdot F^{(1)}) \\ &\quad + 2(\bar{G}^{(1)}G^{(2)} + \bar{G}^{(2)}G^{(1)})\ , \cdots\ , \end{aligned} \tag{11}$$

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where an overbar stands for the complex conjugate. Now we begin to solve Eqs.(9)–(11) successively. According to the usual practice, we choose $G^{(1)} = \mathrm{e}^{\mathrm{i}\lambda(x-\lambda t)}, F^{(1)} = 0$ (where $\lambda = a + \mathrm{i}b$ is a complex number), and then we obtain $G^{(2)} = 0$, $F^{(2)} = \frac{1}{4b^2}\mathrm{e}^{-2b(x-2at)}$, $G^{(n)} = F^{(n)} = 0$, $(n \geq 3)$, and so we obtain the single-soliton solution $u(x,t) = b\ \mathrm{sech}[b(x - x_0 - 2at)]\exp[\mathrm{i}ax - \mathrm{i}(a^2 - b^2)t]$. Similarly, if we start from $G^{(1)} = \mathrm{e}^{\mathrm{i}\lambda_1(x-\lambda_1 t)} + \mathrm{e}^{\mathrm{i}\lambda_2(x-\lambda_2 t)}$, we shall obtain a two-soliton solution, etc. Note that $G^{(1)} = \mathrm{e}^{\mathrm{i}\lambda(x-\lambda t)}$, $F^{(1)} = 0$ is merely one of the special solutions of Eq.(9). If we choose another suitable special solution, perhaps we can gain some new soliton solutions of the NLSE. For this purpose, we choose

$$\begin{aligned} G^{(1)} &= \mathrm{i}[b(x - 2\lambda t) - 1]\exp[\mathrm{i}\lambda(x - \lambda t)]\ , \\ F^{(1)} &= 0\ . \end{aligned} \tag{12}$$

Then Eq.(10) becomes

$$\begin{aligned} (\mathrm{i}\partial_t + \partial_x^2)G^{(2)} =& 0\ , \\ \partial_x^2 F^{(2)} =& \{[b(x - 2at) - 1]^2 + 4b^4t^2\} \\ & \times \exp[-2b(x - 2at)]\ . \end{aligned} \tag{13}$$

It is easy to obtain a special solution of Eq.(13):

$$\begin{aligned} G^{(2)} =& 0\ , \\ F^{(2)} =& \frac{1}{16b^2}[4b^2(x - 2at)^2 + 16b^4t^2 + 2] \\ & \times \exp[-2b(x - 2at)]\ . \end{aligned} \tag{14}$$

Employing Eqs.(12) and (14), we rewrite the third-order equation (11) as

$$\begin{aligned} (\mathrm{i}\partial_t + \partial_x^2)G^{(3)} =& \frac{b}{2}[2bt - \mathrm{i}(x - 2at)] \\ & \times \exp[-3b(x - 2at) \\ & + \mathrm{i}ax - \mathrm{i}(a^2 - b^2)t]\ , \\ \partial_x^2 F^{(3)} =& 0\ . \end{aligned} \tag{15}$$

It can be checked directly that Eq.(15) has the following special solution:

$$\begin{aligned} G^{(3)} =& \frac{1}{16b^2}[2b^2t - \mathrm{i}b(x - 2at) - \mathrm{i}] \\ & \times \exp[-3b(x - 2at) + \mathrm{i}ax - \mathrm{i}(a^2 - b^2)t], \\ F^{(3)} =& 0\ . \end{aligned} \tag{16}$$

Similarly, we can find the solutions of the higher-order equations as

$$\begin{aligned} & G^{(4)} = 0\ , \quad G^{(n)} = 0\ , \ \text{for}\ \ n \geq 5\ , \\ & F^{(4)} = \frac{1}{(16b^2)^2}\exp[-4b(x - 2at)], \\ & F^{(n)} = 0\ , \ \ for\ \ n \geq 5\ . \end{aligned} \tag{17}$$

Consequently, we obtain the following solution of NLSE with the aid of Eqs.(12),(14),(16) and (17):

$$\begin{aligned} u(x,t) =& \frac{G}{F} = \frac{\epsilon G^{(1)} + \epsilon^3 G^{(3)}}{1 + \epsilon^2 F^{(2)} + \epsilon^4 F^{(4)}} \\ =& 2b\ \exp[\mathrm{i}ak - \mathrm{i}(a^2 - b^2)t - \mathrm{i}\varphi] \\ & \times \frac{(2b^2t - \mathrm{i})\cosh[b(x - x_0 - 2at)] + \mathrm{i}b(x - 2at)\sinh[b(x - x_0 - 2at)]}{\cosh^2[b(x - x_0 - 2at)] + b^2[(x - 2at)^2 + 4b^2t]}\ , \end{aligned} \tag{18}$$

where $x_0 = \frac{1}{b}\ln\left(\frac{\epsilon}{4b}\right)$. In a mass-centre system, $a = 0$, then Eq.(18) is reduced to

$$u(x,t) = 2b\mathrm{e}^{\mathrm{i}b^2t - \mathrm{i}\varphi_0}\frac{(2b^2t - \mathrm{i})\cosh[b(x - x_0)] + \mathrm{i}bx\ \sinh[b(x - x_0)]}{\cosh^2[b(x - x_0)] + b^2(x^2 + 4b^2t^2)}\ . \tag{19}$$

We can easily check directly that solution (18) as well as (19) really satisfies the NLSE (1). Obviously, solution (18) (or solution (19)) is different from the usual two-soliton solution given in Refs.[1] and [2]. We shall show later that it is also a special two-soliton solution. To do this, we introduce a logarithmic time τ instead of t:

$$\begin{aligned} & \tau = \ln(2b^2t + \sqrt{4b^4t^2 + 1}) \\ & \text{or}\ \ 2b^2t = \sinh\tau\ , \ \ \sqrt{4b^4t^2 + 1} = \cosh\tau\ . \end{aligned} \tag{20}$$

Then solution (19) can be rewritten as

$$\begin{aligned} u(\xi, \xi_0, \tau) =& 2b\exp\left(\frac{\mathrm{i}}{2}\sinh\tau - \mathrm{i}\varphi\right) \\ & \times \frac{(\sinh\tau - \mathrm{i})\cosh\xi + \mathrm{i}(\xi + \xi_0)\sinh\xi}{\cosh^2\xi + (\xi + \xi_0)^2 + \sinh^2\tau}, \end{aligned} \tag{21}$$

where $\xi = b(x - x_0)$ and $\xi_0 = bx_0$. To clearly show how the two solitons move and interact with each other, we

rewrite solution (21) as a product of two factors:

$$u(\xi,\xi_0,\tau) = b\,[\,\mathrm{sech}(\xi+\tau)+\mathrm{sech}(\xi-\tau)]\,w(\xi,\xi_0,\tau)\,, \tag{22}$$

where

$$w(\xi,\xi_0,\tau) = \exp\left(\frac{\mathrm{i}}{2}\sinh\tau - \mathrm{i}\varphi\right) \times \frac{\tanh\tau + \mathrm{i}[(\xi+\xi_0)\tanh\xi - 1]\,\mathrm{sech}\tau}{1+(\xi+\xi_0)^2\,\mathrm{sech}(\xi+\tau)\,\mathrm{sech}(\xi-\tau)}. \tag{23}$$

Obviously, Eq.(22) means that in the mass-centre system, $u(\xi,\xi_0,\tau)$ is the superposition of two solitons with the same size and velocity but in opposite propagating directions. Function $w(\xi,\xi_0,\tau)$ is a modification factor caused by nonlinear effects. The profiles of wave $|u(\xi,\xi_0,\tau)|$ as functions of ξ at different τ (logarithmic time) for $\xi_0 = 0, 2, 4$ are drawn by computer in Fig.1. They show clearly that the two solitons in a pair which are initially separated by an infinite distance attract and approach each other, and at the time $\tau = 0$ they overlap completely, and afterwards they begin to depart from each other until they are infinitely away from each other. According to Eq.(20), the centre position of each soliton is

$$\xi_c = \pm\tau = \pm\ln[2b^2t + (4b^4t^2+1)^{1/2}]\,. \tag{24}$$

Therefore, the velocity of the soliton is

$$v = \frac{\mathrm{d}\xi_c}{\mathrm{d}t} = \pm 2b^2\mathrm{sech}\xi_c\,. \tag{25}$$

Equation(25) shows that the velocity of solitons decreases to zero exponentially as the separation between them rises infinitely. Equation (25) can be rewritten in the form of energy conservation:

$$\frac{1}{2}v^2 - 2b^4\mathrm{sech}^2\xi_c = E = 0\,, \tag{26}$$

which shows that each soliton moves as if a particle with unit mass does in a potential well $U(\xi) = -2b^4\mathrm{sech}^2\xi$. Therefore, the interaction between the solitons is of short range, which damps exponentially as the separation between them increases. The total energy E being zero means that our two-soliton solution $u(\xi,\tau)$ describes a certain critical state between the scattering state and the boundary state (if it exists) of the two solitons.

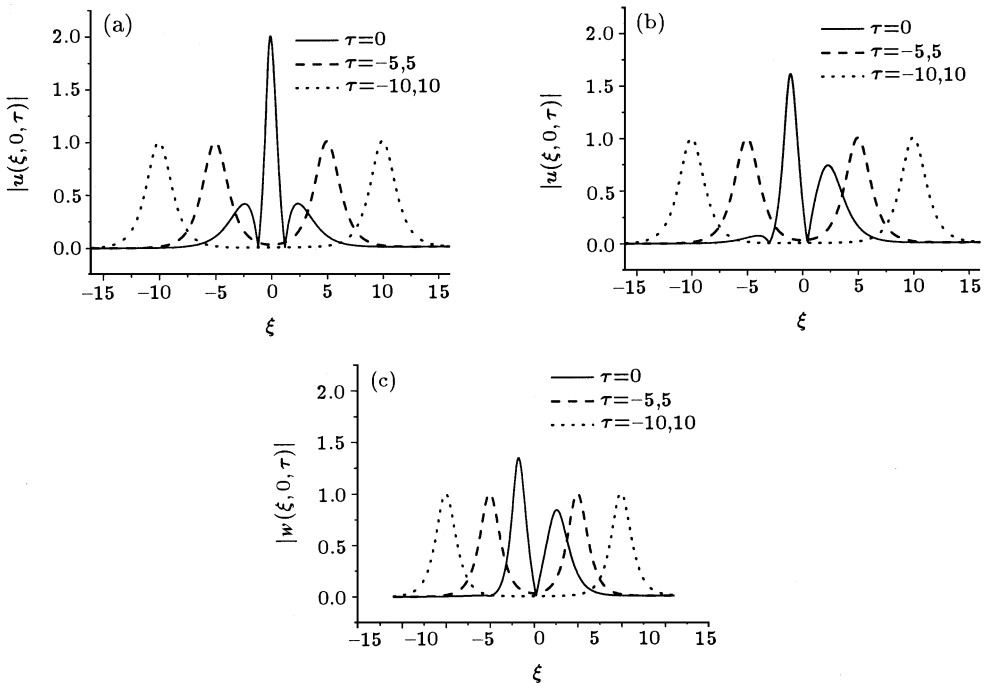


Fig.1. The waveform evolution of the two-soliton solution (19), i.e. the curves $|u(\xi,\xi_0,\tau)|$ versus ξ for different values of $|\tau|$, where (a), (b) and (c) correspond to $\xi_0 = 0$, 2 and 4 respectively, and in all the above figures, the solid, dashed and dotted lines correspond to $\tau = 0$, ∓ 5 and ∓ 10 respectively.

Finally, we study the factor $w(\xi, \xi_0, \tau)$ caused by the nonlinear effects. It affects the two solitons mainly at points $\xi \approx \pm\tau$. The figure of $|w(\xi, \xi_0, \pm\xi)|$ by computer is given in Fig.2, from which we see clearly that nonlinearity affects the solitons mainly at small separations ($\Delta\xi < 4$). Moreover, it is clear from Figs.1 and 2 that $\xi_0 \neq 0$ leads to an asymmetry of the waveform. The larger the $|\xi_0|$, the stronger the asymmetry will be. This is one of the nonlinear effects. Of course, this asymmetry appears clearly only when the two solitons are very near to each other ($\Delta\xi < 4$).

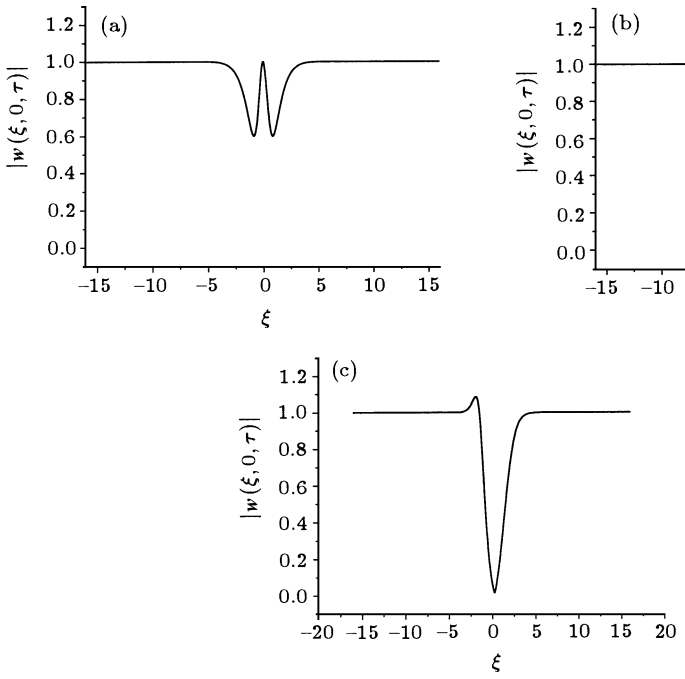


Fig.2. The curves $|w(\xi, \xi_0, \tau = \mp\xi)|$ versus ξ for different of ξ_0, where (a), (b) and (c) correspond to $\tau = 0$, 2 and 4 respectively.

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