Vol 13 No 3, March 2004
1009-1963/2004/13(03)/0317-12

Chinese Physics

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Generation and synchronization of N-scroll chaotic and hyperchaotic attractors in fourth-order systems*

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(Received 6 June 2003; revised manuscript received 21 August 2003)

Based on our previous works and Lyapunov stability theory, this paper studies the generation and synchronization of N-scroll chaotic and hyperchaotic attractors in fourth-order systems. A fourth-order circuit, by introducing additional breakpoints in the modified Chua oscillator, is implemented for the study of generation and synchronization of N-scroll chaotic attractors. This confirms the consistency of theoretical calculation, numerical simulation and circuit experiment. Furthermore, we give a refined and extended study of generating and synchronizing N-scroll hyperchaotic attractors in the fourth-order MCK system and report the new theoretical result, which is verified by computer simulations.

Keywords: fourth-order systems, N-scroll chaotic attractors, N-scroll hyperchaotic attractors, N-scroll system synchronization

PACC: 0545

1. Introduction

In recent years, chaos and its synchronization has been received a great deal of interest.[1−9] One of the important tasks for constructing a chaos communication system is the choice of chaotic oscillators. Chua's circuit is probably the most well-known and commonly used chaotic oscillator in these fields.[10,11] This has led to some extended researches on generating N-scroll chaotic attractors which have a more complicated dynamical behaviour due to introducing additional breakpoints in Chua's circuit.[12−14] Meanwhile, some circuits have been reported for generating double-scroll chaotic and hyperchaotic attractors in fourth-order systems, such as the modified Chua circuit [15,16] and the MCK hyperchaotic circuit.[17] In the modified Chua circuit, a RC parallel circuit is added into the L arm of Chua's circuit. This makes the circuit equation a fourth-order one. The MCK circuit is a typical hyperchaotic oscillator that can generate a double-scroll hyperchaotic attractor.

In this paper, a scheme for generating and synchronizing a new family of N-scroll chaotic and hyperchaotic attractors in fourth-order systems is investigated. The recurrence formula is presented to determine the recurrence parameters, including breakpoints and equilibrium points. The key issue is to construct a piecewise-linear odd function with $2N-1$ segments so as to obtain N-scroll attractors. To observe the generation and synchronization of N-scroll chaotic attractors, an associated electronic circuit is designed and implemented.

This paper is organized as follows. In the first part of the paper, synchronization and circuit implementation of N-scroll chaotic attractors in the modified Chua's circuit is presented. The second part of this paper contributes to a new theoretical analysis

*Project supported by the Natural Science Foundation of Guangdong Province (Grant No 032469), and the National Natural Science Foundation of China (Grant No 69672022), and the Doctoral Foundation of the State Education Ministry of China (Grant No 00056107), and the Natural Science Foundation of the Education Department of Guangdong Province (Grant No 010042).

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with results concerning the generation and synchronizing of a new family of N-scroll hyperchaotic attractors in the MCK system.

2. Unidirectionally coupled synchronization of N-scroll chaotic attractors

According to Ref.[18], taking the 6-scroll chaotic attractor as an example, the dimensionless form of state equations for the driving system can be written as

$$\begin{cases} \mathrm{d}x^{(1)}/\mathrm{d}\tau = \alpha[y^{(1)} - x^{(1)} - g(x^{(1)})], \\ \mathrm{d}y^{(1)}/\mathrm{d}\tau = x^{(1)} - y^{(1)} + z^{(1)}, \\ \mathrm{d}z^{(1)}/\mathrm{d}\tau = -\beta(y^{(1)} - w^{(1)}), \\ \mathrm{d}w^{(1)}/\mathrm{d}\tau = -\gamma_2(z^{(1)} + \gamma_1 w^{(1)}). \end{cases} \tag{1}$$

Associated state equations of the response system are obtained accordingly:

$$\begin{cases} \mathrm{d}x^{(2)}/\mathrm{d}\tau = \alpha[y^{(2)} - x^{(2)} - g(x^{(2)})] + \delta_x(x^{(1)} - x^{(2)}), \\ \mathrm{d}y^{(2)}/\mathrm{d}\tau = x^{(2)} - y^{(2)} + z^{(2)} + \delta_y(y^{(1)} - y^{(2)}), \\ \mathrm{d}z^{(2)}/\mathrm{d}\tau = -\beta(y^{(2)} - w^{(2)}), \\ \mathrm{d}w^{(2)}/\mathrm{d}\tau = -\gamma_2(z^{(2)} + \gamma_1 w^{(2)}), \end{cases} \tag{2}$$

where α=9.934, β=14.334, γ_1=27.333 and γ_2=0.0497. δ_x and δ_y denote the uindirectionally coupled coefficients. $g(\cdot)$ is a 11-segment piecewise-linear odd function used to generate 6-scroll chaotic attractors in the drive and response systems

$$\begin{cases} g(x^{(1)}) = m_5 x^{(1)} + 0.5\sum_{i=1}^{5}(m_{i-1} - m_i) \\ \qquad \cdot(|x^{(1)} + x_i^{(1)}| - |x^{(1)} - x_i^{(1)}|), \\ g(x^{(2)}) = m_5 x^{(2)} + 0.5\sum_{i=1}^{5}(m_{i-1} - m_i) \\ \qquad \cdot(|x^{(2)} + x_i^{(2)}| - |x^{(2)} - x_i^{(2)}|). \end{cases} \tag{3}$$

The breakpoints $x_i^{(j)}$ (j=1, 2, i=1, 2, 3, 4, 5) can be expressed in the following recurrence form:[18]

$$x_{i+1}^{(j)} = \frac{2\sum_{k=1}^{i}(m_k - m_{k-1})x_k^{(j)}}{m_i + \gamma_1/(1+\gamma_1)} - x_i^{(j)} \quad (i = 1,2,3,4, j = 1,2), \tag{4}$$

where the first breakpoint $x_1^{(j)}$ (j=1, 2) of $g(x^{(j)})$ can be arbitrary. For simplicity, we take $x_1^{(1)} = x_1^{(2)} = 1$. By use of Eq.(4), the other breakpoints $x_i^{(1)}$ and $x_i^{(2)}$ are determined. To make the 6-scroll attractors uniform in size and shape, we take $m_0 = m_2 = m_4 = m_a = -1.246$ and $m_1 = m_3 = m_5 = m_b = -0.6724$.

Theorem Suppose that there exist coupling coefficients $\delta_x > 0$ and $\delta_y \geq 0$ such that $\alpha m_a + \delta_x + (\alpha + \alpha m_a + \delta_x)\delta_y > 0$ in the unidirectionally coupled systems (1) and (2), the response system (2) is globally asymptotically synchronized with the drive system (1).

Proof From Eqs.(1)–(4), we obtain the error equations

$$\begin{cases} \dot{e}_x = -(\alpha + \delta_x)e_x + \alpha e_y - \alpha[g(x^{(1)}) - g(x^{(2)})], \\ \dot{e}_y = e_x - (1 + \delta_y)e_y + e_z, \\ \dot{e}_z = -\beta e_y + \beta e_w, \\ \dot{e}_w = -\gamma_2 e_z - \gamma_1\gamma_2 e_w, \end{cases} \tag{5}$$

where $e = [e_x, e_y, e_z, e_w]^{\mathrm{T}}$, $e_x = x^{(1)} - x^{(2)}$, $e_y = y^{(1)} - y^{(2)}$, $e_z = z^{(1)} - z^{(2)}$ and $e_w = w^{(1)} - w^{(2)}$ are error signals. Since $g(x)$ is the 11-segment piecewise-linear continuous odd function with minimum slope m_a and maximum slope m_b, the inequality $m_a(x^{(1)} - x^{(2)}) \leq g(x^{(1)}) - g(x^{(2)}) \leq m_b(x^{(1)} - x^{(2)})$ holds. In this case, we assume $g(x^{(1)}) - g(x^{(2)}) = m(x^{(1)} - x^{(2)}) = me_x$, in which m is the variable parameter $m_a \leq m \leq m_b$.[19] Eq.(5) is rewritten as

$$\begin{cases} \dot{e}_x = -(\alpha + \alpha m + \delta_x)e_x + \alpha e_y, \\ \dot{e}_y = e_x - (1 + \delta_y)e_y + e_z, \\ \dot{e}_z = -\beta e_y + \beta e_w, \\ \dot{e}_w = -\gamma_2 e_z - \gamma_1\gamma_2 e_w. \end{cases} \tag{6}$$

If we consider the Lyapunov function

$$E = \frac{\gamma_2\beta}{2}e_x^2 + \frac{\gamma_2\alpha\beta}{2}e_y^2 + \frac{\gamma_2\alpha}{2}e_z^2 + \frac{\alpha\beta}{2}e_w^2, \tag{7}$$

then its derivative is

$$\begin{aligned} \dot{E} =& \gamma_2\beta e_x\dot{e}_x + \gamma_2\alpha\beta e_y\dot{e}_y + \gamma_2\alpha e_z\dot{e}_z + \alpha\beta e_w\dot{e}_w \\ =& \gamma_2\beta e_x[-(\alpha + \alpha m + \delta_x)e_x + \alpha e_y] \\ & + \gamma_2\alpha\beta e_y[e_x - (1 + \delta_y)e_y + e_z] \\ & + \gamma_2\alpha e_z[-\beta e_y + \beta e_w] + \alpha\beta e_w[-\gamma_2 e_z - \gamma_1\gamma_2 e_w] \\ =& -\gamma_1\gamma_2\alpha\beta e_w^2 + [-\gamma_2\beta(\alpha + \alpha m + \delta_x)e_x^2 \\ & + 2\gamma_2\alpha\beta e_x e_y - \gamma_2\alpha\beta(1 + \delta_y)e_y^2] \\ =& -\gamma_1\gamma_2\alpha\beta e_w^2 + [Ae_x^2 + Be_x e_y + Ce_y^2], \end{aligned} \tag{8}$$

where $A = -\gamma_2\beta(\alpha + \alpha m + \delta_x)$, $B = 2\gamma_2\alpha\beta$ and $C = -\gamma_2\alpha\beta(1 + \delta_y)$ are the real coefficients of the quadratic form $[Ae_x^2 + Be_x e_y + Ce_y^2]$ given by Eq.(8).

We further consider the discriminant

$$B^2-4AC=-4\gamma_2\alpha\beta^2[\alpha m+\delta_x+(\alpha+\alpha m+\delta_x)\delta_y]; \quad (9)$$

if $\alpha m+\delta_x+(\alpha+\alpha m+\delta_x)\delta_y>0$, then $B^2-4AC<0$ and $A<0$ hold. In this case, the quadratic form $[Ae_x^2+Be_xe_y+Ce_y^2]$ is negative definite. According to Eq.(8), $\dot{E}<0$ holds. On the other hand, if $\alpha m_a+\delta_x+(\alpha+\alpha m_a+\delta_x)\delta_y>0$, then $\alpha m+\delta_x+(\alpha+\alpha m+\delta_x)\delta_y>0$ holds for $\forall m\in[m_a,m_b]$. We have $\dot{E}<0$. Based on the Lyapunov stability theorem,[20] the error system (6) is globally asymptotically stable, i.e. $e\to 0$.

Especially, if there exist $\delta_y=0$ and $\delta_x>-\alpha m_a$, then $\alpha m_a+\delta_x+(\alpha+\alpha m_a+\delta_x)\delta_y>0$ holds. The error system (6) is globally asymptotically stable.

3. Simulation of the coupled synchronization for N-scroll chaotic attractors

Let us assume that the drive and response systems have the same parameters α=9.934, β=14.334, γ_1=27.333, γ_2=0.0497, $m_0=m_2=m_4=m_a=-1.246$ and $m_1=m_3=m_5=m_b=-0.6724$. We choose two sets of the coupled coefficients as examples: (1)δ_x=10, δ_y=1.5; (2)δ_x=18, δ_y=0. Since these coupled coefficients satisfy $\alpha m_a+\delta_x+(\alpha+\alpha m_a+\delta_x)\delta_y>0$, the N-scroll chaotic attractors defined by Eqs.(1) and (2) can be synchronized. Simulation results are shown in Fig.1.

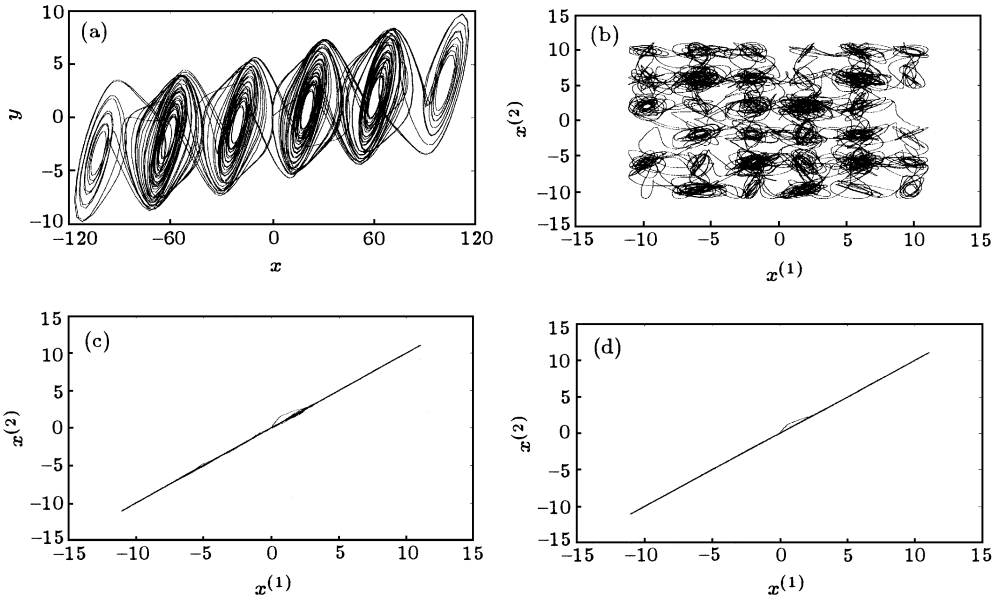


Fig.1. Simulation results of 6-scroll chaotic attractors and their synchronization. (a) 6-scroll chaotic attractor, (b) asynchronous (δ_x=0, $\delta_y=0$), (c) synchronous (δ_x=10, $\delta_y=1.5$), (d) synchronous (δ_x=18, $\delta_y=0$).

4. Circuitry design for generating 4- and 6-scroll chaotic attractors

Substituting the first equation of Eq.(3) into Eq.(1), we can obtain the circuit state equation of the modified Chua circuit utilized to obtain N-scroll chaotic attractors

$$\begin{cases}\dfrac{\mathrm{d}V_{C1}}{\mathrm{d}t}=\dfrac{1}{RC_1}(V_{C2}-V_{C1})-\dfrac{1}{C_1}f(V_{C1}),\\ \dfrac{\mathrm{d}V_{C2}}{\mathrm{d}t}=\dfrac{1}{RC_2}(V_{C1}-V_{C2})+\dfrac{1}{C_2}i_L,\\ \dfrac{\mathrm{d}i_L}{\mathrm{d}t}=\dfrac{1}{L}(V_{C3}-V_{C2}),\\ \dfrac{\mathrm{d}V_{C3}}{\mathrm{d}t}=-\dfrac{1}{C_3}i_L-\dfrac{1}{R_0C_3}V_{C3},\end{cases} \quad (10)$$

where $C_2/C_1=\alpha$, $R^2C_2/L=\beta$, $R/R_0=\gamma_1$, $C_2/C_3=\gamma_2$, $V_{C1}/B_P=x$, $V_{C2}/B_P=y$, $i_LR/B_P=z$, $V_{C3}/B_P=w$, $t/RC_2=\tau$, $B_P=1V$ and $f(V_{C1})=g(V_{C1})/R$. $f(\cdot)$ is defined by

$$\begin{aligned}f(V_{C1})=&\frac{m_5}{R}V_{C1}+0.5\sum_{i=1}^{5}\left(\frac{m_{i-1}}{R}-\frac{m_i}{R}\right)\\&\cdot(|V_{C1}+E_i|-|V_{C1}-E_i|)\\=&G_5V_{C1}+0.5\sum_{i=1}^{5}(G_{i-1}-G_i)\\&\cdot(|V_{C1}+E_i|-|V_{C1}-E_i|),\end{aligned} \quad (11)$$

where $E_i=x_iB_P$ (i=1, 2, 3, 4, 5) are the breakpoint voltages, and $G_i=m_i/R$ (i=0, 1, 2, 3, 4, 5) are

the slope values of the 11-segment piecewise-linear odd function $f(\cdot)$, respectively.

According to Eq.(4), the breakpoint voltages E_i (i=1, 2, 3, 4, 5) of $f(V_{C1})$ can be expressed in the following recurrence form:

$$E_{i+1}=\frac{2\sum_{k=1}^{i}(G_k-G_{k-1})E_k}{G_i+\frac{\gamma_1}{1+\gamma_1}G}-E_i \quad (i=1,2,3,4), \tag{12}$$

where $G=1/R$.

The circuitry used to generate 4- and 6-scroll chaotic attractors is shown in Fig.2, where the block inside the dotted line is equivalent to Chua's diode N_{R} described by $I_N = G_5V_{C1} + 0.5\sum_{i=1}^{5}(G_{i-1}-G_i)\cdot(|V_{C1}+E_i|-|V_{C1}-E_i|)$. The circuitry parameters are selected with C_1=10nF, C_2=99.34nF, C_3=2μF, L=18.64mH, R=1/G=1.64kΩ and R_0=60Ω. All op amps are TL082. Each op amp including its peripheral resistors works as a branch circuit for N_{R}, producing corresponding E_i (i= 1, 2, 3, 4, 5) (except the topside branch circuit). In Fig.2, 4- or 6-scroll chaotic attractors can be generated when the switch K is off or on, respectively.

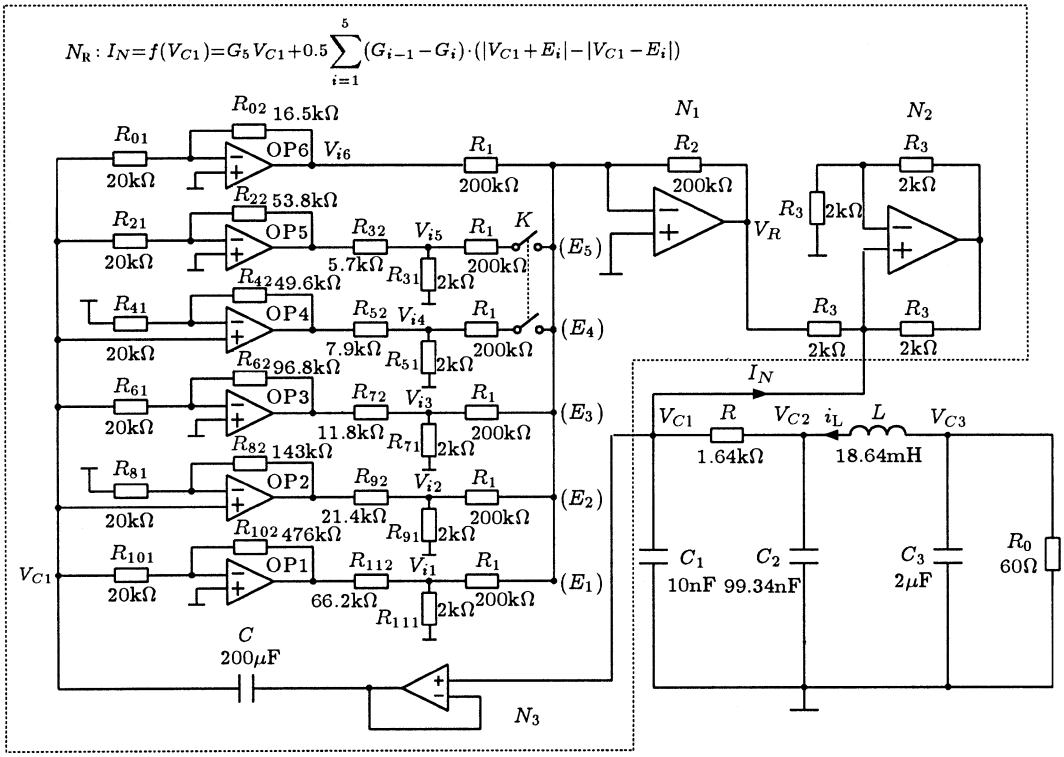


Fig.2. Experimental circuitry for generating 4–6-scroll chaotic attractors.

Let the first breakpoint voltage be E_1=0.6V. Using Eq.(12), we have the other breakpoint voltages E_i (i=1, 2, 3, 4, 5) as follows:

$$\begin{aligned}E_i &=[E_1\ E_2\ E_3\ E_4\ E_5]\\ &=[0.60\mathrm{V}\ 1.75\mathrm{V}\ 2.95\mathrm{V}\ 4.11\mathrm{V}\ 5.31\mathrm{V}].\end{aligned} \tag{13}$$

The ratio $R_{i,2}/R_{i,1}$ (i=2, 4, 6, 8, 10) in Fig.2 are given by

$$\begin{cases} R_{22}/R_{21}=E_{\mathrm{sat}}/E_5,\\ R_{42}/R_{41}=E_{\mathrm{sat}}/E_4-1,\\ R_{62}/R_{61}=E_{\mathrm{sat}}/E_3,\\ R_{82}/R_{81}=E_{\mathrm{sat}}/E_2-1,\\ R_{102}/R_{101}=E_{\mathrm{sat}}/E_1, \end{cases} \tag{14}$$

where E_{sat}=14.3V is the saturation voltage.

In Fig.2, the output–input relationship of the voltage–current converter N_2 is

$$I_N=-V_R/R_3. \tag{15}$$

Based on the superposition theorem, the output–input

relationship of the adder N_1 is given by

$$\begin{aligned} V_R &= -\frac{R_2}{R_1}(V_{i1}+V_{i2}+V_{i3}+V_{i4}+V_{i5}+V_{i6}) \\ &= -(V_{i1}+V_{i2}+V_{i3}+V_{i4}+V_{i5}+V_{i6}), \end{aligned} \quad (16)$$

where $R_1 = R_2$=200kΩ. V_{ij} $(j = 1, 2, \cdots, 6)$ denote the output voltages of each op amp, as shown in Fig.2. From Eqs.(15) and (16), we have

$$\begin{aligned} I_N &= -V_R/R_3 \\ &= (V_{i1}+V_{i2}+V_{i3}+V_{i4}+V_{i5}+V_{i6})/R_3. \end{aligned} \quad (17)$$

According to Eq.(17), each slope value G_i (i=0, 1, 2, 3, 4, 5) in the corresponding linear region can be determined.

(1) In the linear region $|V_{C1}| > E_5$, the op amps OP1–OP5 are saturated, and their outputs are constant. The output–input relationship of the op amp OP6 is linear. Using Eq.(17), we have

$$\begin{cases} I_N = \dfrac{1}{R_3}(V_{i6}+K_1) = \dfrac{1}{R_3}\left(-\dfrac{R_{02}}{R_{01}}V_{C1}+K_1\right) \\ \quad (K_1 \text{ is constant}), \\ G_5 = \dfrac{\mathrm{d}I_N}{\mathrm{d}V_{C1}} = -\dfrac{1}{R_3}\dfrac{R_{02}}{R_{01}}. \end{cases}$$

(2) In the linear region, $E_4 < |V_{C1}| < E_5$, the op amps OP1–OP4 are saturated, and their outputs are constant. The output–input relationship of the op amps OP5 and OP6 is linear. Using Eq.(17), we have

$$\begin{cases} I_N = \dfrac{1}{R_3}(V_{i5}+V_{i6}+K_2) = \dfrac{1}{R_3}\left(-\dfrac{R_{22}}{R_{21}}\dfrac{R_{31}}{R_{31}+R_{32}}V_{C1}-\dfrac{R_{02}}{R_{01}}V_{C1}+K_2\right), (K_2 \text{ is constant}), \\ G_4 = \dfrac{\mathrm{d}I_N}{\mathrm{d}V_{C1}} = -\dfrac{1}{R_3}\dfrac{R_{02}}{R_{01}} - \dfrac{1}{R_3}\dfrac{R_{22}}{R_{21}}\dfrac{R_{31}}{R_{31}+R_{32}} = G_5 - \dfrac{1}{R_3}\dfrac{R_{22}}{R_{21}}\dfrac{R_{31}}{R_{31}+R_{32}}. \end{cases}$$

(3) In the linear region, $E_3 < |V_{C1}| < E_4$, the op amps OP1–OP3 are saturated, and their outputs are constant. The output–input relationship of the op amps OP4–OP6 is linear. Using Eq.(17), we have

$$\begin{cases} I_N = \dfrac{1}{R_3}(V_{i4}+V_{i5}+V_{i6}+K_3) = \dfrac{1}{R_3}\left[\left(1+\dfrac{R_{42}}{R_{41}}\right)\dfrac{R_{51}}{R_{51}+R_{52}} - \dfrac{R_{22}}{R_{21}}\dfrac{R_{31}}{R_{31}+R_{32}}V_{C1} - \dfrac{R_{02}}{R_{01}}V_{C1}+K_3\right], \\ \quad (K_3 \text{ is constant}), \\ G_3 = \dfrac{dI_N}{\mathrm{d}V_{C1}} = \dfrac{1}{R_3}\left(1+\dfrac{R_{42}}{R_{41}}\right)\dfrac{R_{51}}{R_{51}+R_{52}} - \dfrac{1}{R_3}\dfrac{R_{02}}{R_{01}} - \dfrac{1}{R_3}\dfrac{R_{22}}{R_{21}}\dfrac{R_{31}}{R_{31}+R_{32}} = G_4 + \dfrac{1}{R_3}\left(1+\dfrac{R_{42}}{R_{41}}\right)\dfrac{R_{51}}{R_{51}+R_{52}}. \end{cases}$$

(4) In the linear regions $E_2 < |V_{C1}| < E_3$, $E_1 < |V_{C1}| < E_2$ and $|V_{C1}| < E_1$, the expressions for G_2, G_1 and G_0 can be obtained by using the same procedure, respectively. Finally, the recurrence formula for ratios $R_{i,2}/R_{i,1} (i = 0, 3, 5, \cdots, 11)$ is given by

$$\begin{cases} \dfrac{R_{02}}{R_{01}} = -R_3 G_5, \\ \dfrac{R_{32}}{R_{31}} = \dfrac{R_{22}/R_{21}}{R_3(G_5-G_4)} - 1, \\ \dfrac{R_{52}}{R_{51}} = -\dfrac{1+R_{42}/R_{41}}{R_3(G_4-G_3)} - 1, \\ \dfrac{R_{72}}{R_{71}} = \dfrac{R_{62}/R_{61}}{R_3(G_3-G_2)} - 1, \\ \dfrac{R_{92}}{R_{91}} = -\dfrac{1+R_{82}/R_{81}}{R_3(G_2-G_1)} - 1, \\ \dfrac{R_{112}}{R_{111}} = \dfrac{R_{102}/R_{101}}{R_3(G_1-G_0)} - 1, \end{cases} \quad (18)$$

where $G_0 = G_2 = G_4 = m_a/R$, $G_1 = G_3 = G_5 = m_b/R$ and R_3=2kΩ. According to Eqs.(14) and (18), by choosing $R_{01} = R_{21} = R_{41} = R_{61} = R_{81} = R_{101}$= 20kΩ and $R_{31} = R_{51} = R_{71} = R_{91} = R_{111}$=2kΩ, we obtain R_{02}=16.48kΩ, R_{22}=53.8kΩ, R_{42}=49.6kΩ, R_{62}=96.8kΩ, R_{82}=143kΩ, R_{102}=476kΩ, R_{32}=5.7kΩ, R_{52}=7.94kΩ, R_{72}=11.84kΩ, R_{92}=21.4kΩ and R_{112}= 66.2kΩ, as shown in Fig.2.

5. Circuitry experimental results

The circuitry experimental results of generating 4- and 6-scroll chaotic attractors are shown in Fig.3. When the switch K is on, a 6-scroll chaotic attractor is obtained. When the switch K is off, a 4-scroll chaotic attractor is generated.

The drive and response systems are set to take the same parameters. The experimental circuitry of the unidirectionally coupled synchronization is shown in Fig.4, where N_{R} is the Chua diode, that part inside the dotted line in Fig.2. In the circuit experiment, we choose the coupling resistor R_E=0.7kΩ, the coupling coefficients $\delta_x = C_2R/C_1R_E = 23$ and $\delta_y = 0$. As a result, the inequality $\alpha m_a + \delta_x + (\alpha + \alpha m_a + \delta_x)\delta_y > 0$ holds. We see that the synchronization of two 6-scroll attractors can be achieved, as shown in Fig.5.

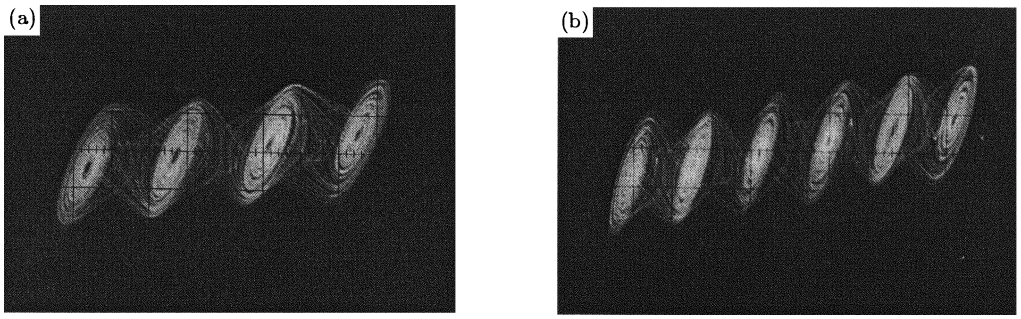


Fig.3. Experimental results of generating 4- and 6-scroll chaotic attractors. (a) 4-scroll chaotic attractor, (b) 6-scroll chaotic attractor.

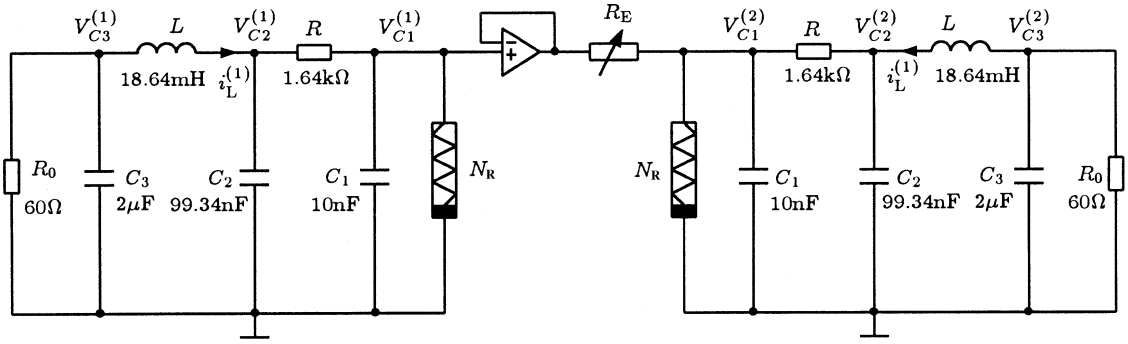


Fig.4. A circuitry used for unidirectionally coupled synchronization between two 6-scroll chaotic attractors.

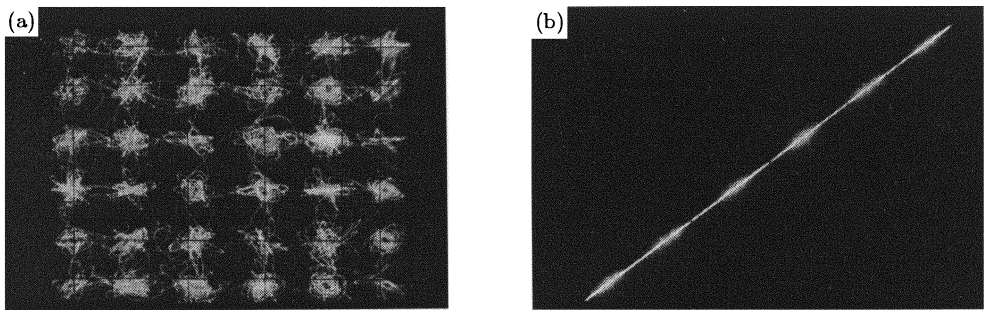


Fig.5. Experimental results of unidirectionally coupled synchronization between two 6-scroll attractors. (a) Asynchronous when not coupled, (b) synchronous when R_E=0.7kΩ.

6. Generation of N-scroll hyperchaotic attractors in the MCK system

A hyperchaotic version of Chua's circuit is the MCK circuit proposed in Ref.[17], which can generate a double-scroll hyperchaotic attractor. Its dimensionless state equation is

$$\begin{cases} \mathrm{d}x/\mathrm{d}\tau = \alpha[-g(x-y)-z], \\ \mathrm{d}y/\mathrm{d}\tau = \beta[g(x-y)-w], \\ \mathrm{d}z/\mathrm{d}\tau = x+z, \\ \mathrm{d}w/\mathrm{d}\tau = \gamma y, \end{cases} \tag{19}$$

where α=2, β=20, γ=1.5, $m_0 = -0.2$ and m_1=3.

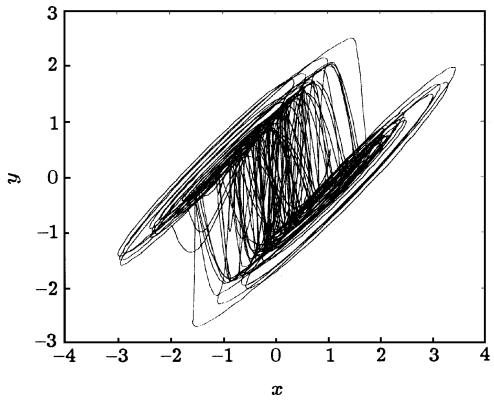


Fig.6. Double-scroll hyperchaotic attractor.

$g(x-y) = m_1(x-y) + 0.5(m_0 - m_1)[|x-y+x_1| - |x-y-x_1|]$ denotes a 3-segment piecewise-linear odd function and satisfies $g(x-y) = -g(y-x)$. The Lyapunov exponents are $[\mu_1, \mu_2, \mu_3, \mu_4]$=[0.24, 0.06, 0.00, –53.8]. A double-scroll hyperchaotic attractor is observed, as shown in Fig.6.

6.1. Derivation of the recurrence formula

Based on Ref.[17], by constructing a piecewise-linear odd function $g(\cdot)$ with $2N-1$ segments, we can generate N-scroll hyperchaotic attractors (N=3, 4, 5, 6, $\cdots$) in the MCK circuit. $g(\cdot)$ is defined by

$$g(x-y) = m_{N-1}(x-y) + 0.5\sum_{i=1}^{N-1}(m_{i-1}-m_i)\cdot(|x-y+x_i| - |x-y-x_i|),$$

$$= \begin{cases} m_0(x-y), & (|x-y| < x_1), \\ m_i(x-y) + \sum_{j=1}^{i}(m_{j-1}-m_j)x_j\dfrac{|x-y|}{x-y}, & (x_i \le |x-y| < x_{i+1}, i = 1,2,\cdots,N-2), \\ m_{N-1}(x-y) + \sum_{j=1}^{N-1}(m_{j-1}-m_j)x_j\dfrac{|x-y|}{x-y}, & (|x-y| \ge x_{N-1}). \end{cases} \tag{20}$$

Substituting Eq.(20) into Eq.(19), we obtain the recurrence formula of the equilibrium points $x_{\mathrm{eq},i}$ ($i = 1,2,\cdots,N-2$) for the x variable

$$\begin{cases} x_{\mathrm{eq},1} = \dfrac{\sum_{j=1}^{1}(m_j - m_{j-1})x_j}{m_1 - 1}, \\ x_{\mathrm{eq},2} = \dfrac{\sum_{j=1}^{2}(m_j - m_{j-1})x_j}{m_2 - 1}, \\ \cdots\cdots \\ x_{\mathrm{eq},N-3} = \dfrac{\sum_{j=1}^{N-3}(m_j - m_{j-1})x_j}{m_{N-3} - 1}, \\ x_{\mathrm{eq},N-2} = \dfrac{\sum_{j=1}^{N-2}(m_j - m_{j-1})x_j}{m_{N-2} - 1}, \end{cases} \tag{21}$$

where the first breakpoint x_1 can be arbitrary, and the other breakpoints x_i ($i = 2,3,\cdots,N-1$) are determined by use of the following recurrence formulae:

$$\begin{cases} \dfrac{x_2 - x_{\mathrm{eq},1}}{x_{\mathrm{eq},1} - x_1} = k_1, \\ \dfrac{x_3 - x_{\mathrm{eq},2}}{x_{\mathrm{eq},2} - x_2} = k_2, \\ \cdots\cdots \\ \dfrac{x_{N-2} - x_{\mathrm{eq},N-3}}{x_{\mathrm{eq},N-3} - x_{N-3}} = k_{N-3}, \\ \dfrac{x_{N-1} - x_{\mathrm{eq},N-2}}{x_{\mathrm{eq},N-2} - x_{N-2}} = k_{N-2}, \end{cases} \tag{22}$$

where k_i ($i = 1,2,\cdots,N-2$) denote the set of appropriate parameters used to determine the breakpoints x_i ($i = 2,3,\cdots,N-1$). Substituting Eq.(21) into Eq.(22), we obtain the recurrence formulae for the breakpoints x_i ($i = 2,3,\cdots,N-1$)

$$\begin{cases} x_2 = \dfrac{(1+k_1)\sum_{i=1}^{1}(m_i - m_{i-1})x_i}{m_1 - 1} - k_1 x_1, \\ x_3 = \dfrac{(1+k_2)\sum_{i=1}^{2}(m_i - m_{i-1})x_i}{m_2 - 1} - k_2 x_2, \\ \cdots\cdots \\ x_{N-2} = \dfrac{(1+k_{N-3})\sum_{i=1}^{N-3}(m_i - m_{i-1})x_i}{m_{N-3} - 1} \\ \qquad -k_{N-3}x_{N-3}, \\ x_{N-1} = \dfrac{(1+k_{N-2})\sum_{i=1}^{N-2}(m_i - m_{i-1})x_i}{m_{N-2} - 1} \\ \qquad -k_{N-2}x_{N-2}. \end{cases} \tag{23}$$

It is noticed that choosing a set of appropriate parameters k_i ($i = 1,2,\cdots$) and changing the position of equilibrium points $x_{\mathrm{eq},i}$ ($i = 1,2,\cdots$) in each piecewise-linear domain is the key step of generating N-scroll hyperchaotic attractors in the MCK system.

6.2. Calculation of the recurrence parameters

To generate 3–10-scroll hyperchaotic attractors, we should select a set of appropriate parameters k_i ($i = 1,2,\cdots,N-2$) and m_i ($i = 1,2,\cdots,N-1$). By use of Eq.(23), we can obtain the recurrence parameters for generating 3–10-scroll hyperchaotic attractors as follows.

(1) The recurrence parameters for generating 3-scroll hyperchaotic attractor (N=3) are

$$\begin{cases} [k_1] = [1.4], \\ [m_0, m_1, m_2] = [3, -0.8, 3], \\ [x_1, x_2] = [1.0000, 3.6667]. \end{cases} \tag{24}$$

(2) The recurrence parameters for generating 4-scroll hyperchaotic attractor (N=4) are

$$\begin{cases} [k_1, k_2] = [1.3, 1.3], \\ [m_0, m_1, m_2, m_3] = [-0.7, 2.9, -0.7, 2.9], \\ [x_1, x_2, x_3] = [1.0000, 3.0579, 6.0479]. \end{cases} \tag{25}$$

(3) The recurrence parameters for generating 5-scroll hyperchaotic attractor (N=5) are

$$\begin{cases} [k_1, k_2, k_3] = [1.1, 1.1, 1.3], \\ [m_0, m_1, m_2, m_3, m_4] = [3, -0.6, 3, -0.6, 3], \\ [x_1, x_2, x_3, x_4] = [1.0000, 3.6250, 5.9350, 9.4138]. \end{cases} \tag{26}$$

(4) The recurrence parameters for generating 6-scroll hyperchaotic attractor (N=6) are

$$\begin{cases} [k_1, k_2, k_3, k_4] = [1.1, 1.1, 1.3, 1.3], \\ [m_0, m_1, m_2, m_3, m_4, m_5] = [-0.9, 2.9, -0.9, 2.9, -0.8, 2.9], \\ [x_1, x_2, x_3, x_4, x_5] = [1.0000, 3.1000, 5.4100, 8.1930, 12.0019]. \end{cases} \tag{27}$$

(5) The recurrence parameters for generating 7-scroll hyperchaotic attractor (N=7) are

$$\begin{cases} [k_1, k_2, k_3, k_4, k_5] = [1.1, 1.1, 1.1, 1.4, 1.4], \\ [m_0, m_1, m_2, m_3, m_4, m_5, m_6] = [3, -0.6, 3, -0.6, 3, -0.6, 3], \\ [x_1, x_2, x_3, x_4, x_5, x_6] = [1.0000, 3.6250, 5.9350, 9.1113, 12.3057, 17.8958]. \end{cases} \tag{28}$$

(6) The recurrence parameters for generating 8-scroll hyperchaotic attractor (N=8) are

$$\begin{cases} [k_1, k_2, k_3, k_4, k_5, k_6] = [1.1, 1.1, 1.1, 1.1, 1.3, 1.3], \\ [m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7] = [-0.9, 2.9, -0.9, 2.9, -0.9, 2.9, -0.8, 2.9], \\ [x_1, x_2, x_3, x_4, x_5, x_6, x_7] = [1.0000, 3.1000, 5.4100, 7.9510, 10.7461, 14.1135, 18.7544]. \end{cases} \tag{29}$$

(7) The recurrence parameters for generating 9-scroll hyperchaotic attractor (N=9) are

$$\begin{cases} [k_1, k_2, k_3, k_4, k_5, k_6, k_7] = [1.1, 1.1, 1.1, 1.1, 1.1, 1.2, 1.4], \\ [m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8] = [3, -0.9, 3, -0.9, 3, -0.9, 3, -0.8, 3], \\ [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8] = [1.0000, 3.2105, 5.5205, 8.1953, 10.9904, 14.2268, 17.7699, 22.9435]. \end{cases} \tag{30}$$

(8) The recurrence parameters for generating 10-scroll hyperchaotic attractor (N=10) are

$$\begin{cases} [k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8] = [1.1, 1.1, 1.1, 1.1, 1.1, 1.1, 1.3, 1.3], \\ [m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9] = [-0.9, 2.9, -0.9, 2.9, -0.9, 2.9, -0.9, 2.9, -0.8, 2.9], \\ [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9] = [1.0000, 3.1000, 5.4100, 7.9510, 10.7461, 13.8207, 17.2028, 21.2774, 26.9686]. \end{cases} \tag{31}$$

6.3. Simulation results of 3–10-scroll hyperchaotic attractors

According to the recurrence parameters provided in section 6.2, the generation of 3–10-scroll hyperchaotic attractors can be observed by computer simulation, as shown in Fig.7.

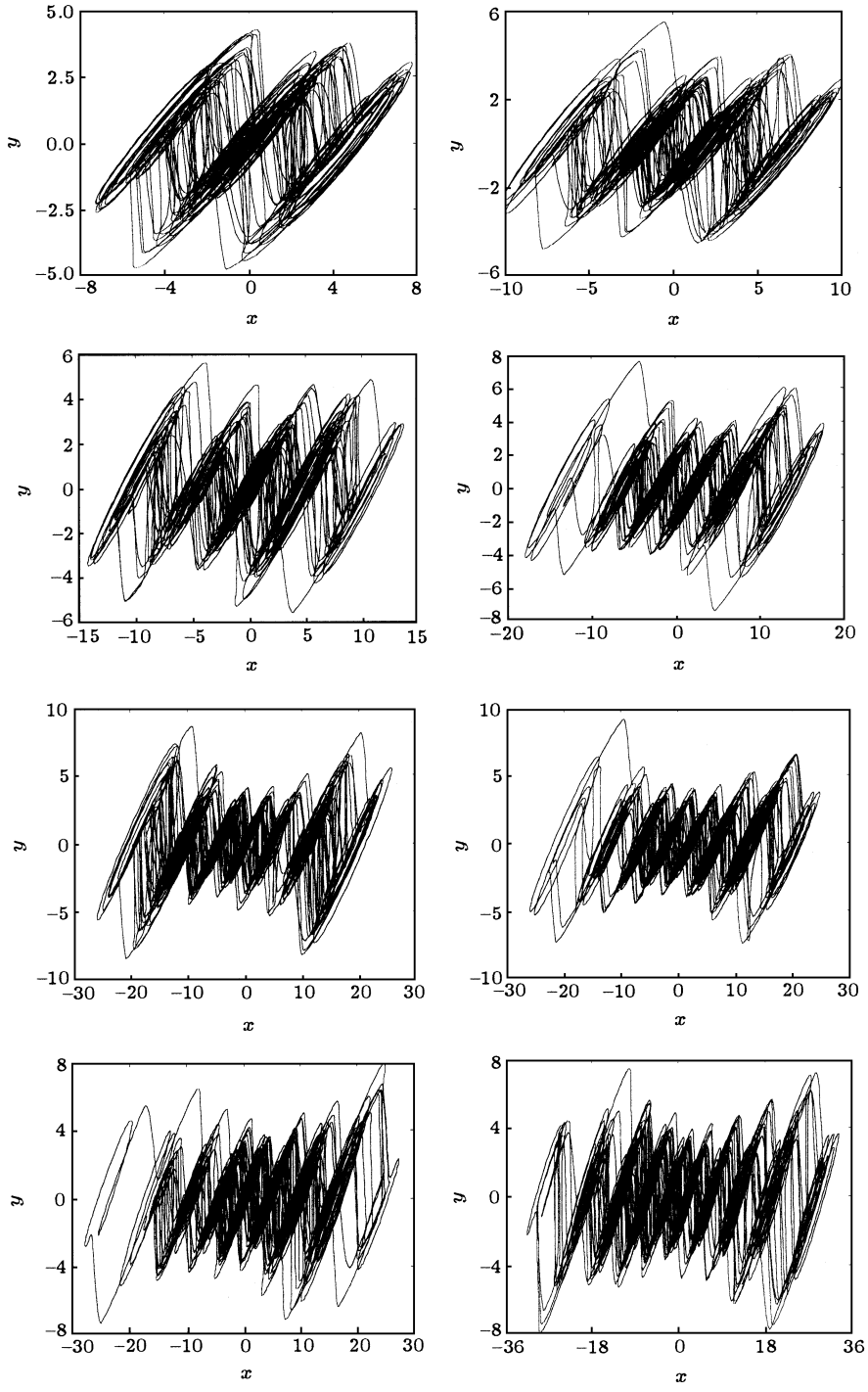


Fig.7. Computer simulation results for 3–10-scroll hyperchaotic attractors.

7. Unidirectionally coupled synchronization of N-scroll hyperchaotic attractors

The synchronization of two N-scroll hyperchaotic attractors is investigated in this section. Here we consider the synchronization of two 6-scroll hyperchaotic attractors. According to Eqs.(19), (20) and (27), the state equation of the drive system is written in the dimensionless form

$$\begin{cases} \mathrm{d}x^{(1)}/\mathrm{d}\tau = \alpha[g(y^{(1)} - x^{(1)}) - z^{(1)}], \\ \mathrm{d}y^{(1)}/\mathrm{d}\tau = \beta[-g(y^{(1)} - x^{(1)}) - w^{(1)}], \\ \mathrm{d}z^{(1)}/\mathrm{d}\tau = x^{(1)} + z^{(1)}, \\ \mathrm{d}w^{(1)}/\mathrm{d}\tau = \gamma y^{(1)}, \end{cases} \tag{32}$$

The associated state equation of the response system

is written as

$$\begin{cases} \mathrm{d}x^{(2)}/\mathrm{d}\tau = \alpha[g(y^{(2)} - x^{(2)}) - z^{(2)}] + \delta_x(x^{(1)} - x^{(2)}), \\ \mathrm{d}y^{(2)}/\mathrm{d}\tau = \beta[-g(y^{(2)} - x^{(2)}) - w^{(2)}] + \delta_y(y^{(1)} - y^{(2)}), \\ \mathrm{d}z^{(2)}/\mathrm{d}\tau = x^{(2)} + z^{(2)} + \delta_z(z^{(1)} - z^{(2)}), \\ \mathrm{d}w^{(2)}/\mathrm{d}\tau = \gamma y^{(2)} + \delta_w(w^{(1)} - w^{(2)}), \end{cases} \tag{33}$$

where $[\delta_x,\ \delta_y,\ \delta_z,\ \delta_w]$ denote the coupling coefficients and $g(\cdot)$ is the 11-segment piecewise-linear function defined by Eq.(20). The slope values m_i $(i = 1, 2, \cdots, 5)$ and the breakpoint values x_i $(i = 2, 3, \cdots, 5)$ are given by Eq.(27).

Since $g(y - x)$ is an 11-segment piecewise-linear function, we have

$$\begin{aligned} & g(y^{(1)} - x^{(1)}) - g(y^{(2)} - x^{(2)}) \\ =& m[(y^{(1)} - x^{(1)}) - (y^{(2)} - x^{(2)})] \\ =& m[(y^{(1)} - y^{(2)}) - (x^{(1)} - x^{(2)})] \\ =& m[e_y - e_x], \end{aligned} \tag{34}$$

where m is a variable parameter [19] whose value satisfies $m_a \le m \le m_b$ with minimum slope $m_a = -0.9$ and maximum slope $m_b = 2.9$. Subtracting Eq.(33) from Eq.(32), we have the error equation in the matrix form

$$\dot{e} = (A - K + M)e, \tag{35}$$

where

$$\dot{e} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \\ \dot{e}_w \end{bmatrix}, \quad e = \begin{bmatrix} e_x \\ e_y \\ e_z \\ e_w \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\beta \\ 1 & 0 & 1 & 0 \\ 0 & \gamma & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} \delta_x & 0 & 0 & 0 \\ 0 & \delta_y & 0 & 0 \\ 0 & 0 & \delta_z & 0 \\ 0 & 0 & 0 & \delta_w \end{bmatrix}$$

and

$$M = \begin{bmatrix} -\alpha m & \alpha m & 0 & 0 \\ \beta m & -\beta m & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$e_x = x^{(1)} - x^{(2)}$, $e_y = y^{(1)} - y^{(2)}$, $e_z = z^{(1)} - z^{(2)}$ and $e_w = w^{(1)} - w^{(2)}$ are the error signals for the drive system (32) and the response systems (33).

Theorem 1 [19] If there exists a positive definite symmetric constant matrix P such that

$$(A - K + M)^{\mathrm{T}}P + P(A - K + M) \le \mu I < 0$$

uniformly for all $x^{(1)}$, $x^{(2)}$, $y^{(1)}$ and $y^{(2)}$ in the phase space, where μ denotes a negative constant, and I is the identity matrix, then the error system (35) is globally exponentially stable.

According to Theorem 1, the conditions of the unidirectionally coupled synchronization between the drive system (32) and the response system (33) can be mathematically obtained.

By choosing $P = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix}$, $p_i > 0 (i = 1, 2, 3, 4)$, $\mu I = \begin{bmatrix} \mu & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix}$ and $\mu < 0$, we have

$$\begin{aligned} & (A - K + M)^{\mathrm{T}}P + P(A - K + M) - \mu I \\ =& \begin{bmatrix} -2p_1\left(\alpha m + \delta_x + \dfrac{\mu}{2p_1}\right) & \alpha m p_1 + \beta m p_2 & -\alpha p_1 + p_3 & 0 \\ \alpha m p_1 + \beta m p_2 & -2p_2\left(\beta m + \delta_y + \dfrac{\mu}{2p_2}\right) & 0 & -\beta p_2 + \gamma p_4 \\ -\alpha p_1 + p_3 & 0 & -2p_3\left(\delta_z - 1 + \dfrac{\mu}{2p_3}\right) & 0 \\ 0 & -\beta p_2 + \gamma p_4 & 0 & -2p_4\left(\delta_w + \dfrac{\mu}{2p_4}\right) \end{bmatrix}, \end{aligned} \tag{36}$$

to ensure $(A-K+M)^{\mathrm{T}}P+P(A-K+M)-\mu I<0$, the matrix $(A-K+M)^{\mathrm{T}}P+P(A-K+M)-\mu I$ should be negative definite, and its N-order (N=1, 2, 3, 4) subdeterminants satisfy

$$\Delta_1 = -2p_1\left(\alpha m+\delta_x+\frac{\mu}{2p_1}\right)<0, \tag{37}$$

$$\Delta_2 = \det\begin{bmatrix} -2p_1\left(\alpha m+\delta_x+\dfrac{\mu}{2p_1}\right) & \alpha m p_1+\beta m p_2 \\ \alpha m p_1+\beta m p_2 & -2p_2\left(\beta m+\delta_y+\dfrac{\mu}{2p_2}\right)\end{bmatrix}>0, \tag{38}$$

$$\Delta_3 = \det\begin{bmatrix} -2p_1\left(\alpha m+\delta_x+\dfrac{\mu}{2p_1}\right) & \alpha m p_1+\beta m p_2 & -\alpha p_1+p_3 \\ \alpha m p_1+\beta m p_2 & -2p_2\left(\beta m+\delta_y+\dfrac{\mu}{2p_2}\right) & 0 \\ -\alpha p_1+p_3 & 0 & -2p_3\left(\delta_z-1+\dfrac{\mu}{2p_3}\right)\end{bmatrix}<0, \tag{39}$$

$$\Delta_4 = \det\begin{bmatrix} -2p_1\left(\alpha m+\delta_x+\dfrac{\mu}{2p_1}\right) & \alpha m p_1+\beta m p_2 & -\alpha p_1+p_3 & 0 \\ \alpha m p_1+\beta m p_2 & -2p_2\left(\beta m+\delta_y+\dfrac{\mu}{2p_2}\right) & 0 & -\beta p_2+\gamma p_4 \\ -\alpha p_1+p_3 & 0 & -2p_3\left(\delta_z-1+\dfrac{\mu}{2p_3}\right) & 0 \\ 0 & -\beta p_2+\gamma p_4 & 0 & -2p_4\left(\delta_w+\dfrac{\mu}{2p_4}\right)\end{bmatrix}>0. \tag{40}$$

Taking $\alpha p_1=p_3$ and $\beta p_2=\gamma p_4$, from Eqs.(37)–(40), we can obtain the inequalities of the coupling coefficients $[\delta_x,\delta_y,\delta_z,\delta_w]$

$$\delta_x > -\alpha m-\frac{\mu}{2p_1}, \tag{41}$$

$$\delta_y > \frac{(\alpha m p_1+\beta m p_2)^2}{4p_1p_2(\alpha m+\delta_x+\frac{\mu}{2p_1})}-\beta m-\frac{\mu}{2p_2}, \tag{42}$$

$$\delta_z > 1-\frac{\mu}{2p_3}+\frac{1}{2p_3}\left[4p_1p_2\left(\alpha m+\delta_x+\frac{\mu}{2p_1}\right)\times\left(\beta m+\delta_y+\frac{\mu}{2p_2}\right)-(\alpha m p_1+\beta m p_2)^2\right], \tag{43}$$

$$\delta_w > -\frac{\mu}{2p_4}. \tag{44}$$

If the coupling coefficients $[\delta_x,\ \delta_y,\ \delta_z,\ \delta_w]$ are large enough, then the inequalities (41)–(44) hold for $\forall m\in[m_a,m_b]$. We obtain $(A-K+M)^{\mathrm{T}}P+P(A-K+M)-\mu I<0$. According to Theorem 1, the error system (35) is globally exponentially stable, i.e. $e\to 0$. Thus, the response system (33) is synchronized globally with the drive system (32).

As an example, we choose

$$\begin{aligned}\mu &= -0.2,\\ P &= \begin{bmatrix} p_1 & 0 & 0 & 0\\ 0 & p_2 & 0 & 0\\ 0 & 0 & p_3 & 0\\ 0 & 0 & 0 & p_4\end{bmatrix}\\ &= \begin{bmatrix} 0.15 & 0 & 0 & 0\\ 0 & 0.15 & 0 & 0\\ 0 & 0 & 0.3 & 0\\ 0 & 0 & 0 & 2\end{bmatrix},\end{aligned}$$

δ_x=5, δ_y=60, δ_z=60 and δ_w=6. Inequalities (41)–(44) hold for $\forall m\in[m_a,m_b]$ with minimum slope $m_a=-0.9$ and maximum slope $m_b=2.9$. Based on Theorem 1, the response system (33) is synchronized globally with the drive system (32). Simulation results are shown in Fig.8.

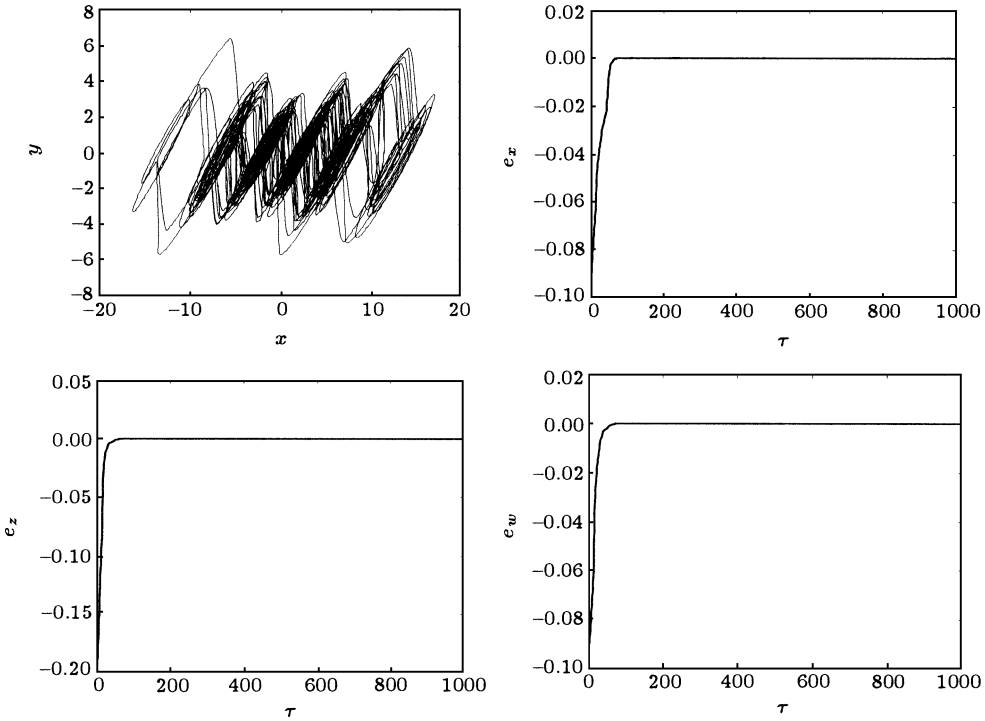


Fig.8. Numerical simulation results of unidirectionally coupled synchronization for two 6-scroll hyperchaotic attractors (δ_x=5, δ_y=60, δ_z=60 and δ_w=6).

8. Conclusions

The results of theoretical analysis, computer simulation and circuitry implementation for the generation and synchronization of N-scroll chaotic and hyperchaotic attractors in fourth-order systems are given in this paper. Based on our previous works and Lyapunov stability theory, the generation and synchronization of N-scroll chaotic attractors in the fourth-order modified Chua circuit are investigated. The circuit experiment results accord with the theoretical analysis and computer simulation rather satisfactorily, which confirms the effectiveness of the proposed method. Furthermore, by introducing a set of appropriate parameters k_i $(i = 1, 2, 3, \cdots)$ in Eq.(22) and using the derived recurrence formula given by Eq.(23), we can present a novel scheme for producing a new family of N-scroll hyperchaotic attractors in the fourth-order MCK system. This is an important part of our studies. In addition, computer simulations for 3–10-scroll hyperchaotic attractors are also shown in this paper. More complicated N-scroll hyperchaotic attractors are potentially useful for enhancing the security of chaotic communication schemes.

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