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Noise reduction method based on weighted manifold decomposition*

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A noise reduction method based on weighted manifold decomposition is proposed in this paper, which does not need knowledge of the chaotic dynamics and choosing number of eigenvalues. The simulation indicates that the performance of this method can increase the signal-to-noise ratio of noisy chaotic time series.

Keywords: noise reduction, manifold decomposition, project filtering, eigenvector
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1. Introduction

In recent years, the study of nonlinear dynamical systems and chaos has been motivated by the relationship between chaotic signals and random processes. One of the applications of chaotic signal processing is the secure communication problem [1] that exploits the self-synchronization property of the chaotic system. But, if the level of additive noise due to the transmitting channel exceeds 10% of the driving signal, synchronization will fail. Therefore, research on the noise reduction algorithm is very important.

Many researchers have proposed different methods for enhancing the contaminated chaotic signals. There are mainly three classes of methods in these algorithms. The first is a version exploiting cost function,[2−4] the second is a version exploiting projective filtering,[5−12] and the third is a version exploiting the manifold decomposition method.[13] The projective filtering method projects each point on the trajectory in the phase space onto the approximation to the noisy attractor to produce a cleaned vector by using the Singular Value Decomposition (SVD) technique, and the manifold decomposition method is based on that the phase space can be decomposed into an unstable manifold subspace and a stable manifold subspace. The unstable manifold subspace is spanned by eigenvectors of eigenvalues corresponding to positive Lyapunov exponents of the Jacobian form, and these eigenvectors denote the expanding directions of the attractor. The stable manifold subspace is spanned by eigenvectors of eigenvalues corresponding to negative Lyapunov exponents of the Jacobian form, and these eigenvectors denote the contracting directions of the attractor.

The algorithm proposed in this paper exploits the above SVD and manifold decomposition idea. If an observed noisy scalar chaotic time series has been decomposed into stable manifold subspace and unstable manifold subspace, then it is easily contaminated by noise in stable directions and noise more rarely contaminates the unstable directions than stable directions of noise-free points. However, the true chaotic attractor consists of stable and unstable subsections. Therefore, we can get a cleaned chaotic time series by weighted manifold decomposition that gives a larger weight in the stable directions and smaller weight in the unstable directions so as to preserve the true attractor information and clean the noise as much as possible. The smaller the eigenvalue is, the larger is the weight given. This is the core idea of noise reduction in this paper.

In Ref.[13] about manifold decomposition, it is assumed that the dynamics of the chaotic system is known, but our method allows the dynamics of the chaotic system to be unknown by exploiting SVD. Exploiting SVD [5−12,14,15] requires preserving larger eigenvalues and throwing off smaller ones, but our

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method can preserve all eigenvalues by weighting to avoid missing information of the true attractor.

2. Noise reduction

Suppose that there is a discrete time chaotic dynamical series $\{x_n,\ n{=}1,\ 2,\ \cdots, N\}$ which is described by a difference equation

$$X_{n+1} = f(X_n), n = 0, 1, 2, \cdots, N, \tag{1}$$

where $X_n \in \mathbf{R}^m$ is the state vector at time n, and f: $\mathbf{R}^m \mapsto \mathbf{R}^m$ describes the dynamics of the system. In a more general context, the relation between x_n and X_n is that the scalar time series $\{x_n,\ n{=}1,\ 2,\cdots, N\}$ can be unfolded in a m-dimensional effective space using delay coordinates

$$X_n = [x_n \cdots x_{n-(m-1)\tau}], \tag{2}$$

where τ is the delay time, and the set of reconstructed points X_n is an embedding of the original attractor of the chaotic dynamical system under certain conditions,[16] and the delay vector set $\{X_n\}$ can fill the available m-dimensional space in an inhomogeneous way. The shadowing lemma [13] indicates that we can find a deterministic chaotic orbit that is arbitrarily close to an observed noisy orbit. Therefore the noise reduction can be regarded as an approximation to the true orbit by finding the optimal shadowing orbit.

The manifold decomposition method is one of those approximate ones. If the chaotic dynamics is linearized as

$$\begin{aligned}&(X_{n+1} - f(X))\\ =&J_n f(X_n)(X_n - X) + o(\|X_n - X\|^2),\end{aligned} \tag{3}$$

where $J_n f\ (.)$ is the Jacobian of $f(.)$ at x_n and X is in the vicinity of point x_n, then the true manifold, that is the attractor, can be locally decomposed into stable subspace and unstable subspace. The unstable manifold subspace is spanned by eigenvectors of eigenvalues corresponding to positive Lyapunov exponents of the Jacobian, and these eigenvectors denote the expanding directions of the attractor. The stable manifold subspace is spanned by eigenvectors of eigenvalues corresponding to negative Lyapunov exponents of the Jacobian, and these eigenvectors denote the contracting directions of the attractor. That is,

$$\begin{aligned}f(X) \approx& J_n f(x_n)X + \varepsilon_n\\ =&\sum_{i=1}^{p} \alpha_n^i u_i + \sum_{j=1}^{q} \beta_n^j v_j + \varepsilon_n, p + q = m,\end{aligned} \tag{4}$$

where u_i and v_j are unit vectors in the expanding (unstable) and contracting (stable) directions of point x_n, respectively, p and q are numbers of unit vectors in the expanding and contracting directions of point x_n, respectively, α and β are coefficients in the expanding and contracting directions of point x_n, respectively, and ε_n is the error.

In the noise-free case, the q smallest principal directions which locally span the stable manifold would have the smallest eigenvalues corresponding to Lyapunov exponents less than or equal to zero, while the remaining p directions locally span the unstable manifold corresponding to Lyapunov exponents greater than zero. In the presence of noise, we identify the q directions with smallest eigenvalues as directions dominated by the noise, and the noise has less effect on the remaining p directions corresponding to larger eigenvalues. For the noise reduction we would exploit an algorithm similar to projective filtering [10] to remove the nullspace spanned by several smallest eigenvectors among the m eigenvectors and to preserve attractor subspace spanned by the remaining large eigenvalues at point x_n in the m-dimensional phase space. If we do not know the accurate numbers of dimensions of the removed nullspace, then some information of the attractor is likely to be omitted. Because noise is more distinguished in the stable directions and less in the unstable directions, we can introduce large weight for the directions with small eigenvalues and less weight for the directions with large eigenvalues to promote the effect of noise reduction. The smaller the eigenvalue, the larger is the weight given.

3. Proposed algorithm

The algorithm proposed in this paper consists of the following steps. First, find a m-dimensional approximation to the attractor described by the trajectory $\{X_n\}$ in the m-dimensional phase space based on the SVD. Secondly, produce a cleaned vector X_n by using weighted combination of unstable and stable subspaces. Thirdly, convert the sequence of cleaned vectors X_n back into the scalar time domain to produce a cleaned time series x_n.

The m-dimensional approximation in the m-dimensional phase space based on the SVD is not completely similar to a local version of principal component analysis, and takes the whole version instead of a

local version in delay coordinate space. The ε neighbourhood in delay coordinate space around a point x_n, that is a set of nearby points, can be defined as

$$U_n^\varepsilon = \{X_n' : \|X_n' - X_n\| < \varepsilon, n' \leq n\}, \tag{5}$$

where $\|\cdot\|$ denotes the Euclidean distance in delay coordinate space, and we use the second norm to compute distance. Let $|U_n^\varepsilon|$ denote the number of vectors in the neighbourhood. If the local centres-of-mass can be described as

$$X^n = \frac{1}{|U_n^\varepsilon|} \sum_{X_{n'} \in U_n^\varepsilon} X_{n'}, \tag{6}$$

then the tangent point in a linear subspace can be described as

$$\bar{X}^n = 2X^n - \frac{1}{|U_n^\varepsilon|} \sum_{X_{n'} \in U_n^\varepsilon} X_{n'}. \tag{7}$$

Equation (7) could only be evaluated after one complete sweep through the data set in which all local centres-of-mass were determined.

Based on Eq.(7), we can compute the local weighted covariance matrix

$$C^n = \sum_{X_{n'} \in U_n^\varepsilon} [R(X_{n'} - \bar{X}^n][R(X_{n'} - \bar{X}^n)]^{\mathrm{T}}, \tag{8}$$

where the weight matrix R is chosen to be diagonal with R_{11} and R_{mm} large and all other diagonal entries R_{ii}=1, and superscript T denotes the transposition operator. By using SVD we can get the orthonormal eigenvectors u_i and v_j and corresponding eigenvalues of C^n. A m-dimensional manifold is then locally approximated by m eigenvectors. The cleaned vector X_n^c is expressed as

$$\begin{aligned} X_n^c =& \bar{X}^n + R^{-1} \sum_{i=1}^{p} \alpha_i \{u_i [u_i R(X_n - \bar{X}^n)]\} \\ &+ R^{-1} \sum_{j=1}^{q} \beta_j \{v_j [v_j R(X_n - \bar{X}^n)\}, p + q = m, \end{aligned} \tag{9}$$

where α_i and β_j are weight constants, and $0 \leq \alpha_i$, $\beta_j \leq 1$.

In order to translate the cleaned vectors back into a scalar time series, the corrected scalar values x_n are obtained by averaging the corresponding components of $X_n^c, \cdots, X_{n+m-1}^c$.

The above algorithm is different from project filtering in using all eigenvectors instead of part of the vectors with largest eigenvalues. The reason is that we cannot neglect the subspace spanned by the stable directions. Though the effect of noise is greater in the stable directions, yet the attractor contains the stable directions, therefore we can take larger weight in stable directions and smaller weight in unstable directions instead of omitting the stable direction to obtain an enhanced scalar time series.

There are two very important parameters that have to be chosen. The first is the embedding dimension, and the second is the neighbourhood size. Based on the embedding theorem the dimension should be greater than or equal to twice the dimension m_0 of the noise-free attractor. But greater dimension can result in a bad noise reduction effect due to more unstable directions, and smaller dimension cannot span the true attractor. The appropriate embedding dimension is greater than or equal to, but smaller than twice, the dimension m_0 of noise-free attractor. Too small neighbourhoods are insufficient for stable linear approximation, either if too few points are available or if all directions are dominated by noise. Too large neighbourhoods can induce strong curvature effects and lead to a distortion of the attractor. The appropriate diameter of a neighbourhood must be greater than the noise level, and the appropriate number of neighbours cannot be less than $m + 1$ points.

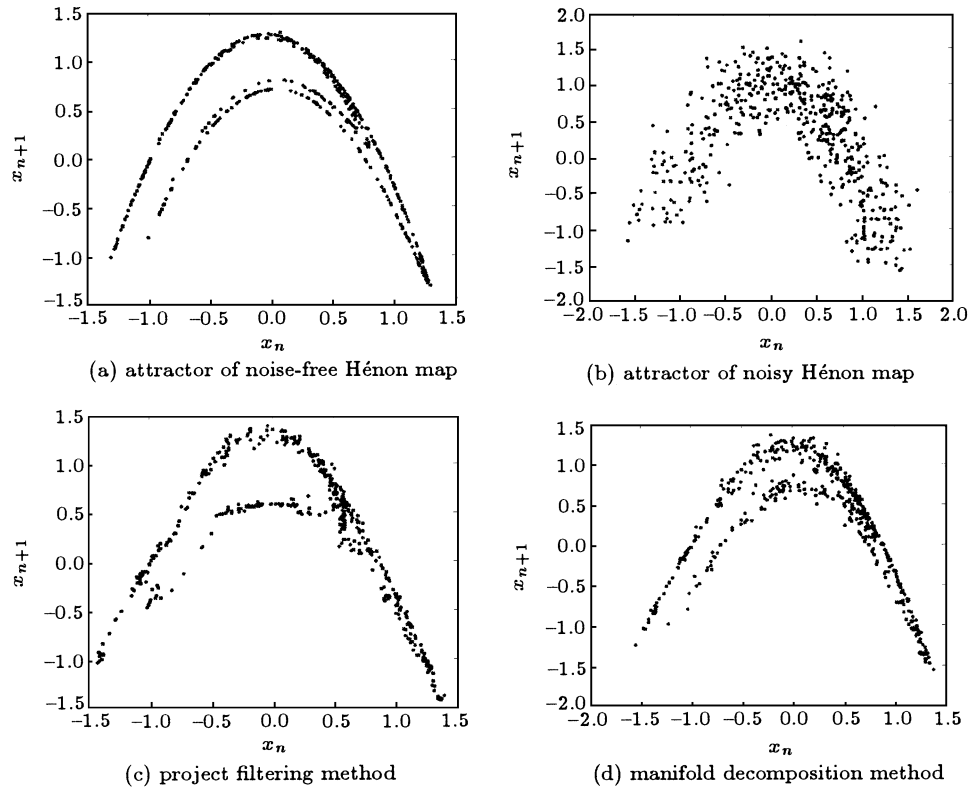
4. Numerical experiment

In this section the results of a numerical experiment are presented for a two-dimensional discrete chaotic time series called the Hénon map described by

$$\begin{cases} x_{n+1} = 1 - 1.4x_n^2 + y_n, \\ y_{n+1} = 0.3x_n. \end{cases} \tag{10}$$

We discuss only the x component of the Hénon map, and consider two cases: the first is a case when there is only additive Gaussian white noise, and the second is a case when there is uniform noise. In the following numerical experiment the embedding dimension is 3, and number of sample points is 500, and time delay is 1.

Figure 1 shows the true attractor (a) and noisy attractor (b) and the cleaned attractors (c) and (d) where the project filtering method and manifold decomposition method are iterated for 3 times, respectively. The additive Gaussian white noise is generated by randn-function in Matlab software and subject to normal distribution N (0,0.4472) and destroys the attractor of noise-free Hénon map (b). The weight vector [α, β_1, β_2] is taken as [0.001, 0.199, 0.8].

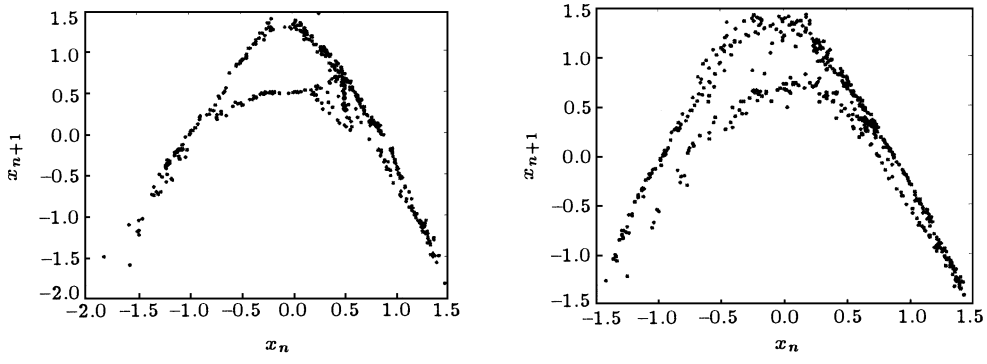


(a) attractor of noise-free Hénon map (b) attractor of noisy Hénon map

(c) project filtering method (d) manifold decomposition method

Fig.1. Original, noisy, and enhanced attractors.

Figures 1(c) and (d) show that we have obtained noise reduction after iteration for three times, but some information of the attractor has been missed by using project filtering,[10] and the weighted manifold decomposition method has better effect than project filtering. Though neither result is completely satisfactory, we can see that the processing method proposed in this paper may be helpful for reducing the signal-to-noise ratio (SNR) of noisy chaotic signals.



(a) project filtering method (b) manifold decomposition method

Fig.2. Enhanced attractors with uniform noise.

Figure 2 shows noise reduction effect on uniformly distributed random noise generated by rand-function in interval [0,1] in Matlab software by using project filtering and the weighted manifold decomposition method, respectively. As in Fig.1, distortion of the attractor occurs badly by using project filtering and the weighted manifold decomposition method gives much better results.

5. Conclusion

A new noise reduction method based on weighted manifold decomposition for treating noisy chaotic time series has been proposed in this paper. In view of the manifold decomposition we can take larger weight in the stable directions and smaller weight in the unstable directions instead of omitting the stable directions for obtaining an enhanced scalar time series. The performance of the numerical experiment indicated that the algorithm can increase the SNR of the noisy chaotic time series and has better noise reduction effect than the project filtering method.

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