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A generalized SHGI integrable hierarchy and its expanding integrable model

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An anti-symmetric loop algebra $\overline{A}_2$ is constructed. It follows that an integrable system is obtained by use of Tu's scheme. The eminent feature of this integrable system is that it is reduced to a generalized Schrödinger equation, the well-known heat-conduction equation and a Gerdjkov–Ivanov (GI) equation. Therefore, we call it a generalized SHGI hierarchy. Next, a new high-dimensional subalgebra $\tilde{G}$ of the loop algebra $\tilde{A}_2$ is constructed. As its application, a new expanding integrable system with six potential functions is engendered.

Keywords: integrable system, Hamiltonian structure, loop algebra
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1. Introduction

It is quite interesting and important in the study of integrable systems to find integrable systems associated with some isospectral problems.[1−3] In recent years, some typical integrable Hamiltonian systems, such as the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, the Kaup–Newell (KN) hierarchy, etc, have been reported in various journals of physics and mathematics.[1−7] In general, constructing new loop algebras could produce new integrable Hamiltonian hierarchies of evolution equations by making use of Tu's scheme.[4] In this paper, we construct an anti-symmetric loop algebra $\overline{A}_2$, which is a subalgebra of the loop algebra $\tilde{A}_2$. It follows that an isospectral problem is established. In particular, V is constructed, which contains two spectral potential functions and their derivatives of b, c with respect to the independent variable x. By using Tu's scheme, a new integrable system is worked out, possessing bi-Hamiltonian structure and infinite commutation symmetries. As its reduced case, a generalized Schrödinger equation, the heat-conduction equation and a generalized Gerdjikov–Ivanov (GI) equation are presented, respectively. Hence, we call the integrable system the SHGI hierarchy. In addition, we construct a new high-dimensional loop algebra $\tilde{G}$, which is different from any one in Refs.[8-11], to produce an integrable coupling of the above integrable systems obtained.

2. A new integrable Hamiltonian system

Consider an anti-symmetric loop algebra $\overline{A}_2$:

$$h(n) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\lambda^{2n+1} \\ 0 & \mathrm{i}\lambda^{2n+1} & 0 \end{pmatrix},$$

$$e(n) = \begin{pmatrix} 0 & 0 & \lambda^{2n+1} \\ 0 & 0 & 0 \\ -\lambda^{2n+1} & 0 & 0 \end{pmatrix},$$

$$f(n) = \begin{pmatrix} 0 & -\mathrm{i}\lambda^{2n} & 0 \\ \mathrm{i}\lambda^{2n} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1a}$$

$$\begin{aligned} &[h(m), e(n)] = f(m+n+1), \\ &[h(m), f(n)] = e(m+n), \\ &[f(m), e(n)] = h(m+n), \end{aligned} \tag{1b}$$

$$\begin{aligned} &\deg h(n) = \deg e(n) = 2n+1, \\ &\deg f(n) = 2n, \mathrm{i}^2 = -1. \end{aligned} \tag{1c}$$

From Eqs.(1a), we establish an isospectral problem as follows:

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$$\phi_x = U\phi, U = \begin{pmatrix} 0 & -\mathrm{i}\lambda^2 & r\lambda \\ \mathrm{i}\lambda^2 & 0 & -\mathrm{i}q\lambda \\ -r\lambda & \mathrm{i}q\lambda & 0 \end{pmatrix}. \tag{2}$$

Let

$$V = \begin{pmatrix} 0 & -\mathrm{i}a & (c_x + ra)/\lambda \\ ia & 0 & (-\mathrm{i}b_x - \mathrm{i}qa)/\lambda \\ -(c_x + ra)/\lambda & (\mathrm{i}b_x + \mathrm{i}qa)/\lambda & 0 \end{pmatrix},$$
$$a = \sum_{m\geq 0} a_m \lambda^{-2m}, \quad b = \sum_{m\geq 0} b_m \lambda^{-2m}, \quad c = \sum_{m\geq 0} c_m \lambda^{-2m}.$$

Solving the stationary zero curvature equation

$$V_x = [U, V] \tag{3}$$

yields

$$\begin{aligned} &a_{mx} = qc_{mx} - rb_{mx}, \\ &b_{mxx} + (qa_m)_x = c_{m+1x}, \\ &c_{mxx} + (ra_m)_x = -b_{m+1x}, \end{aligned} \tag{4a}$$

$$\begin{aligned} &b_0 = c_0 = 0, \\ &a_0 = \alpha, \quad a_1 = \frac{\alpha}{2}(q^2 + r^2), \\ &b_1 = -\alpha r, \quad c_1 = \alpha q. \end{aligned} \tag{4b}$$

Set

$$\begin{aligned} V_+^{(n)} = &\sum_{m=0}^{n} (a_m f(n-m) + b_{mx} h(-1+n-m) \\ &+ c_{mx} e(-1+n-m) + qa_m h(-1+n-m) \\ &+ ra_m e(-1+n-m)), \\ V_-^{(n)} = &\lambda^{2n} V - V_+^{(n)}, \end{aligned}$$

then Eq.(3) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{5}$$

In terms of the method presented in Refs.[3,4], a direct calculation gives

$$\begin{aligned} &-V_{+x}^{(n)} + [U, V_+^{(n)}] \\ =&(b_{n+1xx} + (qa_{n+1})_x)h(0) + (c_{n+1xx} + (ra_{n+1})_x)e(0). \end{aligned}$$

Taking $V^{(n)} = V_+^{(n)}$, the zero curvature equation,

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0,$$

determines the following Lax integrable system:

$$\begin{aligned} u_{t_n} = X_n &= \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} \\ &= \begin{pmatrix} -\partial & 0 \\ 0 & -\partial \end{pmatrix} \begin{pmatrix} b_{n+1x} + qa_{n+1} \\ c_{n+1x} + ra_{n+1} \end{pmatrix} = JG_n. \end{aligned} \tag{6}$$

We find

$$G_{n+1} = N \begin{pmatrix} b_{n+1} \\ c_{n+1} \end{pmatrix},$$
$$N = \frac{1}{2} \begin{pmatrix} \partial - q\partial^{-1}r\partial & q\partial^{-1}q\partial \\ -r\partial^{-1}r\partial & \partial + r\partial^{-1}q\partial \end{pmatrix}.$$

From Eq.(4a), we have

$$\begin{pmatrix} b_{n+1} \\ c_{n+1} \end{pmatrix} = M_1 M_2 \begin{pmatrix} b_n \\ c_n \end{pmatrix},$$
$$M_1 = \begin{pmatrix} r\partial^{-1}r & -1 - r\partial^{-1}q \\ 1 - q\partial^{-1}r & q\partial^{-1}q \end{pmatrix},$$
$$M_2 = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix}.$$

Thus, Eq.(6) can be written as

$$\begin{aligned} u_{t_n} &= -M_2 N \begin{pmatrix} b_{n+1} \\ c_{n+1} \end{pmatrix} \\ &= -M_2 N (M_1 M_2)^n \begin{pmatrix} -\alpha r \\ \alpha q \end{pmatrix}. \end{aligned} \tag{7}$$

When taking $n = 0$ in Eq.(7), we have a generalized Schrödinger equation, which is similar to the result in Ref[11]:

$$\begin{cases} q_{t_0} = \alpha r_{xx} - \dfrac{\alpha}{2}(q^3 + qr^2)_x, \\ r_{t_0} = -\alpha q_{xx} - \dfrac{\alpha}{2}(r^3 + rq^2)_x. \end{cases} \tag{8}$$

Specifically, taking $\alpha = i$, $t_0 = t$, $q = \mathrm{i}r$, the celebrated heat-conduction equation is presented as follows:

$$r_t = r_{xx}. \tag{9}$$

When taking $n = \alpha = 1$, Eq.(7) reduces to a generalized GI equation:[3]

$$\begin{cases} q_{t_1} = q_{xxx} + \frac{1}{2}(r(q^2+r^2))_x - \left[q\left(rq_x - r_xq + \frac{3}{8}(q^4+r^4) + \frac{3}{4}q^2r^2\right)\right]_x, \\ r_{t_1} = r_{xxx} - \frac{1}{2}(q(q^2+r^2))_x - \left[r\left(rq_x - r_xq + \frac{3}{8}(q^4+r^4) + \frac{3}{4}q^2r^2\right)\right]_x. \end{cases}$$

Hence, we call the system (7) a SHGI hierarchy.

A direct calculation gives

$$\begin{aligned} &\left\langle V, \frac{\partial U}{\partial q}\right\rangle = \frac{1}{2}(b_x + qa), \\ &\left\langle V, \frac{\partial U}{\partial r}\right\rangle = \frac{1}{2}(c_x + ra), \\ &\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = \frac{1}{2\lambda}(qb_x + rc_x + (q^2+r^2)a). \end{aligned} \quad (10)$$

Inserting Eqs.(10) into trace identity[4] and comparing the coefficients of λ^{-2m-3} gives rise to

$$\begin{aligned} &\frac{\delta}{\delta u}(qb_{n+2x} + rc_{n+2x} + (q^2+r^2)a_{n+2}) \\ =&(-2n-2+\gamma)G_{n+1}. \end{aligned}$$

From Eqs.(4b), we find $\gamma = -1$. Therefore, we have

$$\begin{aligned} &\frac{\delta H_{n+1}}{\delta u}G_{n+1}, H_{n+1} \\ =&-\frac{qb_{n+2x} + rc_{n+2x} + (q^2+r^2)a_{n+2}}{2n+3}. \end{aligned}$$

The Hamiltonian structure of the system (7) is manifested as

$$u_{t_n} = X_n = J\frac{\delta H_{n+1}}{\delta u} = JL\frac{\delta H_n}{\delta u} = K\frac{\delta H_n}{\delta u}. \quad (11)$$

According to the approach in Refs.[12,13], we can prove that an arbitrary linear combination of the operator pair $\{J, K\}$ is a Hamiltonian operator. Hence, Eq.(11) is a bi-Hamiltonian structure of the system (7).

We find

$$G_{n+1} = N\begin{pmatrix} b_{n+1} \\ c_{n+1} \end{pmatrix} = NM_1M_2N^{-1}G_n = LG_n,$$

$L = NM_1M_2N^{-1}$ and $JL^k (k = 0, 1, \cdots, n)$ are all skew-symmetric operators. In fact, suppose that JL^{k-1} be skew symmetric, then we have

$$\begin{aligned} (JL^k)^* =&(JL^{k-1}L)^* = L^*(JL^{k-1})^* \\ =&-L^*JL^{k-1} = L^*J^*L^{k-1} \\ =&-JLL^{k-1} = -JL^k. \end{aligned}$$

In terms of Ref.[3], we find

$$\begin{aligned} \{H_{m+1}, H_{n+1}\}_J =& \int \left\langle \frac{\delta H_{m+1}}{\delta u}, J\frac{\delta H_{n+1}}{\delta u}\right\rangle \mathrm{d}x \\ =& \int \left\langle \frac{\delta H_{m+1}}{\delta u}, JL\frac{\delta H_n}{\delta u}\right\rangle \mathrm{d}x \\ =& \int \left\langle \frac{\delta H_{m+1}}{\delta u}, L^*J\frac{\delta H_n}{\delta u}\right\rangle \mathrm{d}x \\ =& \int \left\langle L\frac{\delta H_{m+1}}{\delta u}, J\frac{\delta H_n}{\delta u}\right\rangle \mathrm{d}x \\ =&\{H_{m+2}, H_n\}_J = \cdots \\ =&\{H_{n+1}, H_{m+1}\}_J. \end{aligned}$$

Thus, $\{H_n\}$ are in involution under the Poisson bracket: that is, $\{H_m, H_n\} = 0$.

In addition, we can also prove $[X_n, X_m] = J\frac{\delta}{\delta u}\{H_n, H_m\} = 0$. Thus, the system (7) has infinite commutation symmetries $\{X_n\}_{n=0}^{\infty}$; it is Liouville integrable.

3. A new loop algebra and an expanding integrable system

We expand the loop algebra (1) into a high-dimensional loop algebra $\tilde{G}$ as follows:

$$e_1(n) = \frac{1}{2}\begin{pmatrix} \lambda^{2n+1} & 0 & 0 \\ 0 & -\lambda^{2n+1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_2(n) = \frac{1}{2}\begin{pmatrix} 0 & \lambda^{2n+1} & 0 \\ \lambda^{2n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3(n) = \frac{1}{2}\begin{pmatrix} 0 & \lambda^{2n+1} & 0 \\ -\lambda^{2n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4(n) = \begin{pmatrix} 0 & 0 & \lambda^{2n} \\ 0 & 0 & \lambda^{2n-1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_5(n) = \begin{pmatrix} 0 & 0 & \lambda^{2n+1} \\ 0 & 0 & -\lambda^{2n} \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_6(n)=\begin{pmatrix}0&0&\lambda^{2n}\\0&0&\lambda^{2n+1}\\0&0&0\end{pmatrix},\quad e_7(n)=\begin{pmatrix}0&0&\lambda^{2n-1}\\0&0&-\lambda^{2n}\\0&0&0\end{pmatrix},$$

$$[e_1(m),e_2(n)]=e_3(m+n+1),[e_1(m),e_3(n)]=e_2(m+n),[e_3(m),e_2(n)]=e_1(m+n),$$
$$[e_1(m),e_4(n)]=\frac{1}{2}e_5(m+n),[e_2(m),e_4(n)]=\frac{1}{2}e_6(m+n),[e_3(m),e_4(n)]=\frac{1}{2}e_7(m+n),\cdots.$$

In terms of necessity, we choose different commutation relations from the above loop algebra.

Let

$$\phi_x=U\phi,\lambda_t=0,\quad U=e_3(1)+u_1e_1(0)+u_2e_2(0)+u_3e_4(1)+u_4e_5(0)+u_5e_6(0)+u_6e_7(0),\tag{12}$$

and

$$V=ae_3(0)+b_xe_1(-1)+c_xe_2(-1)+u_1ae_1(-1)+u_2ae_2(-1)+de_4(0)+fe_5(0)+ge_6(0)+le_7(0),$$
$$a=\sum_{m=0}^{\infty}a_m\lambda^{-2m},\quad b=\sum_{m=0}^{\infty}b_m\lambda^{-2m},\cdots$$

Solving the equation similar to Eq.(3) yields

$$a_{m+1x}=u_1c_{mx}-u_2b_{mx},b_{mxx}+(u_1a_m)_x=c_{m+1x},c_{mxx}+(u_2a_m)_x=-b_{m+1x},\tag{13a}$$
$$2d_{mx}=-l_{m+1}+u_1f_{m+1}+u_2g_{m+1}-u_4b_{mx}-u_1u_4a_m-u_5c_{mx}-u_2u_5a_m+u_6a_{m+1},\tag{13b}$$
$$2f_{mx}=g_{m+1}+u_1d_m-u_2l_m-u_3b_{mx}-u_1u_3a_m-u_5a_m+u_6c_{mx}+u_2u_6a_m,\tag{13c}$$
$$2g_{mx}=-f_{m+1}+u_1l_m+u_2d_m-u_3c_{mx}-u_6b_{mx}-u_1u_6a_m-u_2u_3a_m-u_5b_{mx},\tag{13d}$$
$$2l_{mx}=d_{m+1}+u_1g_{m+1}-u_2f_{m+1}-u_3a_{m+1}+u_4c_{mx}+u_2u_4a_m-u_1u_5a_m,\tag{13e}$$
$$b_0=c_0=d_0=f_0=g_0=0,a_0=\alpha,a_1=\frac{\alpha}{2}(u_1^2+u_2^2),b_1=-\alpha u_2,c_1=\alpha u_1,\tag{13f}$$
$$d_1=\alpha(-\frac{1}{2}u_1^2u_3-\frac{1}{2}u_2^2u_3-u_2u_4),l_1=-\alpha\left(\frac{1}{2}u_1^2u_6+\frac{1}{2}u_2^2u_6+u_1u_4\right).\tag{13g}$$

Let

$$V_+^{(n)}=\sum_{m=0}^{n}\lambda^{2n-2m}[a_me_3(1)+b_{mx}e_1(-1)+c_{mx}e_2(-1)+u_1a_me_1(-1)+u_2a_me_2(-1)+d_me_4(0)$$
$$+f_me_5(0)+g_me_6(0)+l_me_7(0)],V_-^{(n)}=\lambda^{2n}V-V_+^{(n)}.$$

Similar to Eq.(5), we have

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=[b_{n+1xx}+(u_1a_{n+1})_x]e_1(0)+[c_{n+1xx}+(u_2a_{n+1})_x]e_2(0)-\frac{1}{2}(l_{n+1}+u_1f_{n+1}$$
$$+u_2g_{n+1}+u_6a_{n+1})e_4(0)-\frac{1}{2}g_{n+1}e_5(0)+\frac{1}{2}f_{n+1}e_6(0)$$
$$+\frac{1}{2}(-d_{n+1}-u_1g_{n+1}+u_2f_{n+1}+u_3a_{m+1})e_7(0).$$

Thus the zero curvature equation $U_t-V_{+x}^{(n)}+[U,V_+^{(n)}]=0$ admits the integrable system

$$u_t=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\end{pmatrix}_t=\begin{pmatrix}-\partial^2+\partial u_1\partial^{-1}u_2&\partial u_1\partial^{-1}u_1&0&0&0&0\\\partial u_2\partial^{-1}u_2&-\partial^2-\partial u_2\partial^{-1}u_1&0&0&0&0\\-\frac{1}{2}u_6\partial^{-1}u_2&\frac{1}{2}u_6\partial^{-1}u_1&0&u_1/2&u_2/2&1/2\\0&0&0&0&1/2&0\\0&0&0&-1/2&0&0\\u_3\partial^{-1}u_2&-u_3\partial^{-1}u_1&1&-u_2&u_1&0\end{pmatrix}=J\begin{pmatrix}b_{n+1}\\c_{n+1}\\d_{n+1}\\f_{n+1}\\g_{n+1}\\a_{n+1}\end{pmatrix}.\tag{14}$$

From Eqs.(13a)–(13e), we obtain a recurrence operator

$$L=\begin{pmatrix} u_2\partial^{-1}u_2\partial & -\partial-u_2\partial^{-1}u_1\partial & 0 & 0 & 0 & 0\\ \partial-u_1\partial^{-1}u_2\partial & u_1\partial^{-1}u_1\partial & 0 & 0 & 0 & 0\\ M_1 & N_1 & u_1^2+u_2^2 & 0 & -2u_2\partial & 2\partial\\ (u_1u_6+u_2u_3)\partial^{-1}u_2-u_6\partial & (u_1u_6+u_2u_3)\partial^{-1}u_1-u_3\partial & u_2 & 0 & -2\partial & u_1\\ (u_2u_6-u_1u_5u_3)\partial^{-1}u_2+u_3\partial & (u_1u_3+u_5-u_2u_6)\partial^{-1}u_1+u_6\partial & -u_1 & 2\partial & 0 & u_2\\ M_2 & N_2 & -2\partial & 2u_2\partial & -2u_1\partial & u_1^2+u_2^2 \end{pmatrix},$$

where

$$\begin{aligned} M_1 =&(u_1^2u_3+u_1u_5+u_2^2u_3)\partial^{-1}u_2-u_1u_3\partial-u_2u_6\partial,\\ N_1 =&(2u_1u_2u_6-u_1u_5+u_2^2u_3-u_1^2u_3)\partial^{-1}u_1-u_1u_6\partial-u_2u_3\partial,\\ M_2 =&(u_1^2u_6+u_2^2u_6+u_1u_4)\partial^{-1}u_2-u_1u_6\partial-u_4\partial+u_2u_3\partial-u_6\partial^{-1}u_2^2\partial^{-1}u_2\partial+u_6\partial^{-1}(u_1-u_1^2\partial^{-1}u_2)\partial,\\ N_2 =&(u_1^2u_6+2u_1u_2u_3+u_1u_4-u_2^2u_6)\partial^{-1}u_1-u_1u_3\partial-u_2u_6\partial-u_5\partial+u_6\partial^{-1}u_2(1+u_2\partial^{-1}u_1)\partial\\ &+u_6\partial^{-1}u_1^2\partial^{-1}u_1\partial. \end{aligned}$$

Hence, Eq.(14) can be written as

$$u_t=\begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6 \end{pmatrix}_t=JL^n\begin{pmatrix} b_1\\ c_1\\ d_1\\ f_1\\ g_1\\ l_1 \end{pmatrix}. \tag{15}$$

According to the definition of the integrable couplings in Refs.[14,15], we conclude that Eq.(15) is an integrable coupling of the system (7). Taking $u_1=q$, $u_2=r$, $u_3=u_4=u_5=u_6=0$, then Eq.(15) reduces to Eq.(7). Therefore, the coupling system (15) is also an expanding integrable model of the system (7).

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