Vol 13 No 3, March 2004
1009-1963/2004/13(03)/0297-05

Chinese Physics

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Exact solutions to two coupled nonlinear physical equations

Hu Jian-Lan(胡建兰)†

Institute of Applied Mathematics and Physics, Beijing University of Technology, Beijing 100022, *China*

(Received 24 April 2003; revised manuscript received 18 August 2003)

In this paper, a two-step ansatz is proposed, which leads to some new solutions to two coupled nonlinear physical equations.

Keywords: exact solution, coupled nonlinear differential equations, two-step ansatz method
PACC: 0340K, 0290

1. Introduction

The nonlinear differential equations (NDEs) are a major subject in many fields of physics and nonlinear sciences, and seeking exact solutions to NDEs plays an important role in our understanding of these fields. Much work has been done on this topic,[1−7] but unfortunately there exist no general rules for finding exact solutions to NDEs, especially to coupled NDEs.

In this paper, we discuss the exact solutions to two coupled NDEs which arise in physics of condensed matter, quantum field theory and plasma physics.

The first pair of coupled nonlinear equations [8−10] we shall study is

$$\begin{aligned} u_{xx} &= -u + u^3 + duv^2, \\ v_{xx} &= fv + \lambda v^3 + dv(u^2 - 1), \end{aligned} \tag{1}$$

which is relevant to the quantum field theory of charged soliton. Here u and v are real scalar fields and d, f, λ are parameters.

The second pair of coupled nonlinear equations [10,11] considered here is

$$\begin{aligned} u_{xx} &= a_1 u + a_2 uv, \\ v_{xx} &= b_1 v + b_2 v^2 + b_3 u^2, \end{aligned} \tag{2}$$

which appears in the context of the modulational instability of plasma system. Here u, v are real fields and a_1, a_2, b_1, b_2, b_3 are parameters.

In the next two sections, the analytical solutions to these two coupled equations are given by introducing a new two-step ansatz.

2. Exact solutions to coupled equations (1)

Assume that Eqs.(1) have solutions of the following form:

$$u = a_1 w + a_2 \sqrt{1 - w^2}, \tag{3}$$

$$v = b_1 w + b_2 \sqrt{1 - w^2}, \tag{3}$$

$$w' = r(1 - w^2), \tag{4}$$

with a_1, a_2, b_1, b_2, r are real numbers. Obviously, Eq.(4) has the solution

$$w = \tanh(rx). \tag{5}$$

From (3) it is easy to see that

$$\begin{aligned} u_{xx} =& -2a_1 r^2 w + 2a_1 r^2 w^3 \\ &+ (-a_2 r^2 + 2a_2 r^2 w^2)\sqrt{1 - w^2}, \\ v_{xx} =& -2b_1 r^2 w + 2b_1 r^2 w^3 \\ &+ (-b_2 r^2 + 2b_2 r^2 w^2)\sqrt{1 - w^2}. \end{aligned} \tag{6}$$

Substitute (3) and (6) into Eqs.(1) to obtain

$$\begin{aligned} p_1(w) + p_2(w)\sqrt{1 - w^2} &= 0, \\ p_3(w) + p_4(w)\sqrt{1 - w^2} &= 0, \end{aligned}$$

where $p_i(w)$ (i=1, 2, 3, 4) are polynomials of w. Equating the coefficients of polynomials $p_i(w)$ (i=1,

†E-mail: jianlan0712@sina.com

2, 3, 4) to zero yields the following algebraic system:

$$\begin{aligned}
-2a_1r^2 =& -a_1+3a_1a_2^2+d(a_1b_2^2+2a_2b_1b_2),\\
2a_1r^2 =& a_1^3-3a_1a_2^2+d(a_1b_1^2-a_1b_2^2-2a_2b_1b_2),\\
-a_2r^2 =& -a_2+a_2^3+da_2b_2^2,\\
2a_2r^2 =& 3a_1^2a_2-a_2^3+d(a_2b_1^2-a_2b_2^2+2a_1b_1b_2),\\
-2b_1r^2 =& (f-d)b_1+3\lambda b_1b_2^2+d(a_2^2b_1+2a_1a_2b_2),\\
2b_1r^2 =& \lambda(b_1^3-3b_1b_2^2)+d(a_1^2b_1-a_2^2b_1-2a_1a_2b_2),\\
-b_2r^2 =& (f-d)b_2+\lambda b_2^3+da_2^2b_2,\\
2b_2r^2 =& \lambda(3b_1^2b_2-b_2^3)+d(a_1^2b_2-a_2^2b_2+2a_1a_2b_1).
\end{aligned} \tag{7}$$

In what follows, we first use Mathematica to solve the algebraic system (7) so that we can get the relationship between a_1, a_2, b_1, b_2, r in (3), (4) and d, f, λ in (1). Then we can get solutions to Eqs.(1) by considering the solutions to Eqs.(7) and formula (5). To this end we proceed as follows:

Case 1 When $f=0$, $\lambda=d=1$, from (7) we have

$$a_1=\pm\sqrt{1-b_1^2}, a_2=b_2=0, r=\pm\frac{1}{\sqrt{2}},$$

which gives the following solutions to Eqs.(1)

$$u_{1,2}=\pm\sqrt{1-b_1^2}\tanh\left(\frac{x+c_0}{\sqrt{2}}\right),$$

$$v_{1,2}=b_1\tanh\left(\frac{x+c_0}{\sqrt{2}}\right),$$

with $b_1\neq 0$ real number, c_0 integral constant.

Case 2 When $\lambda\neq d^2, f=d-1$, we have

$$a_1=\pm\sqrt{\frac{d-\lambda}{d^2-\lambda}},\ b_1=\pm\sqrt{\frac{d-1}{d^2-\lambda}},$$
$$a_2=b_2=0, r=\pm\frac{1}{\sqrt{2}}.$$

Here we can get four pairs of solutions to Eqs.(1):

$$u_{3-6}=\pm\sqrt{\frac{d-\lambda}{d^2-\lambda}}\tanh\left(\frac{x+c_0}{\sqrt{2}}\right),$$
$$v_{3-6}=\pm\sqrt{\frac{d-1}{d^2-\lambda}}\tanh\left(\frac{x+c_0}{\sqrt{2}}\right),$$

where any two of u_{3-6}, v_{3-6} with the combination of $(+,+),(+,-),(-,+),(-,-)$ can form a solution to (1) and thus four solutions can be generated.

Case 3 When $d\neq 0, d\neq 1, f=\dfrac{\lambda-d^2}{2(\lambda-d)}$, we have

$$a_1=1, a_2=b_1=0, b_2=\pm\sqrt{\frac{1-d}{d-\lambda}}, r=\sqrt{f}.$$

Here we get

$$u_{7,8}=\tanh[\sqrt{f}(x+c_0)],$$
$$v_{7,8}=\pm\sqrt{\frac{1-d}{d-\lambda}}\operatorname{sech}[\sqrt{f}(x+c_0)].$$

Case 4 When $\lambda<0, d=0, f=\dfrac{1}{2}$, we have

$$a_1=\pm 1, a_2=b_1=0, b_2=\pm\sqrt{-\frac{1}{\lambda}}, r=\pm\frac{1}{\sqrt{2}}.$$

Here we get

$$u_{9-12}=\pm\tanh\left(\frac{x+c_0}{\sqrt{2}}\right),$$
$$v_{9-12}=\pm\sqrt{-\frac{1}{\lambda}}\operatorname{sech}\left(\frac{x+c_0}{\sqrt{2}}\right).$$

Case 5 When $f\neq 0, \lambda=\dfrac{3f^2}{8f-4}, 2d=3f$, we have

$$a_1=\pm 1, b_1=a_2=0, b_2=\pm\sqrt{\frac{1-2f}{d}}, r=\sqrt{f}.$$

Here we get

$$u_{13-16}=\pm\tanh[\sqrt{f}(x+c_0)],$$
$$v_{13-16}=\pm\sqrt{\frac{1-2f}{d}}\operatorname{sech}[\sqrt{f}(x+c_0)],$$

which have been obtained in Refs.[9,10] by a different ansatz.

Case 6 When $d\neq 0, d\neq 2, f\neq 0, d+f-2\neq 0, \lambda=\dfrac{d(d-f)(d-2)}{d+f-2}$, we have

$$a_1=b_2=0, a_2=\pm\sqrt{\frac{d-2-f}{d-2}},$$
$$b_1=\pm\sqrt{\frac{d-2+f}{d(d-2)}}, r=\sqrt{\frac{f}{d-2}}.$$

Here we get

$$u_{17-20}=\pm\sqrt{\frac{d-2-f}{d-2}}\operatorname{sech}\left[\sqrt{\frac{f}{d-2}}(x+c_0)\right],$$

$$v_{17-20}=\pm\sqrt{\frac{d-2+f}{d(d-2)}}\tanh\left[\sqrt{\frac{f}{d-2}}(x+c_0)\right],$$

which are the same as in Refs.[10,12,13] by a different ansatz. Here $u_1-u_6(v_1-v_6)$, u_7-u_{16} and $v_{17}-v_{20}$ are tanh-like kink-type solitary solutions, while v_7-v_{16} and $u_{17}-u_{20}$ are sech-type or bell-shape solitary solutions.

3. Exact solutions to coupled equations (2)

Suppose that Eqs.(2) have solutions of the following form:

$$u = c_0 + c_1 w + c_2 w^2 + d_1\sqrt{1-w^2} + d_2 w\sqrt{1-w^2},$$
$$v = e_0 + e_1 w + e_2 w^2 + f_1\sqrt{1-w^2} + f_2 w\sqrt{1-w^2}, \tag{8}$$

with w as in Eq.(4) and c_0, c_1, c_2, d_1, d_2, e_0, e_1, e_2, f_1, f_2 being real numbers.

From (8), we get

$$\begin{aligned} u_{xx} =& 2c_2r^2 - 2c_1r^2w - 8c_2r^2w^2 + 2c_1r^2w^3 \\ &+ 6c_2r^2w^4 + (-d_1r^2 - 5d_2r^2w \\ &+ 2d_1r^2w^2 + 6d_2r^2w^3)\sqrt{1-w^2}, \\ v_{xx} =& 2e_2r^2 - 2e_1r^2w - 8e_2r^2w^2 + 2e_1r^2w^3 \\ &+ 6e_2r^2w^4 + (-f_1r^2 - 5f_2r^2w \\ &+ 2f_1r^2w^2 + 6f_2r^2w^3)\sqrt{1-w^2}. \end{aligned} \tag{9}$$

Substitute (8) and (9) into (2) to obtain

$$q_1(w) + q_2(w)\sqrt{1-w^2} = 0,$$
$$q_3(w) + q_4(w)\sqrt{1-w^2} = 0,$$

where $q_i(w)$ (i=1,2,3,4) are polynomials of w. Equating the coefficients of polynomials $q_i(w)$ (i=1,2,3,4) to zeroes yields the following algebraic system:

$$\begin{aligned} 2c_2r^2 =& a_1c_0 + a_2(c_0e_0 + d_1f_1), \\ -2c_1r^2 =& a_1c_1 + a_2(c_1e_0 + c_0e_1 + d_1f_2 + d_2f_1), \\ -8c_2r^2 =& a_1c_2 + a_2(c_0e_2 + c_1e_1 + c_2e_0 - d_1f_1 + d_2f_2), \\ 2c_1r^2 =& a_2(c_1e_2 + c_2e_1 - d_1f_2 - d_2f_1), \\ 6c_2r^2 =& a_2(c_2e_2 - d_2f_2), \\ -d_1r^2 =& a_1d_1 + a_2(e_0d_1 + c_0f_1), \\ -5d_2r^2 =& a_1d_2 + a_2(e_1d_1 + c_1f_1 + e_0d_2 + c_0f_2), \\ 2d_1r^2 =& a_2(e_2d_1 + c_2f_1 + e_1d_2 + c_1f_2), \\ 6d_2r^2 =& a_2(e_2d_2 + c_2f_2), \\ 2e_2r^2 =& b_1e_0 + b_2(e_0^2 + f_1^2) + b_3(c_0^2 + d_1^2), \\ -2e_1r^2 =& b_1e_1 + b_2(2e_0e_1 + 2f_1f_2) + b_3(2c_0c_1 + 2d_1d_2), \\ -8e_2r^2 =& b_1e_2 + b_2(e_1^2 + 2e_0e_2 - f_1^2 + f_2^2) \\ &+ b_3(c_1^2 + 2c_0c_2 - d_1^2 + d_2^2), \\ 2e_1r^2 =& b_2(2e_1e_2 - 2f_1f_2) + b_3(2c_1c_2 - 2d_1d_2), \\ 6e_2r^2 =& b_2(e_2^2 - f_2^2) + b_3(c_2^2 - d_2^2), \\ -f_1r^2 =& b_1f_1 + 2b_2e_0f_1 + 2b_3c_0d_1, \\ -5f_2r^2 =& b_1f_2 + b_2(2e_1f_1 + 2e_0f_2) + b_3(2c_1d_1 + 2c_0d_2), \\ 2f_1r^2 =& b_2(2e_2f_1 + 2e_1f_2) + b_3(2c_2d_1 + 2c_1d_2), \\ 6f_2r^2 =& 2b_2e_2f_2 + 2b_3c_2d_2. \end{aligned} \tag{10}$$

This system is very complex, we cannot exhaust all its cases although we use Mathematica. Thus only interesting and special cases are given below.

Case 1 When $c_2 = d_1 = d_2 = e_1 = f_1 = f_2 = 0$, from (10) we have:

(1) $3a_2 = b_2$, $a_1 = 0$, $b_3 = 0$, $c_0 = 0$, $e_0 = -\dfrac{3b_1}{2b_2}$, $e_2 = \dfrac{3b_1}{2b_2}$, $r = \pm\dfrac{\sqrt{b_1}}{2}$.

Here we get

$$u_1 = c_1\tanh\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$
$$v_1 = -\frac{3b_1}{2b_2}\mathrm{sech}^2\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$

with $c_1 \neq 0$ being a real number and c integral number.

(2) $3a_2 = b_2$, $a_1 = \dfrac{b_1}{3}$, $b_3 = 0$, $c_0 = 0$, $e_0 = \dfrac{b_1}{2b_2}$, $e_2 = -\dfrac{3b_1}{2b_2}$, $r = \pm\dfrac{\sqrt{-b_1}}{2}$.

Here we get

$$u_2 = c_1\tanh\left[\frac{\sqrt{-b_1}}{2}(x+c)\right],$$
$$v_2 = -\frac{b_1}{b_2} + \frac{3b_1}{2b_2}\mathrm{sech}^2\left[\frac{\sqrt{-b_1}}{2}(x+c)\right],$$

with $c_1 \neq 0$ being a real number.

Case 2 When $c_1 = d_1 = d_2 = e_1 = f_1 = f_2 = 0$, we have

(1) $a_1 = b_1$, $b_2 = 0$, $c_0 = \mp\dfrac{3b_1}{2\sqrt{a_2b_3}}$, $c_2 = \pm\dfrac{3b_1}{2\sqrt{a_2b_3}}$, $e_0 = -\dfrac{3b_1}{2a_2}$, $e_2 = \dfrac{3b_1}{2a_2}$, $r = \pm\dfrac{\sqrt{b_1}}{2}$.

Here we get

$$u_{3,4} = \mp\frac{3b_1}{2\sqrt{a_2b_3}}\mathrm{sech}^2\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$
$$v_{3,4} = -\frac{3b_1}{2a_2}\mathrm{sech}^2\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$

which corresponds to the result for $b_2 = 0$ case of Ref.[10].

(2) $a_1 = b_1, b_2 = 0, c_0 = \pm\dfrac{b_1}{2\sqrt{a_2b_3}}, c_2 = \mp\dfrac{3b_1}{2\sqrt{a_2b_3}}, e_0 = \dfrac{b_1}{2a_2}, e_2 = -\dfrac{3b_1}{2a_2}, r = \pm\dfrac{\sqrt{-b_1}}{2}$.

Here we get

$$u_{5,6} = \mp\frac{b_1}{\sqrt{a_2b_3}} \pm \frac{3b_1}{2\sqrt{a_2b_3}}\mathrm{sech}^2\left[\frac{\sqrt{-b_1}}{2}(x+c)\right],$$
$$v_{5,6} = -\frac{b_1}{a_2} + \frac{3b_1}{2a_2}\mathrm{sech}^2\left[\frac{\sqrt{-b_1}}{2}(x+c)\right].$$

Case 3 When $c_1 = c_2 = d_2 = e_1 = f_1 = f_2 = 0$, we have

(1) $3a_2 = b_2, 4a_1 = b_1, b_3 = 0, c_0 = 0, e_0 = -\frac{3b_1}{2b_2}, e_2 = \frac{3b_1}{2b_2}, r = \pm\frac{\sqrt{b_1}}{2}$.

Here we get

$$u_7 = d_1 \mathrm{sech}\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$
$$v_7 = -\frac{3b_1}{2b_2}\mathrm{sech}^2\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$

with $d_1 \neq 0$ being a real number.

(2) $3a_2 = b_2, b_1 = 12a_1, c_0 = 0, e_0 = \frac{b_1}{2b_2}, e_2 = -\frac{3b_1}{2b_2} r = \pm\frac{\sqrt{-b_1}}{2}$.

Here we get

$$u_8 = d_1 \mathrm{sech}\left[\frac{\sqrt{-b_1}}{2}(x+c)\right],$$
$$v_8 = -\frac{b_1}{b_2} + \frac{3b_1}{2b_2}\mathrm{sech}^2\left[\frac{\sqrt{-b_1}}{2}(x+c)\right],$$

with $d_1 \neq 0$ being a real number.

(3) $3a_2 = b_2$, $c_0 = 0$, $r = \pm\sqrt{a_1 - \frac{b_1}{3}}$, $d_1 = \pm\sqrt{\frac{2(12a_1 - b_1)(b_1 - 3a_1)}{3b_2 b_3}}$, $e_0 = -\frac{6a_1 - b_1}{b_2}$, $e_2 = \frac{6a_1 - 2b_1}{b_2}$

Here we get

$$u_{9,10} = \pm\sqrt{\frac{2(12a_1 - b_1)(b_1 - 3a_1)}{3b_2 b_3}} \times \mathrm{sech}\left[\sqrt{a_1 - \frac{b_1}{3}}(x+c)\right],$$
$$v_{9,10} = -\frac{b_1}{b_2} - \frac{6a_1 - 2b_1}{b_2}\mathrm{sech}^2\left[\sqrt{a_1 - \frac{b_1}{3}}(x+c)\right].$$

Rao obtained a similar solution of this type in Ref.[11].

Case 4 When $c_1 = c_2 = d_1 = e_1 = f_1 = f_2 = 0$, we have

(1) $a_2 = b_2, 4a_1 = b_1, b_3 = 0, c_0 = 0, e_0 = -\frac{3b_1}{2b_2}, e_2 = \frac{3b_1}{2b_2}, r = \pm\frac{\sqrt{b_1}}{2}$.

Here we get

$$u_{11} = d_2 \tanh\left[\frac{\sqrt{b_1}}{2}(x+c)\right]\mathrm{sech}\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$
$$v_{11} = -\frac{3b_1}{2b_2}\mathrm{sech}^2\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$

with $d_2 \neq 0$ being a real number.

(2) $a_2 = b_2, 4a_1 = 3b_1, b_3 = 0, c_0 = 0, e_0 = \frac{b_1}{2b_2}, e_2 = -\frac{3b_1}{2b_2}, r = \pm\frac{\sqrt{-b_1}}{2}$.

Here we get

$$u_{12} = d_2 \tanh\left[\frac{\sqrt{-b_1}}{2}(x+c)\right]\mathrm{sech}\left[\frac{\sqrt{-b_1}}{2}(x+c)\right],$$
$$v_{12} = -\frac{b_1}{b_2} + \frac{3b_1}{2b_2}\mathrm{sech}^2\left[\frac{\sqrt{-b_1}}{2}(x+c)\right].$$

(3) $b_1 = -2a_1$, $b_2 = 0$, $c_0 = 0$, $d_2 = \pm\frac{6a_1}{\sqrt{-a_2 b_3}}$, $r = \pm\sqrt{a_1}$, $e_0 = -\frac{6a_1}{a_2}$, $e_2 = \frac{6a_1}{a_2}$.

Here we get

$$u_{13,14} = \pm\frac{6a_1}{\sqrt{-a_2 b_3}}\tanh\left[\sqrt{a_1}(x+c)\right]\mathrm{sech}\left[\sqrt{a_1}(x+c)\right],$$
$$v_{13,14} = -\frac{6a_1}{a_2}\mathrm{sech}^2\left[\sqrt{a_1}(x+c)\right].$$

(4) $2a_1 = b_1$, $2a_2 = 3b_2$, $c_0 = 0$, $d_2 = \pm\frac{\sqrt{2}b_1}{\sqrt{b_2 b_3}}$, $e_0 = -\frac{2b_1}{b_2}$, $e_2 = \frac{2b_1}{b_2}$, $r = \pm\sqrt{\frac{b_1}{2}}$.

Here we get

$$u_{15,16} = \pm\frac{\sqrt{2}b_1}{\sqrt{b_2 b_3}}\tanh\left[\sqrt{\frac{b_1}{2}}(x+c)\right]\mathrm{sech}\left[\sqrt{\frac{b_1}{2}}(x+c)\right],$$
$$v_{15,16} = -\frac{2b_1}{b_2}\mathrm{sech}^2\left[\sqrt{\frac{b_1}{2}}(x+c)\right].$$

(5) $a_1 = b_1, 2a_2 = 3b_2, c_0 = 0, d_2 = \pm\frac{\sqrt{2}b_1}{\sqrt{b_2 b_3}}, e_0 = \frac{b_1}{b_2}, e_2 = -\frac{2b_1}{b_2}, r = \pm\sqrt{\frac{-b_1}{2}}$.

Here we get

$$u_{17,18} = \pm\frac{\sqrt{2}b_1}{\sqrt{b_2 b_3}}\tanh\left[\sqrt{\frac{-b_1}{2}}(x+c)\right] \times \mathrm{sech}\left[\sqrt{\frac{-b_1}{2}}(x+c)\right],$$
$$v_{17,18} = -\frac{b_1}{b_2} + \frac{2b_1}{b_2}\mathrm{sech}^2\left[\sqrt{\frac{-b_1}{2}}(x+c)\right].$$

Rao obtained a similar solution of this type in Ref.[11].

Here it is obviously seen that $u_1 - u_2$, $u_3 - u_6$, $u_7 - u_{12}$, and $u_{13} - u_{18}$ are tanh, sech^2, sech, and sech×tanh types, respectively, while v_i is sech^2 type.

It should be mentioned that since $\coth(r(x+c))$ is also a solution to Eq.(4), Eqs.(1) and (2) can have solutions of coth-type, csch-type, csch^2-type and coth×csch-type, which are omitted for the sake of space.

Incidentally, for Eqs.(2), we can also make the following ansatz:

$$v = au^h, h \neq 1, 2, a \neq 0.$$

Thus, when

$$b_3 = 0, 3a_2b_1 = (\sqrt{a_1b_1} + 2a_1)b_2,$$

we get

$$u_{19} = \left(-\frac{3b_1}{2b_2a}\right)^{\sqrt{\frac{a_1}{b_1}}} \mathrm{sech}^{2\sqrt{\frac{a_1}{b_1}}}\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$

$$v_{19} = -\frac{3b_1}{2b_2}\mathrm{sech}^2\left[\frac{\sqrt{b_1}}{2}(x+c)\right],$$

with $a \neq 0$ being a real number, which yield another solution to Eqs.(2). If the assumption $w' = r(1+w^2)$ is made, then we can obtain singular periodic solutions.

In conclusion, by introducing a two-step ansatz we succeeded in deriving many solutions to Eqs.(1) and (2). It is obvious that our ansatz can produce more solutions than those used before for these two equations. Further work will be devoted to wave front solutions to coupled systems of reaction–diffusion type by using the same method.

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