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Form invariance and conserved quantities of Nielsen equations of relativistic variable mass nonholonomic systems

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In this paper, the definition and criterion of the form invariance of Nielsen equations for relativistic variable mass nonholonomic systems are given. The relation between the form invariance and the Noether symmetry is studied. Finally, we give an example to illustrate the application of the result.

Keywords: nonholonomic system, relativity, variable mass, Nielsen equation, form invariance, Noether symmetry

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1. Introduction

The conserved quantity of a mechanical system not only is of mathematical significance, but also reflects its profound physical nature. Furthermore, it has become one of the most important directions in modern analytical mechanics.

In 1918, while carrying out research on the invariance of the Hamilton action under infinitesimal transformations of groups, Noether disclosed the relation between the conserved quantity and the symmetry of the system, and put forth the renowned Noether theory.[1] In recent years, study of the Noether symmetries has made great progress.[2−9]

Symmetry method is a modern method in seeking conservation. A form invariance is a new symmetry. It is different from the Noether symmetry and can also lead to a conserved quantity under certain conditions.

In Refs.[10]–[15], the form invariance of the Lagrange equations, the Appell equations, the Gibbs–Appell equations, the Nielsen equations and the Chaplygin equations have been studied, but many of the research works are restricted to the scope of the classical mechanics.

The relativistic analytical mechanics is the foundation of modern physics. In this paper, we study the form invariance and conserved quantities of Nielsen equations of a relativistic variable mass nonholonomic system by means of the symmetrical theory of the holonomic system corresponding to the nonholonomic system and the symmetrical theory of the nonholonomic system respectively.[16,17] An example is given to illustrate the application of the result.

2. Nielsen equations of the corresponding relativistic variable mass holonomic system

Consider a mechanical system composed of N particles, and its configuration determined by n generalized coordinates q_1, q_2, ..., q_n. The motion of the system is subjected to the following g ideal bilateral nonholonomic constraints of Chetaev type:

$$f_\beta(t,q,\dot{q})=0, \quad (\beta=1,2,\cdots,g). \tag{1}$$

The mass of a representative point i of the system is

$$m_i=\frac{m_{0i}}{\sqrt{1-\dfrac{\dot{\vec{r}}_i^{\,2}}{c^2}}}, \tag{2}$$

where the static mass is

$$m_{0i}=m_{0i}(t,q). \tag{3}$$

Then, the Nielsen equations of a relativistic variable mass nonholonomic system can be written in Routh's

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form:

$$\frac{\partial \dot{L}_r}{\partial \dot{q}_s} - 2\frac{\partial L_r}{\partial q_s} = Q_s + P_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s},$$
$$(\beta = 1, 2, \cdots, g;\ s = 1, 2, \cdots, n). \tag{4}$$

Here L_r is the Lagrangian of the relativistic system, Q_s are nonpotential generalized forces, P_s are the generalized counter thrusts. According to Ref.[15], we have

$$P_s = \sum_{i=1}^{N} \left(\bar{\Phi}_i + \frac{\dot{m}_{0i}\, \dot{\bar{r}}_i}{\sqrt{1 - \frac{\dot{\bar{r}}_i^2}{c^2}}} \right) \cdot \frac{\partial \bar{r}_i}{\partial q_s} - \sum_{i=1}^{N} \frac{\partial m_{0i}}{\partial q_s} c^2 \left(1 - \sqrt{1 - \frac{\dot{\bar{r}}_i^2}{c^2}} \right),$$
$$\bar{\Phi}_i = \dot{m}_{0i} \left(\frac{\bar{u}_i}{\sqrt{1 - \frac{\bar{u}_i^2}{c^2}}} - \frac{\bar{v}_i}{\sqrt{1 - \frac{\bar{v}_i^2}{c^2}}} \right), \tag{5}$$

where λ_β are the constrained multipliers. Before we integrate the differential equations of motion, we can obtain [16] λ_β from Eqs.(1) and (4), and express it as a function of t,q and $\dot{q}$. Thus, Eqs.(4) can be expressed as

$$\frac{\partial \dot{L}_r}{\partial \dot{q}_s} - 2\frac{\partial L_r}{\partial q_s} = Q_s + P_s + \Lambda_s,$$
$$(s = 1, 2, \cdots, n). \tag{6}$$

where

$$\Lambda_s = \Lambda_s(t, q, \dot{q}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \tag{7}$$

are the generalized nonholonomic reactive forces of the constraints.

Equations (6) are called differential equations of motion of the holonomic system corresponding to the relativistic variable mass nonholonomic system (1) and (4). The integrals of the nonholonomic system (1) and (4) can be found according to the corresponding holonomic system (6), only if we apply the restriction of the nonholonomic constraints (1) to the initial conditions.[16,17]

3. Form invariance of Nielsen equations of the corresponding relativistic variable mass holonomic system

Introduce the infinitesimal transformations of time and coordinates as

$$t^* = t + \varepsilon \xi_0(t, q),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, q), \tag{8}$$

where ε is an infinitesimal parameter, and ξ_0, ξ_s are infinitesimal generators. Under transformation (8), L_r, Q_s, P_s and Λ_s become

$$L_r^* = L_r\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right), \quad Q_s^* = Q_s\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right),$$
$$P_s^* = P_s\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right), \quad \Lambda_s^* = \Lambda_s\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right). \tag{9}$$

3.1. Definition

If the form of Nielsen equations (6) remains invariant under the infinitesimal transformations (8), i.e.

$$\frac{\partial \dot{L}_r^*}{\partial \dot{q}_s} - 2\frac{\partial L_r^*}{\partial q_s} = Q_s^* + P_s^* + \Lambda_s^*,\ (s = 1, 2, \cdots, n), \tag{10}$$

then the invariance is called the form invariance of Nielsen equations (6) of the corresponding holonomic system.

3.2. Criterion

Expanding L_r^*, Q_s^*, P_s^* and Λ_s^*, we have [17,18]

$$L_r^* = L_r\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right) = L_r(t, q, \dot{q}) + \varepsilon \left[\frac{\partial L_r}{\partial t}\xi_0 + \frac{\partial L_r}{\partial q_s}\xi_s + \frac{\partial L_r}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) \right] + O(\varepsilon^2),$$
$$(s, j = 1, 2. \cdots, n). \tag{11}$$

We can obtain the formulae for Q_s^*, P_s^* and Λ_s^* which are similar to Eq.(11) in form.

Introducing a Nielsen operator

$$N_s = \frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s}, \tag{12}$$

then the following criteria can be obtained.

Criterion 1. If L_r, Q_s, P_s, Λ_s and the infinitesimal generators ξ_0, ξ_s satisfy the following relations:

$$N_s \left[\frac{\partial L_r}{\partial t}\xi_0 + \frac{\partial L_r}{\partial q_s}\xi_s + \frac{\partial L_r}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) \right] = 0; \tag{13}$$

$$\frac{\partial Q_s}{\partial t}\xi_0 + \frac{\partial Q_s}{\partial q_j}\xi_j + \frac{\partial Q_s}{\partial \dot{q}_j}(\dot{\xi}_j - \dot{q}_j\dot{\xi}_0) = 0,$$
$$\frac{\partial P_s}{\partial t}\xi_0 + \frac{\partial P_s}{\partial q_j}\xi_j + \frac{\partial P_s}{\partial \dot{q}_j}(\dot{\xi}_j - \dot{q}_j\dot{\xi}_0) = 0,$$
$$\frac{\partial \Lambda_s}{\partial t}\xi_0 + \frac{\partial \Lambda_s}{\partial q_j}\xi_j + \frac{\partial \Lambda_s}{\partial \dot{q}_j}(\dot{\xi}_j - \dot{q}_j\dot{\xi}_0) = 0, \tag{14}$$

then the Nielsen equations (6) of the corresponding holonomic system are form invariant under the infinitesimal transformations (8).

Proof:

Using Eqs.(11), (13) and (14), and ignoring ε^2 and the higher-order infinitesimal terms, we have

$$\begin{aligned} &N_s(L_r^*) - Q_s^* - P_s^* - \Lambda_s^* \\ =&N_s(L_r) - Q_s - P_s - \Lambda_s = 0. \end{aligned} \tag{15}$$

Criterion 2. If there exist a constant k and a function $G_L = G_L(t,q)$ such that the infinitesimal generators ξ_0 and ξ_s satisfy Eq.(14) and the following relation:

$$\frac{\partial L_r}{\partial t}\xi_0 + \frac{\partial L_r}{\partial q_s}\xi_s + \frac{\partial L_r}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) = kL_r - \dot{G}_L, \tag{16}$$

then the Nielsen equations (6) are form invariant under the infinitesimal transformations (8).

Proof:

Substituting Eq.(16) into Eq.(13), we obtain

$$N_s(kL_r - \dot{G}_L) = 0. \tag{17}$$

Ignoring ε^2 and the higher-order infinitesimal terms, then

$$\begin{aligned} &N_s(L_r^*) - Q_s^* - P_s^* - \Lambda_s^* \\ =&N_s(L_r) + \varepsilon N_s(kL_r - \dot{G}_L) - Q_s - P_s - \Lambda_s \\ =&N_s(L_r) - Q_s - P_s - \Lambda_s = 0. \end{aligned} \tag{18}$$

3.3. Form invariance and Noether symmetry

The Noether theory points out [18] that for a system determined by Lagrangian L_r, if the infinitesimal generators ξ_0, ξ_s satisfy the Noether identity

$$\begin{aligned} &\frac{\partial L_r}{\partial t}\xi_0 + \frac{\partial L_r}{\partial q_s}\xi_s + \frac{\partial L_r}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) \\ &+ (Q_s + P_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) \\ &+ L_r\dot{\xi}_0 = -\dot{G}_N, \quad (s = 1, 2, \cdots, n), \end{aligned} \tag{19}$$

where $G_N = G_N(t,q)$, then the symmetry is called Noether symmetry. The Noether symmetry leads to the following conserved quantity:

$$I = L_r\xi_0 + \frac{\partial L_r}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_N = \text{const.} \tag{20}$$

The form invariance of Nielsen equations can be either a Noether symmetry or not. We have the following proposition.

3.4. Proposition

If Nielsen equations (6) are form invariant under the infinitesimal transformations (8), and there is a gauge function $G_N = G_N(t,q)$ satisfying

$$\begin{aligned} &kL_r - \dot{G}_L + L_r\dot{\xi}_0 + (Q_s + P_s + \Lambda_s) \\ &\cdot(\xi_s - \dot{q}_s\xi_0) = -\dot{G}_N; \end{aligned} \tag{21}$$

then the form invariance is a Noether symmetry; or else it is not.

Proof:

Substituting Eq.(16) into Eq.(19), we can obtain Eq.(21)

4. Form invariance of Nielsen equations of a relativistic nonholonomic system

Under the transformations (8), L_r, Q_s, P_s, Λ_s, λ_β and f_β become

$$L_r^* = L_r\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right). \tag{22}$$

We can obtain the other formulae for Q_s, P_s, Λ_s, λ_β and f_β which are similar to Eq.(22) in form.

4.1. Definition

If the form of the constraint equations (1) and the differential equations (6) remain invariant under infinitesimal transformations (8), i.e.

$$f_\beta^* = f_\beta\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right) = 0, \quad (\beta = 1, 2, \cdots, g), \tag{23}$$

$$\frac{\partial \dot{L}_r^*}{\partial \dot{q}_s} - 2\frac{\partial L_r^*}{\partial q_s} = Q_s^* + P_s^* + \Lambda_s^*, \quad (s = 1, 2, \cdots, n), \tag{24}$$

then the invariance is called the form invariance of Nielsen equations (4) of the relativistic variable mass nonholonomic system.

Expanding f_β^*, we have [17]

$$\begin{aligned} f_\beta^* =&f_\beta\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}\right) = f_\beta(t, q, \dot{q}) \\ &+ \varepsilon\left[\frac{\partial f_\beta}{\partial t}\xi_0 + \frac{\partial f_\beta}{\partial q_s}\xi_s + \frac{\partial f_\beta}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\right] \\ &+ O(\varepsilon^2). \end{aligned} \tag{25}$$

Substituting Eq.(25) into Eq.(23), and ignoring ε^2 and the higher-order infinitesimal terms, using Eq.(1), we obtain

$$\frac{\partial f_\beta}{\partial t}\xi_0 + \frac{\partial f_\beta}{\partial q_s}\xi_s + \frac{\partial f_\beta}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) = 0. \tag{26}$$

4.2. Criterion

For the relativistic variable mass nonholonomic systems (1) and (4), if the infinitesimal generators ξ_0 and ξ_s satisfy Eqs.(14), (16) and (26), then the nonholonomic system is form invariant under the infinitesimal transformations (8).

Proof:

Substituting Eq.(26) into Eq.(25), using Eq.(1), we can obtain Eq.(23).

Considering the Nielsen operator (13), using Eqs.(11), (14), and (16), ignoring ε^2 and the higher-order infinitesimal terms, then as regard to Eq.(6), we have

$$\begin{aligned} &N_s(L_r^*) - Q_s^* - P_s^* - \Lambda_s^* \\ =&N_s(L_r) + \varepsilon N_s(kL_r - \dot{G}_L) - Q_s - P_s - \Lambda_s \\ =&N_s(L_r) - Q_s - P_s - \Lambda_s = 0. \end{aligned} \tag{27}$$

4.3. Form invariance and Noether symmetry

For the nonholonomic systems (1) and (4), if the infinitesimal generators ξ_0, ξ_s satisfy the Noether identity (19) and additional restriction conditions

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0,$$

$$(\beta = 1, 2, \cdots, g; s = 1, 2, \cdots, n), \tag{28}$$

then the Noether symmetry leads to the same conserved quantity as Eq.(20).

The form invariance of Nielsen equations of the nonholonomic system can be either a Noether symmetry or not. We have the following proposition.

4.4. Proposition

If the nonholonomic system given by Eqs.(1) and (4) is form invariant under infinitesimal transformations (8), and there is a gauge function $G_N = G_N(t, q)$ satisfying the following equation:

$$kL_r - \dot{G}_L + L_r\dot{\xi}_0 + (Q_s + P_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) = -\dot{G}_N, \tag{29}$$

and the infinitesimal generators ξ_0, ξ_s satisfying the additional restriction conditions

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0,$$

$$(\beta = 1, 2, \cdots, g; s = 1, 2, \cdots, n), \tag{30}$$

then the form invariance is Noether symmetry; or else it is not.

Through the above, the form invariance of holonomic system (6) corresponding to the nonholonomic systems (1) and (4) should satisfy the conditions (13), (14) or (14), (16).

The form invariance of the nonholonomic systems (1) and (4), needs to satisfy not only conditions (14), (16) but also (26). It seems that the form invariance of the nonholonomic system must be the form invariance of the holonomic system corresponding to the nonholonomic system. On the contrary, it is uncertainly.

5. Example

We consider a particle moves on a plane, its Lagrangian is

$$L_r = B\mathrm{e}^{-\alpha t}c^2\left(1 - \sqrt{1 - \frac{(\dot{q}_1^2 + \dot{q}_2^2)}{c^2}}\right) - V,$$

$$(B = \text{const}, \alpha = \text{const}, \alpha > 0), \tag{31}$$

and the nonpotential generalized forces are $Q_1 = \dot{q}_1$, $Q_2 = -\dot{q}_2$; potential energy V=0; $\bar{u} = 0$.

The motion of the system is subjected to the following ideal bilateral nonholonomic constraint of Chetaev type:

$$f = \dot{q}_1 - \dot{q}_2 + tq_1 = 0. \tag{32}$$

Let us study the form invariance and the Noether symmetry of Nielsen equations.

Because

$$\begin{aligned} \dot{L}_r =& - B\alpha\mathrm{e}^{-\alpha t}c^2\left(1 - \sqrt{1 - \frac{(\dot{q}_1^2 + \dot{q}_2^2)}{c^2}}\right) \\ &+ B\mathrm{e}^{-\alpha t}\frac{(\dot{q}_1\ddot{q}_1 + \dot{q}_2\ddot{q}_2)}{\sqrt{1 - \frac{(\dot{q}_1^2 + \dot{q}_2^2)}{c^2}}}, \end{aligned} \tag{33a}$$

$$\frac{\partial \dot{L}_r}{\partial \dot{q}_1} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_1),\ \frac{\partial \dot{L}_r}{\partial \dot{q}_2} = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_2), \tag{33b}$$

$$\frac{\partial L_r}{\partial q_1} = \frac{\partial L_r}{\partial q_2} = 0, \tag{33c}$$

where

$$m = \frac{B\mathrm{e}^{-\alpha t}}{\sqrt{1 - \frac{(\dot{q}_1^2 + \dot{q}_2^2)}{c^2}}},$$

according to Eq.(5) and considering $\bar{u} = 0$, we obtain

$$P_1 + P_2 = 0. \tag{34}$$

Equations (4) give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_1) &= \dot{q}_1 + \lambda, \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_2) &= -\dot{q}_2 - \lambda. \end{aligned} \tag{35}$$

From Eqs.(32) and (35), we obtain

$$\lambda = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) - \dot{q}_1 + \frac{q_1t}{2}. \tag{36}$$

Therefore, we have

$$\begin{aligned} Q_1 + P_1 + \Lambda_1 &= \dot{q}_1 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) - \dot{q}_2 + \frac{q_1t}{2}, \\ Q_2 + P_2 + \Lambda_2 &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) - \frac{q_1t}{2}. \end{aligned} \tag{37}$$

Taking

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = 1, \tag{38}$$

and substituting these into Eq.(18), we obtain

$$kL_r - \dot{G}_L = 0. \tag{39}$$

Taking

$$k = 0, \; G_L = 0, \tag{40}$$

formulae (14), (16) and (26) hold true, which corresponds to a form invariance of Nielsen equations. Substituting Eqs.(38) and (40) into Eq.(29), we obtain

$$G_N = q_1 - q_2 \tag{41}$$

and Eq.(38) satisfying the additional restriction conditions (30). So, the form invariance leads to a Noether symmetry. In this case, conserved quantity (20) gives

$$I = B\mathrm{e}^{-\alpha t}\frac{(\dot{q}_1 + \dot{q}_2)}{\sqrt{1 - \dfrac{(\dot{q}_1^2 + \dot{q}_2^2)}{c^2}}} + (q_2 - q_1) = \text{const.} \tag{42}$$

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