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Velocity-dependent symmetries and conserved quantities of the constrained dynamical systems*

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In this paper, we have extended the theorem of the velocity-dependent symmetries to nonholonomic dynamical systems. Based on the infinitesimal transformations with respect to the coordinates, we establish the determining equations and restrictive equation of the velocity-dependent system before the structure equation is obtained. The direct and the inverse issues of the velocity-dependent symmetries for the nonholonomic dynamical system is studied and the non-Noether type conserved quantity is found as the result. Finally, we give an example to illustrate the conclusion.

Keywords: velocity-dependent symmetry, Lie group, determining equation, non-Noether type conserved quantity

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1. Introduction

The symmetries of mechanical and physical systems have a close relation with the associated conserved quantities. There are three important methods of symmetries to seek the conserved quantities of the dynamical systems: Noether symmetry, Lie symmetry, and Mei's form invariance. The Noether symmetry is an invariance of the Hamiltonian action under the infinitesimal transformation of a continuous group.[1] In 1975, Djukić and Vujanović gave the Noether symmetries and conserved quantities for the classical non-conservative mechanical systems.[2,3] Li extended Noether theory to the classical and quantum constraint systems.[4,5] Mei advanced Noether theorem from Hamiltonian systems to Birkhoffian systems.[6] Sarlet studied the pseudo-symmetries, Noether's theorem and the adjoint equation.[7] The Lie symmetry preserves the equation of motion under the infinitesimal transformation of a continuous group. In 1979, Lutzky applied the Lie theory to mechanical systems, and obtained Lie symmetries and conserved quantities of dynamical systems.[8] The research on Lie symmetries has gained many important results.[9−18] Mei pointed out that it is an invariant of the dynamical equation of the system, of which the infinitesimal transformation of a continuous group is taken from a physical quantity such as Lagrangian, non-potential generalized force and generalized constrained force etc.[19−21] With the development of science and technologies, this study of form invariance becomes more and more important.[22−24]

For Lie symmetries and Mei's form invariance, however, only the Noether-type conserved quantity is obtained. In 1992, Hojman gave a theorem regarding the conserved quantities as second-order dynamical systems.[25] González-Gascón [26] considered the theorem in generalized coordinates, and the result may be stated as follows: Let the function $\eta_l(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ determine a symmetry generator $E=\eta_l\dfrac{\partial}{\partial q_l}+\dot{\eta}_l\dfrac{\partial}{\partial\dot{q}_l}$ for the equations of motion $\ddot{q}_l=\alpha_l(\boldsymbol{q},\dot{\boldsymbol{q}},t)$; if a function $\lambda(\boldsymbol{q},\dot{\boldsymbol{q}},t)$ can be found to make

$$\Omega=\frac{\partial\alpha_l}{\partial\dot{q}_l}+\frac{\mathrm{d}}{\mathrm{d}t}\ln\lambda$$

vanishing, a constant of the motion can be given by

$$\phi=E(\ln\lambda)+\frac{\partial\eta_l}{\partial q_l}+\frac{\partial\dot{\eta}_l}{\partial\dot{q}_l}.$$

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Lately, Lutzky proved that a conserved quantity is associated with a velocity-dependent Lagrangian symmetry.[27,28] The theorem referred to above results holds whether or not the equations of motion are derivable from a Lagrangian. ϕ is a non-Noether type conserved quantity and it is dependent on a Lagrangian. Recently, based on the relationships among motion, nonconservative forces and Lagrangian, we present conservation laws on non-Noether symmetries for nonconservative dynamical systems by the authors of Ref.[29].

In this paper, we extend the theorem of velocity-dependent symmetries to nonholonomic dynamical systems to derive the non-Noether type conserved quantities. The direct and inverse problems are studied separately.

2. Conserved quantities and velocity-dependent symmetries of nonholonomic systems

2.1. Direct issue of the velocity-dependent symmetries

A mechanical system of n degree of freedom with Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, subjected to the g ideal bilateral nonholonomic constraints of the form

$$f_\beta(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = 0 \quad (\beta = 1, \cdots, g), \tag{1}$$

has the equations of motion as [30,31]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s^{"} + \Lambda_s, \quad \left(\Lambda_s = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}\right). \tag{2}$$

The equations of motion can be put in the form

$$\ddot{q}_s = g_s(q_k, \dot{q}_k, t) \ (s, k = 1, \cdots, n). \tag{3}$$

Let us introduce the infinitesimal transformations with respect to generalized coordinates

$$q_s^{\bullet}(t) = q_s(t) + \varepsilon \eta_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, t), \tag{4}$$

where ε is a small parameter, and η_s are infinitesimal generators.

Furthermore, we introduce the generator vector of the infinitesimal transformation

$$E^0 = \eta_s \frac{\partial}{\partial q_s}, \tag{5}$$

and its first expansion

$$E^1 = \eta_s \frac{\partial}{\partial q_s} + \dot{\eta}_s \frac{\partial}{\partial \dot{q}_s}, \tag{6}$$

and the second expansion

$$E^2 = \eta_s \frac{\partial}{\partial q_s} + \dot{\eta}_s \frac{\partial}{\partial \dot{q}_s} + \ddot{\eta}_s \frac{\partial}{\partial \ddot{q}} = E^1 + \ddot{\eta}_s \frac{\partial}{\partial \ddot{q}}. \tag{7}$$

where functions $\eta_s(q, \dot{q}, t)$ determine a symmetry generator.

The invariance of Eqs.(3) and Eq.(1) under the infinitesimal transformation (4) leads to the determining equations

$$\ddot{\eta}_s = E^1(g_s), \tag{8}$$

and restrictive equations

$$\eta_s \frac{\partial f_\beta}{\partial q_s} + \dot{\eta}_s \frac{\partial f_\beta}{\partial \dot{q}_s} = E^1(f_\beta) = 0. \tag{9}$$

Equations (8) and (9) can be regarded as the criterion of velocity-dependent symmetries.

Criterion 1 The symmetries are called the velocity-dependent symmetries corresponding to holonomic dynamical system (3) if the infinitesimal generators η_s satisfy the determining equations (8).

Criterion 2 The symmetries are called the velocity-dependent symmetries corresponding to nonholonomic system (3) and (1) if the infinitesimal generators η_s satisfy the determining equations (8) and the restrictive equations (9).

The velocity-dependent symmetry does not always imply the conserved quantity, and so one has the following theorem.

Theorem 1 If the infinitesimal generations $\eta_s = \eta_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ are of velocity-dependent symmetry for Eqs.(1) and (3), and a gauge function G can be found to satisfy

$$\frac{\partial g_s}{\partial \dot{q}_s} + \dot{G} = \psi, \tag{10}$$

then a constant of the motion is given by

$$\phi = E^1[G] + \frac{\partial \eta_s}{\partial q_s} + \frac{\partial \dot{\eta}_s}{\partial \dot{q}_s}. \tag{11}$$

Equation (10) is called the structural equation of the velocity-dependent symmetry, and Eq.(11) expresses a non-Noether symmetry-type conserved quantity associated with the symmetry. In fact, $E^2(\psi) = 0$ just implies $\dot{\phi} = 0$ for ϕ can be demonstrated as

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}[E^1(G)] + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \eta_s}{\partial q_s} + \frac{\partial \dot{\eta}_s}{\partial \dot{q}_s}\right]. \tag{12}$$

For any function $f = f(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial q_s} = \frac{\partial \dot{f}}{\partial q_s} - \frac{\partial g_k}{\partial q_s}\frac{\partial f}{\partial \dot{q}_k}, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}_s}=\frac{\partial \dot{f}}{\partial \dot{q}_s}-\frac{\partial f}{\partial q_s}-\frac{\partial g_k}{\partial \dot{q}_s}\frac{\partial f}{\partial \dot{q}_k}, \tag{14}$$

$$E\left(\frac{\partial f}{\partial \dot{q}_s}\right)=\frac{\partial}{\partial \dot{q}_s}E^1(f)-\frac{\partial \eta_k}{\partial \dot{q}_s}\frac{\partial f}{\partial q_k}-\frac{\partial \dot{\eta}_k}{\partial \dot{q}_s}\frac{\partial f}{\partial \dot{q}_k}. \tag{15}$$

Given $\ddot{\eta}_s = E^1(g_s)$, η_s can define the symmetry (7) for $\ddot{q}_s = g_s$. So we know that if E^1 is a symmetry, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E^1(f)=E^1(\dot{f}). \tag{16}$$

From Eqs.(13) and (14) we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \eta_s}{\partial q_l}+\frac{\partial \dot{\eta}_s}{\partial \dot{q}_s}\right)=\frac{\partial \ddot{\eta}_s}{\partial \dot{q}_s}-\frac{\partial g_k}{\partial \dot{q}_s}\frac{\partial \dot{\eta}_s}{\partial \dot{q}_k}-\frac{\partial g_k}{\partial q_s}\frac{\partial \eta_s}{\partial \dot{q}_k}. \tag{17}$$

Furthermore, Eq.(15) makes possible $E^1\left(\frac{\partial g_s}{\partial \dot{q}_s}\right)$ equal to the right-hand sides of Eq.(17), so that we have

$$E^1\left(\frac{\partial g_s}{\partial \dot{q}_s}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial \eta_s}{\partial q_s}+\frac{\partial \dot{\eta}_s}{\partial \dot{q}_s}\right]. \tag{18}$$

Substituting Eqs.(16) and (18) into Eq.(12) yields

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=E^1\left(\frac{\partial g_s}{\partial \dot{q}_s}+G\right)=0. \tag{19}$$

Consequently Eq.(11) leads to the fact that ϕ should be a conserved quantity.

2.2. Inverse issue of the velocity-dependent symmetries

Finding the velocity-dependent symmetries from the given conserved quantities is called the inverse problem, which can be solved by the following steps.

First, we find out the corresponding velocity-dependent symmetry from the known conserved quantity. Suppose that there exists some conserved quantity in the nonholonomic system

$$\phi=\phi(\boldsymbol{q},\dot{\boldsymbol{q}},t)=\mathrm{const}, \tag{20}$$

then we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\frac{\partial \phi}{\partial t}+\frac{\partial \phi}{\partial q_s}\dot{q}_s+\frac{\partial \phi}{\partial \dot{q}_s}\ddot{q}_s=0. \tag{21}$$

Multiplying Eq.(2) by η_s and summing up for s leads to

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}-Q_s-\Lambda_s\right)\eta_s=0. \tag{22}$$

Adding Eq.(21) to Eq.(22), and expanding the resulting equation yields

$$\frac{\partial \phi}{\partial t}+\frac{\partial \phi}{\partial q_s}\dot{q}_s+\frac{\partial \phi}{\partial \dot{q}_s}\ddot{q}_s-\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\ddot{q}_k+\frac{\partial^2 L}{\partial \dot{q}_s\partial q_k}\dot{q}_k+\frac{\partial^2 L}{\partial \dot{q}_s\partial t}-\frac{\partial L}{\partial q_s}-Q_s-\Lambda_s\right)\eta_s=0. \tag{23}$$

Let the coefficients of the $\ddot{q}_k$ be zero, and we have

$$\eta_s=\bar{h}_{sk}\frac{\partial \phi}{\partial \dot{q}_k},\ \bar{h}_{sk}h_{kr}=\delta_{sr}=\begin{cases}1 & s=r,\\ 0 & s\neq r,\end{cases} \tag{24}$$

where

$$h_{sk}=\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}. \tag{25}$$

Letting the conserved quantity in Eq.(11) be equal to Eq.(20) yields

$$E^1(G)+\frac{\partial \eta_s}{\partial q_s}+\frac{\partial \dot{\eta}_s}{\partial \dot{q}_s}=\phi, \tag{26}$$

and then we can obtain generators η_s and gauge function G from Eqs.(24) and (26).

Secondly, substituting generators η_s and gauge G into determining equation (8) and restrictive equation (9), we can determine the velocity-dependent symmetry of the system.

Theorem 2 The symmetry is called the velocity-dependent symmetry for the holonomic systems (3) when the infinitesimal generators η_s determined by Eqs.(24) and (26) satisfy the determining equation (8).

Theorem 3 The symmetry is called the velocity-dependent symmetry for the nonholonomic system (1) and (3) when the infinitesimal generators η_s determined by Eqs.(24) and (26) satisfy the determining equation (8) and restrictive equation (9).

This study of the velocity-dependent symmetry is confined to the nonholonomic system, but the velocity-dependent symmetry for the non-conservative dynamical system can be studied similarly.[2,3]

3. An example: Appell–Hamel problem [32,33]

The Lagrangian of the Appell–Hamel problem is

$$L=\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-mgq_3, \tag{27}$$

subjected to the nonholonomic constraints of the form

$$f=\dot{q}_3-(\dot{q}_1^2+\dot{q}_2^2)^{\frac{1}{2}}=0. \tag{28}$$

First we study the direct issue of the velocity-dependent symmetry. The dynamical equations of the system are

$$m\ddot{q}_1=2\lambda\dot{q}_1,\ m\ddot{q}_2=2\lambda\dot{q}_2,\ m\ddot{q}_3=-mg-2\lambda\dot{q}_3. \tag{29}$$

Using Eqs.(28) and (29), the constrained multiplier is given by

$$\lambda = -\frac{mg\dot{q}_3}{2(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)} = -\frac{mg}{4\dot{q}_3}. \tag{30}$$

So we have

$$\begin{aligned} Q_1'' + \Lambda_1 &= -\frac{1}{2}mg\frac{\dot{q}_1}{\dot{q}_3}, \\ Q_2'' + \Lambda_2 &= -\frac{1}{2}mg\frac{\dot{q}_2}{\dot{q}_3}, \\ Q_3'' + \Lambda_3 &= \frac{1}{2}mg. \end{aligned} \tag{31}$$

The equations of motion can be expressed as

$$\begin{aligned} \ddot{q}_1 &= -\frac{1}{2}g\frac{\dot{q}_1}{\dot{q}_3} = h_1, \\ \ddot{q}_2 &= -\frac{1}{2}g\frac{\dot{q}_2}{\dot{q}_3} = h_2, \\ \ddot{q}_3 &= -\frac{1}{2}g = h_3. \end{aligned} \tag{32}$$

Substituting Eq.(32) into Eq.(8) leads to the following determining equations:

$$\begin{aligned} \ddot{\eta}_1 &= \dot{\eta}_1\left(-\frac{1}{2}g\frac{1}{\dot{q}_3}\right) + \dot{\eta}_3\left(\frac{1}{2}g\frac{\dot{q}_1}{\dot{q}_3^2}\right), \\ \ddot{\eta}_2 &= \dot{\eta}_2\left(-\frac{1}{2}g\frac{1}{\dot{q}_3}\right) + \dot{\eta}_3\left(\frac{1}{2}g\frac{\dot{q}_2}{\dot{q}_3^2}\right), \\ \ddot{\eta}_3 &= 0. \end{aligned} \tag{33}$$

Equation (33) has the following solutions:

$$\eta_1^1 = 1;\ \eta_2^1 = 1,\ \eta_3^1 = 1,\ E_1 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}; \tag{34}$$

$$\begin{aligned} \eta_1^2 &= \int \frac{\dot{q}_1}{\dot{q}_3}\mathrm{d}t, \\ \eta_2^2 &= \int \frac{\dot{q}_2}{\dot{q}_3}\mathrm{d}t, \\ \eta_3^2 &= 1; \\ E_2 &= \int \frac{\dot{q}_1}{\dot{q}_3}\mathrm{d}t\frac{\partial}{\partial q_1} + \int \frac{\dot{q}_1}{\dot{q}_3}\mathrm{d}t\frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}. \end{aligned} \tag{35}$$

We can obtain the velocity-dependent symmetry corresponding to the holonomic system from the above equations.

Substituting Eq.(32) into Eq.(9) yields

$$\dot{\eta}_1\dot{q}_1 + \dot{\eta}_2\dot{q}_2 - \dot{\eta}_3\dot{q}_3 = 0. \tag{36}$$

Substituting the generator (34) into Eq.(36) can hold, and then Eq.(34) is a velocity-dependent symmetry associated with the nonholonomic system.

Substituting Eqs.(32) and (31) into the structural equation (10) yields

$$\begin{aligned} \psi = &-\frac{g}{2\dot{q}_3} - \frac{g}{2\dot{q}_3} - \frac{1}{2}mg\frac{\dot{q}_1}{\dot{q}_3} \\ &-\frac{1}{2}mg\frac{\dot{q}_2}{\dot{q}_3} + \frac{1}{2}mg + \dot{G}, \quad E^2(\psi) = 0. \end{aligned} \tag{37}$$

Letting $\psi = 0$, we cannot find the gauge function G. But when we take

$$\begin{aligned} \psi = &-\frac{g}{\dot{q}_3} - \frac{1}{2}mg\frac{\dot{q}_1}{\dot{q}_3} - \frac{1}{2}mg\frac{\dot{q}_2}{\dot{q}_3} \\ &+ q_2\dot{q}_1 - q_1\dot{q}_1, \quad E^2(\psi) = 0, \end{aligned} \tag{38}$$

where ψ is an invariant of the symmetry (34), we can obtain a gauge function

$$G = q_2q_1 - \frac{1}{2}q_1^2 - \frac{1}{2}mgt. \tag{39}$$

Substituting Eqs.(34) and (39) into Eq.(11) leads to

$$\begin{aligned} \phi_1 = &\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}\right) \\ &\times \left(q_2q_1 - \frac{1}{2}q_1^2 - \frac{1}{2}mgt\right) = q_2 = \text{const.} \end{aligned} \tag{40}$$

Therefore, we obtain a conserved quantity associated with the velocity-dependent symmetry for nonholonomic system (28) and (32). Certainly, it is a non-Noether type conserved quantity.

The generators (35) and (36) cannot both hold, and so the generator (35) is the only velocity-dependent symmetry associated with the holonomic system (32). Substituting Eqs.(35) and (39) into Eq.(11) leads to the following conserved quantity:

$$\begin{aligned} \phi_2 = &\left(\int \frac{\dot{q}_1}{\dot{q}_3}\mathrm{d}t\frac{\partial}{\partial q_1} + \int \frac{\dot{q}_2}{\dot{q}_3}\mathrm{d}t\frac{\partial}{\partial q_2} + \frac{\partial}{\partial q_3}\right) \\ &\times \left(q_2q_1 - \frac{1}{2}q_1^2 - \frac{1}{2}mgt\right) \\ &+ \frac{\partial}{\partial q_1}\int \frac{\dot{q}_1}{\dot{q}_3}\mathrm{d}t + \frac{\partial}{\partial q_2}\int \frac{\dot{q}_2}{\dot{q}_3}\mathrm{d}t + \frac{1}{\dot{q}_3} + \frac{1}{\dot{q}_3} \\ = &(q_2 - q_1)\int \frac{\dot{q}_1}{\dot{q}_3}\mathrm{d}t + q_1\int \frac{\dot{q}_2}{\dot{q}_3}\mathrm{d}t + \frac{\partial}{\partial q_1}\int \frac{\dot{q}_1}{\dot{q}_3}\mathrm{d}t \\ &+ \frac{\partial}{\partial q_2}\int \frac{\dot{q}_2}{\dot{q}_3}\mathrm{d}t - \frac{4}{gt} = \text{const.} \end{aligned} \tag{41}$$

We can conclude as follows. The velocity-dependent symmetries can lead directly to the non-Noether type conserved quantities. The velocity-dependent symmetries and Noether symmetries of the dynamical systems are two different symmetries. The gauge function of the velocity dependent symmetries is determined by choosing invariant ψ of the symmetry group and the conserved quantities given by Eqs.(10) and (11). The conserved quantities of the Noether symmetries are obtained by the Noether theorem. The velocity-dependent symmetries and Lie symmetries of the dynamical systems are also two different symmetries, and so the gauge function and the conserved quantities of the Lie symmetries can be obtained by using the structural equation and the formula of conserved quantities of the Noether symmetry.

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