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Bound states of two-dimensional relativistic harmonic oscillators

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We give the exact normalized bound state wavefunctions and energy expressions of the Klein–Gordon and Dirac equations with equal scalar and vector harmonic oscillator potentials in the two-dimensional space.

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1. Introduction

The harmonic oscillator is a basic model both in classical physics and modern physics. It is widely used in many fields in physics. In non-relativistic quantum mechanics, the harmonic oscillator potential is one of the solvable potentials for Schrödinger's equation and has been adequately studied whether it is one-dimensional (1D), two-dimensional (2D) or three-dimensional (3D). In contrast, the study of this problem is somewhat insufficient in relativistic quantum mechanics. Though Su and Ma [1] discussed 3D and 1D Dirac equations with both scalar and vector harmonic oscillator potentials, we also solved the Klein–Gordon and Dirac equations with equal scalar and vector harmonic oscillator potentials, then obtained the normalized wavefunction and the formula of energy,[2] yet we have never seen any report on the research of 2D relativistic harmonic oscillators. The importance of 2D quantum systems is never less than that of 1D or 3D quantum systems. For example, a 2D quantum system can probe the connection between classical and quantum chaos, and it also has application in a number of surface systems, such as atomic corrals.[3] In this paper, we investigate a typical 2D quantum system, and 2D relativistic harmonic oscillators. We solve the Klein–Gordon and Dirac equations with equal scalar and vector harmonic oscillator potentials in 2D space, and give the complete normalized wavefunction and the formula of energy.

2. Bound-state solution of the Klein–Gordon equation with scalar and vector potentials equal to each other

According to Ref.[4], the Klein–Gordon equation for the 2D equal scalar and vector central potential is ($\hbar$=c=1)

$$\left[\boldsymbol{P}^2 - \left(E - \frac{V(\rho)}{2}\right)^2\right]\psi = -\left(\mu + \frac{V(\rho)}{2}\right)^2\psi, \quad (1)$$

where $\boldsymbol{P}$ is the momentum operator, E and μ are the energy and rest mass of the particle respectively. For the 2D harmonic oscillator potential we have

$$V(\rho) = \frac{1}{2}k\rho^2, \quad (2)$$

in plane polar coordinates (ρ, ϕ), Eq.(1) is written as

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi(\rho,\phi)\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\psi(\rho,\phi) + \alpha_1\left(\alpha_2 - \frac{1}{2}k\rho^2\right)\psi(\rho,\phi) = 0, \quad (3)$$

where

$$\alpha_1 = E + \mu, \quad \alpha_2 = E - \mu. \quad (4)$$

Considering the periodic boundary condition of coordinate ϕ and letting

$$\psi(\rho,\phi) = g(\rho)\mathrm{e}^{\mathrm{i}m\phi}, \quad (5)$$

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we obtain the equation of $g(\rho)$

$$g''(\rho)+\frac{1}{\rho}g'(\rho)-\left[\frac{m^2}{\rho^2}-\alpha_1(\alpha_2-\frac{1}{2}k\rho^2)\right]g(\rho)=0. \tag{6}$$

To solve Eq.(6) we first study the asymptotic behaviours of $g(\rho)$ for small ρ and large ρ. It is easy to see for large ρ, $g(\rho)\to \mathrm{e}^{-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}$, and $g(\rho)$ is zero when $\rho\to\infty$. On the other hand, for small ρ, the solution of Eq.(6), which satisfies the necessary condition of finiteness, is $g(\rho)\to\rho^{|m|}$. The above analysis suggests that we let

$$x=\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2,\quad g(\rho)=\mathrm{e}^{\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}\rho^{|m|}f(x), \tag{7}$$

then we obtain

$$\begin{aligned}&xf''(x)+(1+|m|-x)f'(x)\\&-\left[\frac{1+|m|}{2}-\frac{\sqrt{k\alpha_1}\alpha_2}{2\sqrt{2}k}\right]f(x)=0.\end{aligned} \tag{8}$$

This is a confluent hypergeometric equation,[5] its solution is

$$f(x)=F\left[\frac{1+|m|}{2}-\frac{\sqrt{k\alpha_1}\alpha_2}{2\sqrt{2}k},1+|m|,x\right], \tag{9}$$

where F represents the confluent hypergeometric function $F[\alpha,\gamma,z]$.[5] Combining Eqs.(7) and (9), we obtain

$$\begin{aligned}g(\rho)=&\mathrm{e}^{\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}\rho^{|m|}F\left[\frac{1+|m|}{2}\right.\\&\left.-\frac{\sqrt{k\alpha_1}\alpha_2}{2\sqrt{2}k},1+|m|,\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2\right].\end{aligned} \tag{10}$$

For bound states, the confluent hypergeometric function $F[\alpha,\gamma,z]$ must terminate as a polynomial, which demands

$$\frac{1+|m|}{2}-\frac{\sqrt{k\alpha_1}\alpha_2}{2\sqrt{2}k}=-n_\rho,\quad n_\rho=0,1,2,\cdots \tag{11}$$

Now we can use n_ρ to express $g(\rho)$

$$\begin{aligned}g(\rho)=&c_{n_\rho,m}\mathrm{e}^{\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}\rho^{|m|}F\\&\times\left[-n_\rho,1+|m|,\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2\right],\end{aligned} \tag{12}$$

where $c_{n_\rho,m}$ is a normalization constant. According to the normalization condition

$$\int_0^\infty\int_0^{2\pi}\rho|\psi(\rho,\phi)|^2\mathrm{d}\phi\ \mathrm{d}\rho=\int_0^\infty\rho|g(\rho)|^2\mathrm{d}\rho=1, \tag{13}$$

and using the relation of the confluent hypergeometric function with Laguerre polynomial and the normalization condition of Laguerre polynomial,[5] finally, we obtain the normalization constant

$$c_{n_\rho,m}=\frac{2^{\frac{1-|m|}{4}}(k\alpha_1)^{\frac{1+|m|}{4}}\sqrt{\Gamma(1+|m|+n_\rho)}}{\sqrt{n_\rho!}\ \Gamma(1+|m|)}. \tag{14}$$

3. Bound-state solution of the Dirac equation with scalar and vector potentials equal to each other

The Dirac equation with a 2D scalar potential $s(\rho)$ and a vector potential $v(\rho)$ is ($\hbar$=c=1)

$$\{\alpha\cdot\boldsymbol{p}+\beta[\mu+\boldsymbol{s}(\boldsymbol{r})]\}\psi=[\boldsymbol{E}-\boldsymbol{v}(\boldsymbol{r})]\psi, \tag{15}$$

or in the explicit form

$$\begin{aligned}&-[\alpha_2-s(\rho)-v(\rho)]\ \Psi_1+P_-\ \Psi_4=0,\\&-[\alpha_2-s(\rho)-v(\rho)]\ \Psi_2+P_+\ \Psi_3=0,\\&-[\alpha_1+s(\rho)-v(\rho)]\ \Psi_3+P_-\ \Psi_2=0,\\&-[\alpha_2+s(\rho)-v(\rho)]\ \Psi_4+P_+\ \Psi_1=0,\end{aligned} \tag{16}$$

where α_1 and α_2 are the same as in Eq.(4), and in plane polar coordinates (ρ,ϕ),

$$P_\pm=P_x\pm\mathrm{i}P_y=-\mathrm{e}^{\pm\mathrm{i}\phi}\left(\mathrm{i}\frac{\partial}{\partial\rho}\mp\frac{1}{\rho}\frac{\partial}{\partial\phi}\right). \tag{17}$$

Taking $s(\rho)=v(\rho)=\frac{1}{2}k\rho^2$ in Eq.(16) and letting

$$\Psi(\rho,\ \phi)=\begin{pmatrix}\Psi_1(\rho,\phi)\\\Psi_2(\rho,\phi)\\\Psi_3(\rho,\phi)\\\Psi_4(\rho,\phi)\end{pmatrix}=\begin{pmatrix}g_1(\rho)\mathrm{e}^{\mathrm{i}m\phi}\\g_2(\rho)\mathrm{e}^{\mathrm{i}m\phi}\\g_3(\rho)\mathrm{e}^{\mathrm{i}(m-1)\phi}\\g_4(\rho)\mathrm{e}^{\mathrm{i}(m+1)\phi}\end{pmatrix}, \tag{18}$$

leads to the equations

$$\begin{aligned}&g_4'(\rho)+\frac{m+1}{\rho}\ g_4(\rho)-\left(\alpha_2-\frac{1}{2}k\rho^2\right)\ g_1(\rho)=0,\\&g_3'(\rho)-\frac{m-1}{\rho}\ g_3(\rho)-\left(\alpha_2-\frac{1}{2}k\rho^2\right)\ g_2(\rho)=0,\\&g_2'(\rho)+\frac{m}{\rho}\ g_2(\rho)+\alpha_1\ g_3(\rho)=0,\\&g_1'(\rho)-\frac{m}{\rho}\ g_1(\rho)+\alpha_1\ g_4(\rho)=0.\end{aligned} \tag{19}$$

From the third and fourth formulae of Eq.(19) we find

$$g_3(\rho)=-\frac{\rho\ g_2'(\rho)+m\ g_2(\rho)}{\alpha_1\ \rho}, \tag{20}$$

$$g_4(\rho)=-\frac{\rho\ g_1'(\rho)-m\ g_1(\rho)}{\alpha_1\ \rho}. \tag{21}$$

Substituting Eqs.(20) and(21) into the first two formulae of Eq.(19), we find $g_1(\rho)$ and $g_2(\rho)$ satisfy the same equation, which is the same as Eq.(6)

$$g''(\rho)+\frac{1}{\rho}\,g'\,(\rho) - \left[\frac{m^2}{\rho^2}-\alpha_1\left(\alpha_2-\frac{1}{2}k\rho^2\right)\right]g(\rho)=0, \qquad (22)$$

where $g(\rho)=g_1(\rho)$ or $g_2(\rho)$. As in Section 2, letting

$$x=\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2, \quad g(\rho)=\mathrm{e}^{-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}\rho^{|m|}f(x), \qquad (23)$$

we obtain the same equation as Eq.(8):

$$xf''(x)+(1+|m|-x)f'(x) - \left(\frac{1+|m|}{2}-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}k}\alpha_2\right)f(x)=0. \qquad (24)$$

The solution of Eq.(24), of course, is the confluent hypergeometric function

$$f(x)=F\left[\frac{1+|m|}{2}-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}k}\alpha_2, 1+|m|, x\right]. \qquad (25)$$

For bound states, the confluent hypergeometric function $F[\alpha,\gamma,z]$ must terminate as a polynomial, which means

$$\frac{1+|m|}{2}-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}k}\alpha_2=-n_\rho, \quad n_\rho=0,1,2,\cdots \qquad (26)$$

Now, we can use n_ρ to express $g_1(\rho)$ and $g_2(\rho)$, and substitute $g_1(\rho)$ and $g_2(\rho)$ into Eqs.(20) and (21) to obtain $g_3(\rho)$ and $g_4(\rho)$:

$$g_1(\rho)=g_2(\rho)=C_{n_\rho,m}\mathrm{e}^{-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}\rho^{|m|}F \times\left[-n_\rho,1+|m|,\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2\right], \qquad (27)$$

$$g_3(\rho)=\frac{C_{n_\rho,m}}{\alpha_1(|m|+1)}\mathrm{e}^{-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}\rho^{|m|-1} \times\left\{n_\rho\sqrt{2k\alpha_1}\rho^2F\left[1-n_\rho,2+|m|,\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2\right] -\left[m^2+|m|+m(1+|m|) -\frac{|m|+1}{\sqrt{2}}\sqrt{k\alpha_1}\rho^2\right] \times F\left[-n_\rho,1+|m|,\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2\right]\right\}, \qquad (28)$$

$$g_4(\rho)=\frac{C_{n_\rho,m}}{\alpha_1(|m|+1)}\mathrm{e}^{-\frac{\sqrt{k\alpha_1}}{2\sqrt{2}}\rho^2}\rho^{|m|-1} \times\left\{n_\rho\sqrt{2k\alpha_1}\rho^2F\left[1-n_\rho,2+|m|,\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2\right] -\left[m^2+|m|-m(1+|m|) -\frac{|m|+1}{\sqrt{2}}\sqrt{k\alpha_1}\rho^2\right] \times F\left[-n_\rho,1+|m|,\frac{\sqrt{k\alpha_1}}{\sqrt{2}}\rho^2\right]\right\}. \qquad (29)$$

According to the normalization condition

$$\int_0^\infty\int_0^{2\pi}\rho\psi^\dagger\,\psi\,\mathrm{d}\phi\,\mathrm{d}\rho =\int_0^\infty\rho\Big(|g_1(\rho)|^2+|g_2(\rho)|^2 +|g_3(\rho)|^2+|g_4(\rho)|^2\Big)\mathrm{d}\rho=1, \qquad (30)$$

and the formula of the integral including two general Laguerre polynomials, exponential function and powers [5]

$$\int_0^\infty z^\lambda\mathrm{e}^{-z}L_n^\mu(z)L_{n'}^{\mu'}(z)\mathrm{d}z=(-)^{n+n'}\Gamma(\lambda+1)\sum_k\begin{pmatrix}\lambda-\mu\\ n-k\end{pmatrix}\begin{pmatrix}\lambda-\mu'\\ n'-k\end{pmatrix}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}, \qquad (31)$$

then by doing very tedious calculations, finally we find that the normalization constant is

$$c_{n_\rho,m}=\frac{\alpha_1(k\alpha_1)^{\frac{|m|+1}{4}}\sqrt{\Gamma(1+|m|+n_\rho)}}{2^{\frac{|m|}{4}}\Gamma(1+|m|)\sqrt{\Gamma(1+n_\rho)\left[\sqrt{2}\alpha_1^2+(1+|m|+2n_\rho)\sqrt{k\alpha_1}\right]}}. \qquad (32)$$

4. Energy equation

It is interesting that Eqs.(11) and (26) are the same, so they give the same energy equation

$$(1+|m|+2n_\rho)\sqrt{2k}-\sqrt{E+\mu}(E-\mu)=0. \qquad (33)$$

This equation has three solutions, the only real solution of which gives the energy

$$E=\frac{1}{3}(\mu+4\mu^2A^{-1/3}+A^{1/3}), \qquad (34)$$

where

$$A=27kn^2-8\mu^3+3n\sqrt{3k(27kn^2-16\mu^3)}, \quad n=1+|m|+2n_\rho=1,2,\cdots \qquad (35)$$

The energy equation can also be written as

$$\frac{1+|m|}{2}-\frac{\sqrt{\alpha_2+2\mu}}{2\sqrt{2k}}\alpha_2+n_\rho=0. \tag{36}$$

If we expand the left-hand side of the above equation as a series of α_2

$$(1+|m|+2n_\rho)-\sqrt{\frac{\mu}{k}}\ \alpha_2 - \frac{1}{4\sqrt{k\mu}}\ \alpha_2^2+O(\alpha_2)^3=0, \tag{37}$$

and only take the terms of α_2, the powers of which are not higher than one, then solve the energy equation, we have

$$\begin{aligned} E' =& E-\mu=\alpha_2=(1+|m|+2n_\rho)\sqrt{\frac{k}{\mu}} \\ =&(1+|m|+2n_\rho)\omega, \end{aligned} \tag{38}$$

where E' is just the non-relativistic energy ($\hbar=1$).[6]

5. The relationship with 3D harmonic oscillator

It is well known that in non-relativistic quantum mechanics, there is a corresponding relationship between 2D and 3D harmonic oscillators.[6] On the other hand, the energy equation (34) is very similar to the energy equation of 3D harmonic oscillator in form.[2] These suggest us to suspect whether there is any corresponding relationship between 2D and 3D harmonic oscillators in the relativistic quantum mechanics. In fact, the radial equation of 3D harmonic oscillators in Klein–Gordon theory is [2]

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR(r)) - \left[\frac{l(l+1)}{r^2}-\alpha_1(\alpha_2-\frac{1}{2}kr^2)\right]R(r)=0, \tag{39}$$

where α_1 and α_2 are the same as in Eq.(4). If we let

$$R(r)=\frac{1}{r}f(r), \tag{40}$$

then we obtain the equation of $f(r)$

$$f''(r)-\left[\frac{l(l+1)}{r^2}-\alpha_1(\alpha_2-\frac{1}{2}kr^2)\right]f(r)=0. \tag{41}$$

From Ref.[2] we also know that two radial wavefunctions of 3D harmonic oscillators in Dirac theory are $\frac{F(r)}{r}$ and $\frac{G(r)}{r}$, and $F(r)$ satisfies the equation

$$F''(r)-\frac{K(K-1)}{r^2}F(r)+\alpha_1\left(\alpha_2-\frac{1}{2}kr^2\right)F(r)=0, \tag{42}$$

where $K=\pm\left(j+\frac{1}{2}\right)$ is a conservative quantity of 3D central field in Dirac theory. On the other hand, if we let

$$g(\rho)=\frac{1}{\sqrt{\rho}}f(\rho), \tag{43}$$

Eqs.(6) and (22) become

$$f''(\rho)-\left[\frac{\left(|m|-\frac{1}{2}\right)\left(|m|+\frac{1}{2}\right)}{\rho^2} - \alpha_1\left(\alpha_2-\frac{1}{2}k\rho^2\right)\right]f(\rho)=0. \tag{44}$$

Comparing Eq.(44) with Eqs.(41) and (42), we find that there are the corresponding relationship between 2D and 3D harmonic oscillators, $\rho\leftrightarrow r$, $|m|-\frac{1}{2}\leftrightarrow l$ in Klein–Gordon theory, and $\rho\leftrightarrow r$, $|m|+\frac{1}{2}\leftrightarrow K$ in Dirac theory.

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