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Bound states of the Klein–Gordon and Dirac equation for scalar and vector pseudoharmonic oscillator potentials*

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The exact bound state solutions of the Klein–Gordon equation and Dirac equation with scalar and vector pseudoharmonic oscillator potentials are obtained in this paper. Furthermore, we have used the supersymmetric quantum mechanics, shape invariance and alternative method to obtain the required results.

Keywords: pseudoharmonic oscillator potential, Klein–Gordon equation, Dirac equation, bound state,
supersymmetric quantum mechanics, shape invariance

PACC: 0300, 0365

1. Introduction

A moving particle in a potential should be described with the Klein–Gordon equation and Dirac equation because of the relativistic effects.[1] Recently, many authors obtained the s-wave bound state solutions of the Klein–Gordon equation and the Dirac equation in some typical potential fields.[2−14] Not only in classical physics, but also in quantum physics, the pseudoharmonic oscillator is an important model. In non-relativistic quantum mechanics, many authors have adequately studied the problem of the pseudoharmonic oscillator.[15−20] But, in relativistic quantum mechanics, it has not yet been investigated. In this paper, we assume that scalar potential and vector potential are equal to each other, then obtain the exact bound state solutions of the Klein–Gordon equation and Dirac equation with scalar and vector pseudoharmonic oscillator potentials. It should be noticed that a new method, which is the supersymmetric quantum mechanics,[21−23] and shape invariance[24] and the alternative method,[25] should be employed firstly to solve the bound state of the Klein–Gordon and Dirac equations with the scalar and vector potentials equal to each other.

2. Bound state solutions of the Klein–Gordon equation with scalar and vector potentials equal to each other

According to Ref.[26], the Klein–Gordon equation for a pseudoharmonic oscillator is ($\hbar = c = 1$)

$$p^2 - \left[E - \left(\frac{1}{4}kr^2 + \frac{A}{4r^2}\right)\right]^2 = -\left[m - \left(\frac{1}{4}kr^2 + \frac{A}{4r^2}\right)\right]^2. \quad (1)$$

From Eq.(1) we have the radial part of the Klein–Gordon equation of a three-dimensional relativistic pseudoharmonic oscillator:

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR(r)) - \left[\frac{l(l+1)}{r^2} + (m^2 - E^2) + \frac{1}{2}(m+E)\left(kr^2 + \frac{A}{r^2}\right)\right]R(r) = 0. \quad (2)$$

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Using the notation

$$l'(l'+1) = l(l+1) + \frac{1}{2}(m+E)A, \tag{3}$$

$$\alpha = \sqrt{\frac{1}{2}(m+E)k}, \tag{4}$$

$$\lambda = E^2 - m^2, \tag{5}$$

$$R(r) = r^{-1}u(r), \tag{6}$$

we can obtain

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r) + \left[\alpha^2 r^2 + \frac{l'(l'+1)}{r^2}\right]u(r) = \lambda u(r), \tag{7}$$

whose ground state wavefunction and ground state energy are, respectively,

$$u_0(r) = r^{l'+1}\exp\left(-\frac{\sqrt{\alpha}}{2}r^2\right), \quad \lambda_0 = (2l'+3)\sqrt{\alpha}. \tag{8}$$

The corresponding superpotential is

$$W(r) = \sqrt{\alpha}\, r - \frac{l'+1}{r^2}. \tag{9}$$

From Eq.(9) we can obtain the partner potentials

$$V_+(r,l') = \alpha r^2 + \frac{(l'+1)(l'+2)}{r^2} - 2\left(l'+\frac{1}{2}\right)\sqrt{\alpha}, \tag{10}$$

$$V_-(r,l') = \alpha r^2 + \frac{l'(l'+1)}{r^2} - 2\left(l'+\frac{3}{2}\right)\sqrt{\alpha}. \tag{11}$$

We find that

$$V_+(r,l') = V_-(r,l'+1) + R(l'+1), \quad R(l'+1) = 4\sqrt{\alpha}, \tag{12}$$

hence the shape invariance condition is satisfied in r dimension, and the complete eigenvalue spectrum is given by

$$\lambda_n^- = \sum_{k=1}^{n} R(l'+k) = 4n\sqrt{\alpha}, \quad n = 1,2,3,\cdots \tag{13}$$

Thus we can have the value of λ_n in Eq.(7)

$$\lambda_n = 2(2n+l'+1)\sqrt{\alpha}. \tag{14}$$

Substituting Eq.(14) into Eq.(7), we can obtain

$$\begin{aligned} &-\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r) + \left[\alpha^2 r^2 + \frac{l'(l'+1)}{r^2}\right]u(r) \\ =&2(2n+l'+1)\sqrt{\alpha}u(r). \end{aligned} \tag{15}$$

In principle, we can use supersymmetric quantum mechanics and shape invariance to find the eigenfunction of Eq.(35), however here we will use an alternative method. Introducing $y = \sqrt{\alpha}r^2$ and letting

$$u(y) = y^{(l'+1)/2}\exp\left(-\frac{y}{2}\right)L_n(y), \tag{16}$$

we can gain

$$y\frac{\mathrm{d}^2}{\mathrm{d}y^2}L_n(y) + [l'+1-y]\frac{\mathrm{d}}{\mathrm{d}y}L_n(y) + nL_n(y) = 0. \tag{17}$$

This is the well-known differential equation satisfied by Laguerre polynomials $L_n^{l'}(y)$, hence we can obtain

$$u(y) = y^{(l'+1)/2}\exp\left(-\frac{y}{2}\right)L_n^{l'}(y). \tag{18}$$

From Eqs.(3)–(5) and (14) we can consequently obtain the energy equation

$$\begin{aligned} &\sqrt{k + 4k[l(l+1) + \frac{1}{2}(m+E)A]} \\ &-(E-m)\sqrt{2(E+m)} = -(4n+2)\sqrt{k}, \end{aligned} \tag{19}$$

and the wavefunction corresponding to the energy levels is (unnormalized)

$$R(r) = r^{l'}\exp\left(-\frac{\sqrt{\alpha}}{2}r^2\right)L_n^{l'}(\sqrt{\alpha}r^2). \tag{20}$$

3. Bound state solutions of the Dirac equation with scalar and vector potentials equal to each other

The Dirac equation with scalar potential $S(r)$ and vector potential $V(r)$is ($\hbar = c = 1$)

$$C\cdot\boldsymbol{P} + D[m + S(r)]\}\psi = [E - V(r)]\psi. \tag{21}$$

In relativistic quantum mechanics, the complete conservative quantity set of a particle in a central field can be taken to be $(H, \boldsymbol{K}, \boldsymbol{J}^2, \boldsymbol{J}_z)$, the eigenfunctions of which are [27]

$$\psi = \frac{1}{r}\begin{pmatrix} f_{n,K}(r)\phi^A_{j\,m_j} \\ \mathrm{i}g_{n,K}(r)\phi^B_{j\,m_j} \end{pmatrix}, \quad \left(K = j + \frac{1}{2}\right), \tag{22}$$

$$\psi = \frac{1}{r}\begin{pmatrix} f_{n,K}(r)\phi^B_{j\,m_j} \\ \mathrm{i}g_{n,K}(r)\phi^A_{j\,m_j} \end{pmatrix}, \quad \left(K = -\left(j + \frac{1}{2}\right)\right), \tag{23}$$

where

$$\phi^A_{jm_j}=\begin{pmatrix}\sqrt{\dfrac{j+m_j}{2j}}Y_{j-\frac{1}{2},m_j-\frac{1}{2}}\\ \sqrt{\dfrac{j-m_j}{2j}}Y_{j-\frac{1}{2},m_j+\frac{1}{2}}\end{pmatrix},\quad \phi^B_{jm_j}=\begin{pmatrix}-\sqrt{\dfrac{j-m_j+1}{2j+2}}Y_{j+\frac{1}{2},m_j-\frac{1}{2}}\\ \sqrt{\dfrac{j+m_j+1}{2j+2}}Y_{j+\frac{1}{2},m_j+\frac{1}{2}}\end{pmatrix}. \tag{24}$$

Substituting Eq.(22) or (23) into Eq.(21) gives the radial part of the Dirac equation

$$\frac{\mathrm{d}f}{\mathrm{d}r}-\frac{K}{r}f=[m+E+S(r)-V(r)]g, \tag{25}$$

$$\frac{\mathrm{d}g}{\mathrm{d}r}+\frac{K}{r}g=[m-E+S(r)+V(r)]f. \tag{26}$$

Taking $S(r)=V(r)=\left(\dfrac{kr^2}{4}+\dfrac{A}{4r^2}\right)$, Eqs.(25) and (26) become

$$\frac{\mathrm{d}f}{\mathrm{d}r}-\frac{K}{r}f=[m+E]g, \tag{27}$$

$$\frac{\mathrm{d}g}{\mathrm{d}r}+\frac{K}{r}g=\left[m-E+\left(\frac{kr^2}{2}+\frac{A}{2r^2}\right)\right]f. \tag{28}$$

Eliminating g from Eqs.(27) and (28) gives

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}f-\frac{K(K-1)}{r^2}f+(E+m)\times\left[(E-m)-\left(\frac{kr^2}{2}+\frac{A}{2r^2}\right)\right]f=0. \tag{29}$$

Using the same procedure for solving Eq.(2), if we put

$$K'(K'-1)=K(K-1)+\frac{1}{2}(E+m)A, \tag{30}$$

we can obtain the equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r)+\left[\alpha^2r^2+\frac{K'(K'-1)}{r^2}\right]f(r)=\lambda f(r), \tag{31}$$

whose ground state wavefunction and ground state energy are

$$f_0(r)=r^{K'}\exp\left(-\frac{\sqrt{\alpha}}{2}r^2\right),\quad \lambda_0=(2K'+1)\sqrt{\alpha}. \tag{32}$$

The corresponding superpotential is

$$W(r)=\sqrt{\alpha}\,r-\frac{K'}{r^2}. \tag{33}$$

From Eq.(33) we can obtain the partner potential

$$V_+(r,K')=\alpha r^2+\frac{K'(K'+1)}{r^2}-2\left(K'-\frac{1}{2}\right)\sqrt{\alpha}, \tag{34}$$

$$V_-(r,K')=\alpha r^2+\frac{K'(K'-1)}{r^2}-2\left(K'+\frac{1}{2}\right)\sqrt{\alpha}. \tag{35}$$

We find that

$$V_+(r,K')=V_-(r,K'+1)+R(K'+1),\quad R(K'+1)=4\sqrt{\alpha}, \tag{36}$$

hence the shape invariance condition is satisfied in r dimension, and the complete eigenvalue spectrum is given by

$$\lambda^-_n=\sum_{k=1}^{n}R(K'+k)=4n\sqrt{\alpha},\quad n=1,2,3,\cdots. \tag{37}$$

Also we can obtain the value of λ_n in Eq.(31)

$$\lambda_n=2\left(2n+K'+\frac{1}{2}\right)\sqrt{\alpha}. \tag{38}$$

Substituting Eq.(14) into Eq.(7), we have

$$-\frac{\mathrm{d}^2}{\mathrm{d}r^2}f(r)+\left[\alpha^2r^2+\frac{K'(K'-1)}{r^2}\right]f(r)=2(2n+K')\sqrt{\alpha}f(r). \tag{39}$$

In principle, we can use supersymmetric quantum mechanics and shape invariance to find eigenfunction of Eq.(39), here we will use an alternative method. Introducing $y=\sqrt{\alpha}r^2$ and letting

$$f(y)=y^{K'/2}\exp\left(-\frac{y}{2}\right)L_n(y), \tag{40}$$

we obtain

$$y\frac{\mathrm{d}^2}{\mathrm{d}y^2}L_n(y)+[K'-y]\frac{\mathrm{d}}{\mathrm{d}y}L_n(y)+nL_n(y)=0. \tag{41}$$

This is the well-known differential equation satisfied by Laguerre polynomials $L_n^{K'-1}(y)$, hence we can obtain

$$f(y)=y^{K'/2}\exp\left(-\frac{y}{2}\right)L_n^{K'-1}(y). \tag{42}$$

From Eqs.(4), (5), (30) and (38) we can consequently obtain the energy equation

$$\sqrt{k+4k[K(K-1)+\frac{1}{2}(m+E)A]}-(E-m)\sqrt{2(E+m)}=-(4n+2)\sqrt{k}, \tag{43}$$

the wavefunction corresponding to the energy levels is (unnormalized)

$$f(r)=r^{K'}\exp\left(-\frac{\sqrt{\alpha}}{2}r^2\right)L_n^{K'-1}(\sqrt{\alpha}r^2). \tag{44}$$

From Eq.(21) we can obtain the wavefunctions g corresponding to f and the energy levels (unnormalized)

$$g(r) = \frac{1}{(E+m)}\left\{\left[\frac{(K'-K)}{r} - \sqrt{\alpha}r^2\right]r^{K'} \times \exp\left(-\frac{\sqrt{\alpha}}{2}r^2\right)L_n^{'K-1}(\sqrt{\alpha}r^2) - 2\sqrt{\alpha}r^{K'+1}\exp\left(-\frac{\sqrt{\alpha}}{2}r^2\right)L_{n-1}^{K'}(\sqrt{\alpha}r^2)\right\}. \quad (45)$$

Substituting Eqs.(44) and (45) into Eq.(22) or (23) gives the spinor wavefunction of the Dirac equation.

4. Conclusion

The above results show that the Klein–Gordon and Dirac equations with equal scalar and vector potentials, each of which is equal to half of the pseudoharmonic oscillator potential, can be exactly solved. Furthermore, supersymmetric quantum mechanics and shape invariance allows one to determine eigenvalues and eigenstates using algebraic operator formalism without having to solve the Klein–Gordon and Dirac radial differential equations by standard series method. With the above considerations the authors hope to stimulate further examples of applications of supersymmetric quantum mechanics and shape invariance in important problems of physics.

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