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Bosonic and fermionic entropy of black holes with different temperatures on horizon surface *

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By using the method of quantum statistics, we derive directly the partition functions of bosonic and fermionic field in the black hole space–time with different temperatures on horizon surface. The statistical entropy of the black hole is obtained by an improved brick-wall method. When we choose a proper parameter in our results, we can obtain that the entropy of the black hole is proportional to the area of horizon. In our result, there do not exist any neglected term or divergent logarithmic term as given in the original brick-wall method. We have avoided the difficulty in solving the wave equation of the scalar and Dirac field. A simple and direct way of studying entropy of the black hole is given.

Keywords: quantum statistics, membrane model, statistical entropy, brick-wall method
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1. Introduction

Since Bekenstein [1] and Hawking and Bardeen [2,3] proved that the entropy of a black hole is proportional to the area of the horizon, the statistical origin of the black hole has remained a key problem to be solved. Many methods for obtaining the black hole's entropy have been given.[4−9] Among these methods, the most frequently used is the brick-wall method advanced by Hooft.[7] This method is used to study the statistical property of scalar and Dirac field in various black holes.[10−14] It is found that the generic expression of the black hole's entropy consists of the term which is proportional to the area of its horizon plus the term which is not proportional to the area of its horizon and logarithmic divergent. However, the following statements remain open questions: (1) the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole; (2) the state density near the event horizon is divergent; (3) the logarithmic term is left out and L^3 is considered as the contribution of distant vacuum surrounding the system in order to make the entropy of black hole proportional to its horizon; (4) the wavefunction of scalar or Dirac field is solved approximately. The above-mentioned problems in the original brick-wall method are unnatural and insurmountable.

It is well known that the entropy of a black hole is proportional to the area of the horizon and the existence of the horizon is a basic property of the black hole. It has been proved that the existence of the horizon generally results in the Hawking effect,[15] and whether there is entropy of the black hole or not is related to the existence of the horizon.[16] Then it can be said that it is a natural disposition that the entropy of the black hole is proportional to the area of the horizon. Its value has nothing to do with the radiation field outside the horizon, and the horizon only has the property of a two-dimensional membrane in the three-dimensional space. Does the number of quantum states of the two-dimensional membrane correspond to the entropy of the black hole? If it does, how to then calculate the entropy of the membrane will be the key problem.

We first derive the bosonic and fermionic partition functions in a black hole with different temperatures on the horizon surface directly by using the quantum statistical method [17] and then obtain the integral expression of the entropy of the system. Then we use the membrane model to calculate the entropy.[17−22] As a result, the left out term and the divergent logarithmic term in the original brick-wall method will not appear any more. The doubt for the statement that

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the entropy of the scalar or Dirac field outside the event horizon is the entropy of the black hole and the problem that the state density near the event horizon is divergent also do not exist any more. We then take into consideration the influence of the spinning degeneracy of particles on the entropy. In the whole process, we avoid the difficulty in solving the wave equation, so that the physical meaning is clear, calculation is simple and the result is reasonable. This offers a neat way of studying the entropy of various complicated black holes. In this article, we use the simplest functional form of the temperature ($C = \eta = G = k_{\rm B} = 1$).

2. A black hole with different temperatures on the horizon surface

The linear element in Kim space–time is given in Ref.[23],

$$\begin{aligned} {\rm d}s^2 =& \Delta^{-2/(2\omega+3)} \sin^{-4/(2\omega+3)} \theta \\ & \times \left[-\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma} \right) {\rm d}t^2 \right. \\ & - \frac{2a \sin^2\theta (r^2 + a^2 - \Delta)}{\Sigma} {\rm d}t{\rm d}\varphi \\ & \left. + \left(\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \right) \sin^2\theta {\rm d}\varphi^2 \right] \\ & + \Delta^{2/(2\omega+3)} \sin^{4/(2\omega+3)} \theta \left[\frac{\Sigma}{\Delta} {\rm d}r^2 + \Sigma {\rm d}\theta^2 \right], \end{aligned} \quad (1)$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta,$$
$$\Delta = r^2 - 2Mr + a^2 + e^2 = (r - r_+)(r - r_-),$$

$r_\pm = M \pm (M - a^2 - e^2)^{1/2}$ are the locations of outer and inner horizons respectively. The Hawking radiation temperature of the black hole is as follows:

$$T_+ = \frac{\kappa_+}{2\pi}. \quad (2)$$

$$\begin{aligned} \kappa_\pm =& \Delta^{-2/(2\omega+3)}(r_\pm) \sin^{-4/(2\omega+3)} \theta \\ & \times \left(\frac{2\omega+1}{2\omega+3} \right) \frac{(r_\pm - r_\mu)}{2(r_\pm^2 + a^2)}. \end{aligned} \quad (3)$$

From Eq.(3) the radiation temperature of the black hole is related to the polar angle θ. On the surface of the horizon of the black hole, the temperature is not constant. The areas of the outer and inner horizons are as follows:

$$A_\pm = 4\pi(r_\pm^2 + a^2). \quad (4)$$

The natural radiation temperature [10,24] felt by the observer at rest at an infinite distance is

$$T = \frac{T_+}{\sqrt{-g'_{tt}}}, \quad (5)$$

where T_+ is the equilibrium temperature and

$$\begin{aligned} g'_{tt} =& \frac{g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} \\ =& -\Delta^{-2/(2\omega+3)} \sin^{-4/(2\omega+3)} \theta \\ & \times \frac{(r - r_+)(r - r_-)(r^2 + a^2\cos^2\theta)}{(r^2+a^2)^2 - (r - r_+)(r - r_-)a^2 \sin^2\theta}. \end{aligned} \quad (6)$$

3. Bosonic entropy

For a bosonic gas, the partition function is as follows:

$$\ln Z = -\sum_i g_i \ln(1 - e^{-\beta\varepsilon_i}). \quad (7)$$

In unit volume, the number of quantum states with the energy between ε and $\varepsilon + {\rm d}\varepsilon$ or the frequency between ν and $\nu + {\rm d}\nu$ is given by

$$g(\nu){\rm d}\nu = j 4\pi \nu^2 {\rm d}\nu, \quad (8)$$

where j is the spin degeneracy of particles. In the space–time (1), the area of a two-dimensional curved surface at a random point r is

$$A(r) = \int {\rm d}A = \int \sqrt{g} {\rm d}\theta {\rm d}\varphi, \quad (9)$$

where $g = \begin{vmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{vmatrix} = g_{\theta\theta} g_{\varphi\varphi}$. The volume of the lamella at a random point r outside the horizon is

$${\rm d}V = A(r)\sqrt{g_{rr}} {\rm d}r. \quad (10)$$

So, the partition function of the system at the lamella with a random thickness at point r outside the horizon is written as

$$\begin{aligned} \ln Z =& \int A(r)\sqrt{g_{rr}} {\rm d}r \sum_i g_i \sum_{n=1} \frac{1}{n} e^{-n\beta\varepsilon_i} \\ =& j 4\pi \int A(r)\sqrt{g_{rr}} {\rm d}r \sum_{n=1} \frac{1}{n} \int_0^\infty e^{-\frac{nh\nu}{T}} \nu^2 {\rm d}\nu \\ =& j \frac{1}{90} \pi^2 \int \frac{A(r)\sqrt{g_{rr}} {\rm d}r}{\beta^3} \\ =& j \frac{\pi^2}{90} \int \frac{\sqrt{g_{\theta\theta} g_{\varphi\varphi} g_{rr}} {\rm d}r {\rm d}\theta {\rm d}\varphi}{\beta^3}, \end{aligned} \quad (11)$$

where $\frac{1}{\beta}=T$. Using the relation between entropy and partition function

$$S=\ln Z-\beta_0\frac{\partial \ln Z}{\partial\beta_0}, \tag{12}$$

we have

$$\begin{aligned}S_{\rm b}=&j\frac{2}{45}\pi^2\int\frac{1}{\beta_0^3}\frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}{\rm d}r}{(-g'_{tt})^{3/2}}{\rm d}\theta{\rm d}\varphi\\ =&j\frac{2\pi^2}{45}\int\frac{\sin^{8/(2\omega+3)}\theta}{\beta_0^3}{\rm d}\theta{\rm d}\varphi\int\Delta^{4/(2\omega+3)}\frac{[(r^2+a^2)^2-(r-r_+)(r-r_-)a^2\sin^2\theta]^2}{(r-r_+)^2(r-r_-)^2(r^2+a^2\cos^2\theta)}\sin\theta{\rm d}r\\ =&j\frac{2\pi^2}{45A}\int\sin^{\frac{2\omega-1}{2\omega+3}}\theta{\rm d}\theta{\rm d}\varphi\int\Delta^{4/(2\omega+3)}\frac{[(r^2+a^2)^2-(r-r_+)(r-r_-)a^2\sin^2\theta]^2}{(r-r_+)^2(r-r_-)^2(r^2+a^2\cos^2\theta)}{\rm d}r,\end{aligned} \tag{13}$$

where

$$A=8\pi^3\Delta^{6/(2\omega+3)}(r_+)\left(\frac{2\omega+3}{2\omega+1}\right)^3\frac{8(r_+^2+a^2)^3}{(r_+-r_-)^3}, \tag{14}$$

with $\beta=\beta_0\sqrt{-g'_{tt}}$. In Eq.(14), using the membrane model,[17−22] we take the integral region $[r_++\zeta,r_++N\zeta]$, where r_+ is the location of the horizon, ζ is a small nonnegative quantity, and N is a constant larger than one. So Eq. (14) can be written as

$$\begin{aligned}S_{\rm b}=&j\frac{2\pi^2}{45A}\int\sin^{\frac{2\omega-1}{2\omega+3}}\theta{\rm d}\theta{\rm d}\varphi\int_{r_++\zeta}^{r_++N\zeta}\Delta^{4/(2\omega+3)}\frac{[(r^2+a^2)^2-(r-r_+)(r-r_-)a^2\sin^2\theta]^2}{(r-r_+)^2(r-r_-)^2(r^2+a^2\cos^2\theta)}{\rm d}r\\ =&j\frac{4\pi^3}{45A}\int_0^\pi\frac{(r_+^2+a^2)^4\Delta^{4/(2\omega+3)}}{(r_+-r_-)^2(r_+^2+a^2\cos^2\theta)}\left[\frac{N-1}{N\zeta}\right]\sin^{\frac{2\omega-1}{2\omega+3}}\theta{\rm d}\theta+G(r_+,N,\zeta).\end{aligned} \tag{15}$$

The relation between ζ and N is

$$\zeta=\frac{T_+}{90}\frac{N-1}{N}\frac{\alpha}{\sin\alpha\cos\alpha}\left(\frac{2\omega+1}{2\omega+3}\right)^2. \tag{16}$$

Then

$$\begin{aligned}S_{\rm b}=&j\frac{\sin\alpha\cos\alpha}{\alpha}\int_0^\pi\frac{\pi(r_+^2+a^2)^2\sin\theta}{2(r_+^2+a^2\cos^2\theta)}{\rm d}\theta+G(r_+,N,\zeta)\\ =&j(r_+^2+a^2)+G(r_+,N,\zeta)=j\frac{A_+}{4}+G(r_+,N,\zeta),\end{aligned} \tag{17}$$

where

$$\begin{aligned}G(r_+,N,\zeta)=&j\frac{4\pi^3}{45A}\int_0^\pi\left[\frac{8r_+(r_+^2+a^2)^3}{(r_+-r_-)^2(r_+^2+a^2\cos^2\theta)}-\frac{2r_+(r_+^2+a^2)^4}{(r_+-r_-)^2(r_+^2+a^2\cos^2\theta)^2}\right.\\ &\left.-\frac{2(r_+^2+a^2)^4}{(r_+-r_-)^3(r_+^2+a^2\cos^2\theta)}-\frac{2(r_+^2+a^2)^2a^2\sin^2\theta}{(r_+-r_-)(r_+^2+a^2\cos^2\theta)}\right]\sin^{\frac{2\omega-1}{2\omega+3}}\theta{\rm d}\theta\ln N\\ &+j\frac{4\pi^3}{45A}\int_0^\pi\sin^{\frac{2\omega-1}{2\omega+3}}\theta{\rm d}\theta\int_{r_++\zeta}^{r_++N\zeta}F(r){\rm d}r,\end{aligned} \tag{18}$$

$$F(r)=\sum_{n=2}^\infty\frac{f_1^{(n)}(r_+)}{n!}(r-r_+)^{n-2}-\sum_{n=1}^\infty\frac{f_2^{(n)}(r_+)}{n!}(r-r_+)^{n-1}+f_3(r). \tag{19}$$

Here

$$f_1(r)=\frac{(r^2+a^2)^4}{(r-r_-)^2(r^2+a^2\cos^2\theta)},$$
$$f_2(r)=\frac{2(r^2+a^2)^2a^2\sin^2\theta}{(r-r_-)(r^2+a^2\cos^2\theta)},$$
$$f_3(r)=\frac{a^4\sin^4\theta}{r^2+a^2\cos^2\theta},$$
$$f_1'(r)=\frac{\mathrm{d}f_1}{\mathrm{d}r},$$
$$f_2'(r)=\frac{\mathrm{d}f_2}{\mathrm{d}r},\quad \alpha=\arctan\frac{a}{r_+}.$$

As $N\to 1$, $\zeta\to 0$ and $N\zeta\to 0$, that is, the ultraviolet cutoff and infrared cutoff both approach the outer horizon of the black hole, and $F(r)$ is an analytic function at $r=r_+$, we have $\lim\limits_{N\to 1}G(r_+,N,\zeta)\to 0$. The entropy of the black hole is

$$S_{\mathrm{b}}=j\frac{1}{4}A_+, \tag{20}$$

where $A_+=4\pi(r_+^2+a^2)$ is the area of the horizon. Since we let the integral upper limit and lower limit tend to the outer horizon, the entropy obtained in Eq.(20) is independent of the radiation field outside the horizon. It only has the property of a one-dimensional membrane in the three-dimensional space. So the obtained entropy has the property of a one-dimensional membrane. The existence of an horizon is the basic property of a black hole. It has already been proved that the general existence of horizon leads to the Hawking effect,[15] and whether there is the black hole's entropy or not directly involves the existence of horizon.[16] So the entropy in Eq.(20) should be of the black hole. When $j=1$ for radiation particles, we obtain that the entropy of the black hole is a quarter of the area of the horizon. When $j\neq 1$, j can be taken into account by the relation between ζ and N given in Eq.(16) to make sure that the entropy is a quarter of the area of the horizon.

4. Fermionic entropy

For a fermionic gas, the grand partition function reads as

$$\ln Z=\sum_i g_i\ln(1+e^{-\beta\varepsilon_i}). \tag{21}$$

From Eq.(8), we obtain

$$\begin{aligned}\ln Z=&\sum_i g_i\sum_{n=1}\frac{(-1)^{n-1}}{n}e^{-n\beta\varepsilon_i}\int A(r)\sqrt{g_{rr}}\mathrm{d}r\\ =&\overline{\omega}\frac{7}{8}\frac{1}{90}\pi^2\int\frac{A(r)\sqrt{g_{rr}}\mathrm{d}r}{\beta^3}\\ =&\overline{\omega}\frac{7}{8}\frac{\pi^2}{90}\int\frac{\sqrt{g_{\theta\theta}g_{\varphi\varphi}g_{rr}}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi}{\beta^3},\end{aligned} \tag{22}$$

where $\overline{\omega}$ is the spin degeneracy of fermionic particles. Using the result of Section 3, we can obtain the fermionic entropy as follows:

$$S_{\mathrm{f}}=\overline{\omega}\frac{7}{8}\frac{A_+}{4}. \tag{23}$$

5. Conclusion

In the above analysis, we have derived the partition functions for various fields in the black hole with different temperatures on the horizon surface directly by using the statistical method. We have avoided the difficulty in solving the wave equation. Since we use the improved brick-wall method, the membrane model, to calculate the entropy of various fields, the problem that the state density is divergent around the horizon does not exist any more. In our calculation, as $N\to 1$, we have $\zeta\to 0$ and $N\zeta\to 0$, that is, the inner and outer "brick walls" both approach the outer horizon of the black hole. From Eqs.(20) and (23), we know that the divergent logarithmic term and L^3 term in the original brick-wall method no longer exist. The obtained entropy is proportional to the area of its horizon, so it can be taken as the entropy of the black hole. From Eq.(16), we know that the relation between ζ and N is a function of temperature, and the temperature of the black hole we discuss is related to the polar angle θ. So the relation given by Eq.(16) is related to θ. When $\omega\to\infty$, the space–time will turn into the Kerr–Newman space–time, and the radiation temperature has nothing to do with angle θ. So the relation between ζ and N given by Eq.(16) also has nothing to do with temperature.

In the above analysis, we know that by using the statistical and membrane model methods, the doubt about the entropy of the scalar or Dirac field outside the event horizon being the entropy of the black hole in the original brick-wall method does not exist any more, and the complicated approximations in the solution are avoided. In the whole process, the physical meaning of the parameters is clear; the calculation is simple; and the result is reasonable. We have also considered the influence of the spin degeneracy of particles on the entropy. We offer a way to calculate the entropy of different black holes.

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