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High order generalized permutational fractional Fourier transforms *

Ran Qi-Wen(冉启文)[a)b)†], Yuan Lin(袁 琳)[a)], Tan Li-Ying(谭立英)[b)],
Ma Jing(马 晶)[b)], and Wang Qi(王 骐)[b)]

[a)] *Department of Mathematics, Harbin Institute of Technology, Harbin* 150001, *China*
[b)] *State Key Laboratory of Tunable Laser Technology, Research Institute of Optic-Electronics, Harbin Institute of Technology, Harbin* 150001, *China*

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We generalize the definition of the fractional Fourier transform (FRFT) by extending the new definition proposed by Shih. The generalized FRFT, called the high order generalized permutational fractional Fourier transform (HGPFRFT), is a generalized permutational transform. It is shown to have arbitrary natural number M periodic eigenvalues not only with respect to the order of Hermite–Gaussian functions but also to the order of the transform. This HGPFRFT will be reduced to the original FRFT proposed by Namias and Liu's generalized FRFT and Shih's FRFT at the three limits with $M = +\infty$, $M = 4k$ (k is a natural number), and $M = 4$, respectively. Therefore the HGPFRFT introduces a comprehensive approach to Shih's FRFT and the original definition. Some important properties of HGPFRFT are discussed. Lastly the results of computer simulations and symbolic representations of the transform are provided.

Keywords: Fourier transform, fractional Fourier transform, permutational fractional Fourier transform

PACC: 4200, 4230, 4230K

1. Introduction

The fractional Fourier transform (FRFT) emerged [1,2] as an extension of the Fourier transform, which, here, is referred to as the ordinary Fourier transform and is one of the most commonly used tools in signal processing. It has recently been demonstrated to be useful in quantum mechanics [3,4] and has also been introduced in bulk optics,[5–7] as well as in fibre optics,[8,9] as a fundamental tool for optical information processing. The FRFT has extended the conventional Fourier transform to a more general time-frequency (or space-frequency) joint representation [10] by allowing the concept of the fractional integral to be applied in the Fourier integrals. Along with their challenging applications, their mathematical properties still attract much attention. Recently, Shih [11] reinvented a technique to fractionalize the Fourier transform with only the weighted composition of four basic functions, i.e. the original function, its first, second and third Fourier transforms, according to the three postulates that the FRFTs should obey. The relationship of Shih's FRFT with the original definition has been discussed in Ref.[12]. In addition, some generalized fractionalization of the conventional Fourier transform can be found in Refs.[13–19].

In this paper, we first discuss the relationship of Shih's FRFT with the conventional Fourier transform. We are amazed to see that Shih's FRFT is the same permutational transform as the conventional Fourier transform, but with different permutation matrices which have an order of 4. We then extend Shih's fractionalization procedure by allowing the order of permutation matrix to be arbitrary natural number $M \geq 3$ and demonstrate that Shih's FRFT, Liu's generalized FRFT (GFRFT) and the original FRFT are three especial cases of high order generalized permutational fractional Fourier transforms (HGPFRFT) proposed in this paper which correspond to $M = 4$, $M = 4k$ (k is a natural number) and $M = \infty$, respectively.

This paper is arranged as follows. In Section 2, we briefly introduce Shih's definition of the FRFT for the reason that we will frequently refer to it in the following sections. We will also find the permutational

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†E-mail: ranlvwen@public.hr.hl.cn

characterization of Shih's FRFT in this section. In Section 3, we introduce our HGPFRFT by extending the conventional Fourier transform based on the complete orthonormal basis of Fourier transform eigenfunctions, i.e. the normalized Hermite–Gaussian functions. Computer simulations are given to compare several definitions with the original FRFT. Some properties of HGPFRFT are briefly discussed in Section 4; there is an application concerning the concept of self-fractional Fourier function.[20−22] Finally, we present our conclusions.

2. Permutational characterization of Shih's FRFT

Shih's definition of the FRFT [11] takes an alternative continuous approach to fractionalize the Fourier transform. He assumed that any transient state of the Fourier transform could be expressed by its first four integer-order states which are analogous to quantum states,[12,13] because the integer-order Fourier transform is a 4-periodic operation. Mathematically, the conventional Fourier transform F can be described by the integral

$$(Ff)(\omega) = \frac{1}{\sqrt{2\pi}} \int f(x)\exp(-\mathrm{i}\omega x)\mathrm{d}x,$$

where $(Ff)(\omega)$ is the Fourier transform of the function $f(x)$. For the integer-order Fourier transform, there are only four different functions,

$$\begin{aligned} f_1(x) &= (Ff)(x), \\ f_2(x) &= (F^2 f)(x) = f(-x), \\ f_3(x) &= (F^3 f)(x) = f_1(-x), \\ f_4(x) &= (F^4 f)(x) = f(x), \end{aligned}$$

if $f_0(x) = (F^0 f)(x) = f(x)$, then $f_4(x) = f_0(x)$ and $f_{4L+l}(x) = (F^{4L+l} f)(x) = f_l(x)$, where L is any integer and $l = 0, 1, 2, 3$. One can construct the α-order fractional Fourier transform as a weighted composition of these four functions as follows:

$$F^\alpha = \sum_{l=0}^{3} P_l(\alpha) F^l, \tag{1}$$

where $P_l(\alpha)$ is the coefficient, $l = 0, 1, 2, 3$. The so-defined fractional Fourier transform operators obey the following axioms:

(1) Continuity axiom: $P_0(\alpha)$, $P_1(\alpha)$, $P_2(\alpha)$, $P_3(\alpha)$ should all be continuous for all real number α.

(2) Boundary axiom: $P_l(m) = \begin{cases} 1, & m = l \\ 0, & m \neq l \end{cases}$ $(0 \leq m, l \leq 3)$.

(3) Periodicity axiom: $P_l(\alpha) = P_l(\alpha + 4)$, $l = 0, 1, 2, 3$, for all α.

(4) Additive axiom: $F^\alpha F^\beta = F^\beta F^\alpha = F^{\alpha+\beta}$ for any α and β.

It is easy to see that (1) means that F^α is a continuous operator; (2) means that F^α reduces to F^l when α is an integer l; in addition, because $F^{4L+l} = F^l$, thus $P_l(k) = P_l(k + 4)$, where $l = 0, 1, 2, 3$, L and k are any integers, i.e. when α is an integer, $P_l(\alpha)$ is periodic with period 4 for α; (3) means that this is also true for any α. By using these properties, one can obtain a set of coupled equations for the coefficients as follows:

$$P_l(\alpha + \beta) = \sum_{\substack{0 \leq m, n \leq 3 \\ m+n \equiv l \bmod(4)}} P_m(\alpha) P_n(\beta), \qquad l = 0, 1, 2, 3. \tag{2}$$

In detail, it can be written as follows:

$$\begin{aligned} P_0(\alpha+\beta) &= P_0(\alpha)P_0(\beta) + P_1(\alpha)P_3(\beta) + P_2(\alpha)P_2(\beta) + P_3(\alpha)P_1(\beta), \\ P_1(\alpha+\beta) &= P_0(\alpha)P_1(\beta) + P_1(\alpha)P_0(\beta) + P_2(\alpha)P_3(\beta) + P_3(\alpha)P_2(\beta), \\ P_2(\alpha+\beta) &= P_0(\alpha)P_2(\beta) + P_1(\alpha)P_1(\beta) + P_2(\alpha)P_0(\beta) + P_3(\alpha)P_3(\beta), \\ P_3(\alpha+\beta) &= P_0(\alpha)P_3(\beta) + P_1(\alpha)P_2(\beta) + P_2(\alpha)P_1(\beta) + P_3(\alpha)P_0(\beta). \end{aligned} \tag{3}$$

The coefficient $P_l(\alpha)$ is solved to be

$$P_l(\alpha) = \frac{1}{4} \frac{1 - \exp(-2\pi(\alpha - l)\mathrm{i})}{1 - \exp(-2\pi(\alpha - l)\mathrm{i}/4)}, \qquad l = 0, 1, 2, 3. \tag{4}$$

The geometrical locus of these coefficients has been illustrated in Fig.1. They are the same closed curve in the complex plane, but when α increases from 0 to 4, their jumping-off points are different from each other as shown therewith.

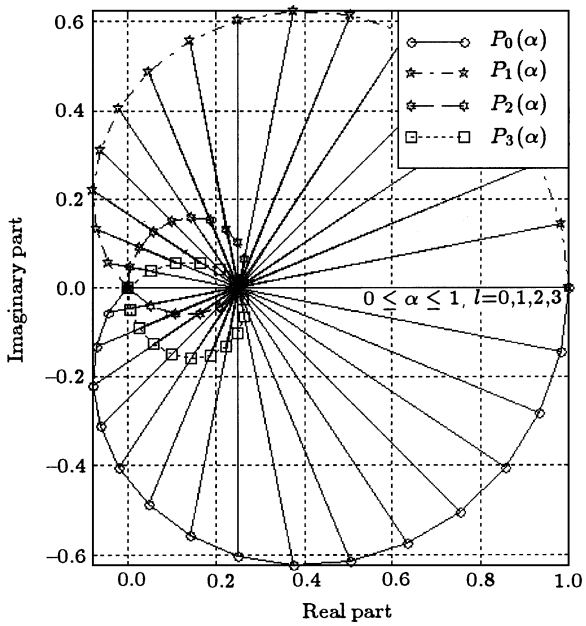


Fig.1. Geometrical locus of coefficients $P_l(\alpha)$, $l = 0, 1, 2, 3$ in the complex plane.

Symbolically, the fractional Fourier transform operator F^α can be more compactly written as

$$F^\alpha = \frac{1}{4}\sum_{l=0}^{3} \frac{1-\exp(-2\pi(\alpha-l)\mathrm{i})}{1-\exp(-2\pi(\alpha-l)\mathrm{i}/4)} F^l. \qquad (5)$$

We define a 4 by 4 matrix $\boldsymbol{P}(\alpha)$ with parameter α as follows:

$$\boldsymbol{P}(\alpha) = \begin{pmatrix} P_0(\alpha) & P_3(\alpha) & P_2(\alpha) & P_1(\alpha) \\ P_1(\alpha) & P_0(\alpha) & P_3(\alpha) & P_2(\alpha) \\ P_2(\alpha) & P_1(\alpha) & P_0(\alpha) & P_3(\alpha) \\ P_3(\alpha) & P_2(\alpha) & P_1(\alpha) & P_0(\alpha) \end{pmatrix}. \qquad (6)$$

It is called a 4 by 4 generalized permutation matrix with the parameter α. In addition, for any α, the notation $\rho^{(\alpha)}$ denotes the vector $(P_0(\alpha)\ P_1(\alpha)\ P_2(\alpha)\ P_3(\alpha))^{\mathrm{T}}$, which is the first column of the matrix $\boldsymbol{P}(\alpha)$.

In fact, it is easy to see that, for $\alpha = 0, 1, 2, 3$,

$$\boldsymbol{P}(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{P}(1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\boldsymbol{P}(2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{P}(3) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

are the 0-step, 1-step, 2-step and 3-step standard permutation matrices, respectively, and $\rho^{(0)} = (1\ 0\ 0\ 0)^{\mathrm{T}}$, $\rho^{(1)} = (0\ 1\ 0\ 0)^{\mathrm{T}}$, $\rho^{(2)} = (0\ 0\ 1\ 0)^{\mathrm{T}}$, $\rho^{(3)} = (0\ 0\ 0\ 1)^{\mathrm{T}}$. Moreover, $\boldsymbol{P}(4) = \boldsymbol{P}(0)$, thus for any integer m, $\boldsymbol{P}(m+4) = \boldsymbol{P}(m)$.

By using the matrix symbol, Eqs.(2) and (3) can be more compactly written as

$$\boldsymbol{P}(\alpha)\boldsymbol{P}(\beta) = \boldsymbol{P}(\alpha+\beta) = \boldsymbol{P}(\beta)\boldsymbol{P}(\alpha). \qquad (7)$$

By observing the first column on both sides of the second equal sign in Eq.(7), we have

$$\rho^{(\alpha+\beta)} = \boldsymbol{P}(\beta)\rho^{(\alpha)}. \qquad (8)$$

Furthermore, for any function $f(x)$, the fractional Fourier transform of $f(x)$ can be rewritten as

$$f_\alpha(x) = (F^\alpha f)(x) = (f_0(x)\ f_1(x)\ f_2(x)\ f_3(x))\rho^{(\alpha)}.$$

Therefore F^α could be determined uniquely by a vector $\rho^{(\alpha)}$ for any fraction α. In detail, for $\alpha = 0, 1, 2, 3$, we could rewrite $(F^\alpha f)(x)$ as follows:

$$\begin{aligned} f_0(x) &= (F^0 f_0)(x) = 1\cdot f_0(x) + 0\cdot f_1(x) + 0\cdot f_2(x) + 0\cdot f_3(x), \\ f_1(x) &= (F^1 f_0)(x) = 0\cdot f_0(x) + 1\cdot f_1(x) + 0\cdot f_2(x) + 0\cdot f_3(x), \\ f_2(x) &= (F^2 f_0)(x) = 0\cdot f_0(x) + 0\cdot f_1(x) + 1\cdot f_2(x) + 0\cdot f_3(x), \\ f_3(x) &= (F^3 f_0)(x) = 0\cdot f_0(x) + 0\cdot f_1(x) + 0\cdot f_2(x) + 1\cdot f_3(x). \end{aligned}$$

Thus $f_l(x) = (Ff_{l-1})(x) = (F^l f_0)(x)$, $l = 0, 1, 2, 3$. Note that if we are given the original signal $f_0(x)$, the four signals on the left side of the above equations, i.e. $f_0(x)$, $f_1(x)$, $f_2(x)$, $f_3(x)$, could be uniquely determined by the vectors (1 0 0 0), (0 1 0 0), (0 0 1 0), (0 0 0 1), respectively. Visually, this can be viewed as

$$\begin{array}{ccccc}
f_0(x) = (F^0 f)(x) & \rightarrow \boxed{F} \rightarrow & f_1(x) = (Ff_0)(x) & & \rho^{(0)} = (1\ 0\ 0\ 0)^{\mathrm{T}} \rightarrow \boxed{P(1)} \rightarrow \rho^{(1)} = P(1)\rho^{(0)} = (0\ 1\ 0\ 0)^{\mathrm{T}} \\
\uparrow & & \downarrow & & \uparrow \qquad\qquad \downarrow \\
\boxed{F} & & \boxed{F} & \Longleftrightarrow & \boxed{P(1)} \qquad\qquad \boxed{P(1)} \\
\uparrow & & \downarrow & & \uparrow \qquad\qquad \downarrow \\
f_3(x) = (Ff_2)(x) & \leftarrow \boxed{F} \leftarrow & f_2(x) = (Ff_1)(x) & & \rho^{(3)} = P(1)\rho^{(2)} = (0\ 0\ 0\ 1)^{\mathrm{T}} \leftarrow \boxed{P(1)} \leftarrow \rho^{(2)} = P(1)\rho^{(1)} = (0\ 0\ 1\ 0)^{\mathrm{T}}
\end{array}$$

Namely, in this case, the order-1 FRFT F, i.e. the conventional Fourier transform, is equivalent to the standard 1-step permutation matrix $\boldsymbol{P}(1)$. In addition, for any α, because of $F^{\alpha+1} = F^0 F^{\alpha+1} = F^\alpha F^1$, and $\rho^{(\alpha+1)} = \boldsymbol{P}(1)\rho^{(\alpha)} = (P_3(\alpha)\ P_0(\alpha)\ P_1(\alpha)\ P_2(\alpha))^{\mathrm{T}}$, one may see that, in general, the classical Fourier transform F is also equivalent to the standard 1-step permutation matrix $\boldsymbol{P}(1)$ that permutes vector $(P_0(\alpha)\ P_1(\alpha)\ P_2(\alpha)\ P_3(\alpha))^{\mathrm{T}}$ to $(P_3(\alpha)\ P_0(\alpha)\ P_1(\alpha)\ P_2(\alpha))^{\mathrm{T}}$. Therefore, F is a permutational transform in nature. Similarly, it is easy to see that, the order-l FRFT F^l is equivalent to matrix $\boldsymbol{P}(l)$ and $\boldsymbol{P}(l) = [\boldsymbol{P}(1)]^l$, $l = 0, 1, 2, 3, 4$, where $\boldsymbol{P}(0) = \boldsymbol{P}(4) = \boldsymbol{I}$ is the identity matrix.

Therefore, the conventional Fourier transform and its successive application are all standard permutational transforms with integral steps.

Generally, by observing the first column on both sides of the first equal sign in Eq.(7), we can obtain $\rho^{(\alpha+\beta)} = \boldsymbol{P}(\alpha)\rho^{(\beta)}$. Let $\beta = 0$, it becomes $\rho^{(\alpha)} = \boldsymbol{P}(\alpha)\rho^{(0)}$, here the additive axiom becomes $F^\alpha = F^{\alpha+0} = F^\alpha F^0$. So from the original signal $f_0(x) = (F^0 f_0)(x)$ which has coefficient vector $\rho^{(0)} = (1\ 0\ 0\ 0)^{\mathrm{T}}$, in order to implement its fraction-α FRFT $f_\alpha(x) = (F^\alpha f_0)(x)$ which has coefficient vector $\rho^{(\alpha)} = (P_0(\alpha)\ P_1(\alpha)\ P_2(\alpha)\ P_3(\alpha))^{\mathrm{T}}$, it is enough to apply F^α to F^0. In other words, we could apply the generalized permutation matrix $\boldsymbol{P}(\alpha)$ to the vector $\rho^{(0)}$. Thus, the fractional Fourier transform F^α is equal to the generalized permutation matrix $\boldsymbol{P}(\alpha)$. Based on the comparability of the additive axiom with Eq.(7), we have discovered that successive application of F^α on F^β is equivalent to the multiplication of $\boldsymbol{P}(\alpha)$ and $\boldsymbol{P}(\beta)$. Therefore, the fractional Fourier transform is a permutational transform not only from intuitive point of view, but it is also true from mathematical operation's point of view. Actually, it is easy to validate that the following theorem is true.

Theorem 1 The set of the FRFTs

$$\mathcal{F} = \{F^\alpha | \alpha \in R\} = \{F^\alpha | 0 \le \alpha < 4\}$$

and the set of the generalized permutation matrices

$$\mathcal{P} = \{\boldsymbol{P}(\alpha) | \alpha \in R\} = \{\boldsymbol{P}(\alpha) | 0 \le \alpha < 4\}$$

are two equivalent multiplicative groups. The latter is called a generalized permutation matrix group. They are also equivalent to the quotient group

$$R/Z(4) = \{\alpha | 0 \le \alpha < 4\},$$

where $R = \{x | -\infty < x < +\infty\}$ is the set of all real numbers, $Z(4) = \{4m | m \in Z\}$ is the set of all integers divisible by 4, Z is the set of all integers, $R/Z(4)$ is the additive group with modulus 4.

Therefore the successive application of F^α on F^β is equivalent to its power addition and is also equivalent to the index addition, and the Fourier transform and the FRFT are the same permutational transform with different permutation matrix, but the former is the standard integer-step one and the latter is the fraction-step one.

3. HGPFRFT

It is well known that the conventional Fourier transform has an eigenvalue $\lambda_m = \exp(-m\pi \mathrm{i}/2)$ corresponding to the standard eigenfunction $\phi_m(x) = 1/\sqrt{2^m m! \sqrt{\pi}} \mathrm{H}_m(x) \exp(-x^2/2)$, i.e. the Hermite–Gaussian function, where $\mathrm{H}_m(x)$ is the mth order Hermite polynomial having the recursive relation $\mathrm{H}_{m+1}(x) = 2x\mathrm{H}_m(x) - 2m\mathrm{H}_{m-1}(x)$, and $\mathrm{H}_0(x) = 1$, $\mathrm{H}_1(x) = 2x$. By using this notation, the conventional Fourier transform F can then expressed by the superposition of Hermite–Gaussian functions, $(F^\alpha f)(x) = \sum_{m=0}^{\infty} d_m \lambda_m^\alpha \phi_m(x)$, where $f(x) = \sum_{m=0}^{\infty} d_m \phi_m(x)$, $d_m = \int_R f(x)\bar{\phi}_m(x)\mathrm{d}x$. For any natural number $M \ge 3$, we

define intentionally "the basic transform" $\mathcal{R}$ as follows:

$$(\mathcal{R}f)(x) = \sum_{m=0}^{\infty} d_m \nu_m(M) \phi_m(x), \tag{9}$$

where $\nu_m(M) = \exp(-2\pi \mathrm{i} m/M)$ is the $\mathcal{R}$ eigenvalue corresponding to the eigenfunction $\phi_m(x)$, m is a nonnegative integer. By recalling the definition of the original FRFT, it is easy to see that the "basic transform" $\mathcal{R}$ is FRFT $\mathcal{F}^{\alpha}$ with $\alpha = 4/M$ which will be referenced in the following Eq.(13). Since $\nu_0(M)$=1, $\nu_1(M) = \exp(-2\pi \mathrm{i}/M), \cdots, \nu_{M-1}(M) = \exp(-2\pi \mathrm{i}(M-1)/M)$ and $\nu_M(M) = \nu_0(M) = 1$, therefore "basic transform" $\mathcal{R}$ is the mathematical operation with a periodicity of M, i.e. $f_0(x) = (\mathcal{R}^0 f)(x) = f(x)$, $f_1(x) = (\mathcal{R}f)(x), \cdots, f_{M-1}(x) = (\mathcal{R}^{M-1}f)(x)$, $f_M(x) = (\mathcal{R}^M f)(x) = f_0(x)$ and $f_{ML+l}(x) = (\mathcal{R}^{ML+l}f)(x) = f_l(x)$, where $f(x)$ is a function, L is an integer and $l = 0, 1, 2, \cdots, (M-1)$. The rest of the mathematics is straightforward as mentioned by Shih in Ref.[12].

We can construct the α-order fractional transform as a weighted composition of these M functions as follows:

$$(\mathcal{R}^{\alpha}f)(x) = \sum_{l=0}^{(M-1)} P_l(\alpha)(\mathcal{R}^l f)(x). \tag{10}$$

The so-defined fractional transform is called the HGPFRFT. It should obey the following axioms:

1) Continuity axiom: $P_0(\alpha), P_1(\alpha), \cdots, P_{M-1}(\alpha)$ should all be continuous for all real α.

2) Boundary axiom: $P_l(m) = \begin{cases} 1, & m = l \\ 0, & m \neq l \end{cases}$ $(0 \leq m, l \leq (M-1))$.

3) Periodicity axiom: $P_l(\alpha) = P_l(\alpha + M)$, $l = 0, 1, \cdots, (M-1)$, for all α.

4) Additive axiom: $\mathcal{R}^{\alpha}\mathcal{R}^{\beta} = \mathcal{R}^{\beta}\mathcal{R}^{\alpha} = \mathcal{R}^{\alpha+\beta}$ for any α and β.

Being similar to the previous section, we define an M by M matrix $\boldsymbol{P}(\alpha)$ as follows:

$$\boldsymbol{P}(\alpha) = \begin{pmatrix} P_0(\alpha) & P_{M-1}(\alpha) & \cdots & P_1(\alpha) \\ P_1(\alpha) & P_0(\alpha) & \cdots & P_2(\alpha) \\ \vdots & \vdots & \ddots & \vdots \\ P_{M-1}(\alpha) & P_{M-2}(\alpha) & \cdots & P_0(\alpha) \end{pmatrix},$$

where $\boldsymbol{P}(\alpha)$ is called the high order generalized permutation matrix and the notation $\rho^{(\alpha)}$ denotes the vector $(P_0(\alpha)\ P_1(\alpha)\ \cdots\ P_{M-1}(\alpha))^{\mathrm{T}}$, which is the first column of the matrix $\boldsymbol{P}(\alpha)$. Postulate 2 means that $\boldsymbol{P}(0)$, $\boldsymbol{P}(1)$, $\cdots$, $\boldsymbol{P}(M-1)$ are the standard 0-step, 1-step, $\cdots$, $(M-1)$-step permutation matrices, respectively. Postulate 3 means that $\boldsymbol{P}(\alpha + M) = \boldsymbol{P}(\alpha)$. The meaning of postulate 4 is $\boldsymbol{P}(\alpha)\boldsymbol{P}(\beta) = \boldsymbol{P}(\beta)\boldsymbol{P}(\alpha) = \boldsymbol{P}(\alpha + \beta)$. In order to simplify the equations of coefficients $P_l(\alpha)$, we choose the M-point discrete Fourier transform $\mu^{(\alpha)} = (\mu_0(\alpha), \mu_1(\alpha), \cdots, \mu_{M-1}(\alpha))^{\mathrm{T}} = \boldsymbol{T}(M)\rho^{(\alpha)}$, where $\boldsymbol{T}(M) = (t_{m,n})$, $t_{m,n} = \exp(-2\pi \mathrm{i}(m-1)(n-1)/M)$, $1 \leq m, n \leq M$. In this way, postulates 2–4 become $\mu_l(m) = \exp(-2\pi m l \mathrm{i}/M)$, $\mu_l(M + \alpha) = \mu_l(\alpha)$ and $\mu_l(\alpha + \beta) = \mu_l(\alpha)\mu_l(\beta)$, respectively, where $0 \leq m, l \leq (M-1)$, α and β are any real numbers. So $\mu_l(\alpha) = \exp(-2\pi \alpha l \mathrm{i}/M)$ for any real fraction α distinctly. Accordingly, the analytical representation of $P_l(\alpha)$ is shown as follows:

$$\begin{aligned} P_l(\alpha) &= \frac{1}{M} \sum_{n=0}^{(M-1)} \exp(-2\pi n(\alpha - l)\mathrm{i}/M) \\ &= \frac{1}{M} \frac{1 - \exp(-2\pi(\alpha - l)\mathrm{i})}{1 - \exp(-2\pi(\alpha - l)\mathrm{i}/M)}, \\ & \qquad l = 0, 1, \cdots, (M-1). \end{aligned}$$

Then HGPFRFT $\mathcal{R}^{\alpha}$ can be more compactly written as

$$\begin{aligned} &(\mathcal{R}^{\alpha}f)(x) \\ &= \frac{1}{M} \sum_{l=0}^{(M-1)} \sum_{n=0}^{(M-1)} \exp(-2\pi n(\alpha - l)\mathrm{i}/M)(\mathcal{R}^l f)(x), \end{aligned} \tag{11}$$

and it is easy to compute that the HGPFRFT $\mathcal{R}^{\alpha}$ can also be written as the superposition of the Hermite–Gaussian functions

$$(\mathcal{R}^{\alpha}f)(x) = \sum_{m=0}^{\infty} d_m \nu_m(M, \alpha) \phi_m(x), \tag{12}$$

where $\nu_m(M, \alpha) = \exp(-2\pi \alpha \mathrm{i} \bmod(m, M)/M)$ is the $\mathcal{R}^{\alpha}$'s eigenvalue corresponding to the eigenfunction $\phi_m(x)$, d_m is defined as above, and m is a nonnegative integer. Incidentally, the eigenvalues $\nu_m(M, \alpha)$ are M-periodic not only with respect to the order of the transform α but also to the order of the Hermite–Gaussian functions m.

From Eq.(12) one can easily recognize that Shih's FRFT and Liu's GFRFT are just two extreme cases of HGPFRFT with $M = 4$ and $M = 4k$ (k is a natural number), respectively. In addition, when $M \to \infty$,

because

$$\begin{aligned}&|(\mathcal{R}^{M\alpha/4}f)(x)-(\mathcal{F}^{\alpha}f)(x)|\\ \le&\sum_{m=M}^{\infty}|d_m(\exp(-\mathrm{mod}(m,M)\pi\alpha\mathrm{i}/2)\\ &-\exp(-m\pi\alpha\mathrm{i}/2))\phi_m(x)|\\ \le&2\sum_{m=M}^{\infty}|d_m\phi_m(x)|,\end{aligned}$$

thus $(\mathcal{R}^{M\alpha/4}f)(x)\to(\mathcal{F}^{\alpha}f)(x)$, for any real number fraction α and any original function $f(x)$, where

$$(\mathcal{F}^{\alpha}f)(x)=\sum_{m=0}^{\infty}d_m\exp(-m\pi\alpha\mathrm{i}/2)\phi_m(x) \qquad (13)$$

is the representation of the original fractional Fourier transform $\mathcal{F}^{\alpha}$ by the superposition of Hermite–Gaussian functions $\phi_m(x)$, which is the eigenfunction according to the eigenvalue $\lambda_m(\alpha)=\exp(-m\pi\alpha\mathrm{i}/2)$ of $\mathcal{F}^{\alpha}$. Therefore the original fractional Fourier transform $\mathcal{F}^{\alpha}$ is the extreme case of HGPFRFT with $M=\infty$. Accordingly, CGPFRFT introduces a complete generalization and extension of Shih's FRFT and the original definition, which is not just the incomplete one proposed in Ref.[13].

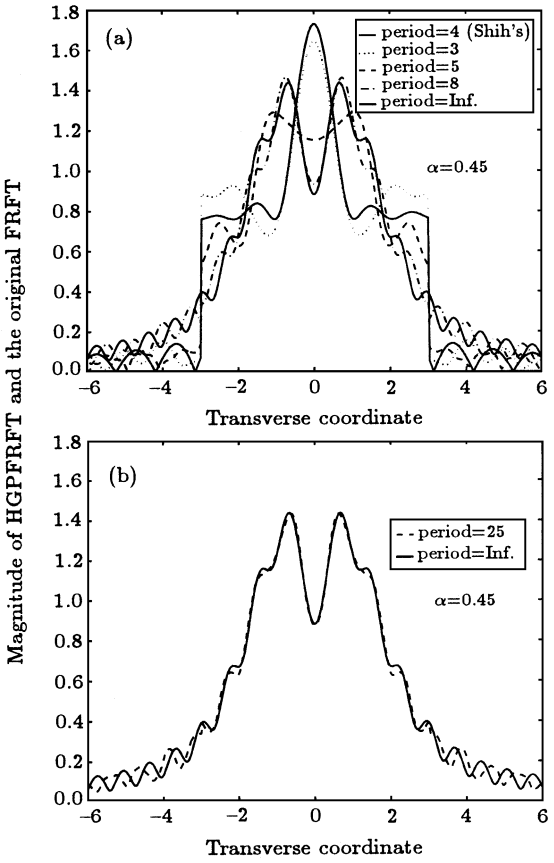


Fig.2. HGPFRFTs of a rectangular function, $\alpha=0.45$, at increasing periods (it is the M in the definition of HGPFRFT) with (a) the periods 3, 4,5,8, and infinity, respectively, (b) finite large period 25 and infinite period.

Here we have made computer simulations of some HGPFRFTs on a rectangular function and compared the results with the plots generated by the original FRFT. Figure 2 gives the simulation results of several HGPFRFTs and comparison with the original FRFT (its period is infinity) and the Shih's FRFT (its period is 4). Figure 2(a) demonstrates the differences between several HGPFRFTs while Fig.2(b) indicates the effect of approximating the conventional FRFT by the HGPFRFT with enough large period of 25. The two plots in Fig.2(b) are almost the same.

4. Properties of HGPFRFT

In this section, we discuss the generalized permutational characteristics of HGPFRFT and the properties of the coefficients in the definition of HGPFRFT.

4.1. Generalized permutational characterization of HGPFRFT

Being similar to the above Section 2, for any integer $M\ge 3$, one can easily validate the following theorem.

Theorem 2 The set of the HGPFRFTs

$$\mathcal{R}=\{\mathcal{R}^{\alpha}|\alpha\in R\}=\{\mathcal{R}^{\alpha}|0\le\alpha<M\}$$

and the set of the generalized permutation matrices

$$\mathcal{P}=\{\boldsymbol{P}(\alpha)|\alpha\in R\}=\{\boldsymbol{P}(\alpha)|0\le\alpha<M\}$$

are two equivalent multiplicative groups. The latter is called a high order generalized permutation matrix group. This is why the new FRFT is called the complete generalized permutational fractional Fourier transform. They are also equivalent to the quotient group

$$R/Z(M)=\{\alpha|0\le\alpha<M\},$$

where $R=\{x|-\infty<x<+\infty\}$ is the set of all real numbers, $Z(M)=\{Mm|m\in Z\}$ is the set of all integers divisible by M, Z is the set of all integers, and $R/Z(M)$ is the additive group with modulus M.

4.2. Properties of coefficients

There are several significant properties associated with the coefficients in the definition of HGPFRFT. Firstly the sum of all coefficients is always one, i.e. $\sum_{l=0}^{(M-1)}P_l(\alpha)=1$, for any real fraction α and any order M of the generalized permutation matrix $\boldsymbol{P}(\alpha)$. One ordinary application of this property is to affirm

that the average position of all coefficients which are M points in the complex plane is $Q_M = (1/M, 0)$. The other is applied to the problem of self-fractional Fourier functions, which is more difficult than the first one.

Similar to the meanings of self-Fourier function and self-fractional-Fourier function introduced in Refs.[13, 20–22], a function is called a self-high-order generalized permutational Fourier function if it transforms to itself after undergoing a high order generalized permutational fractional Fourier transform. From $\sum_{l=0}^{(M-1)} P_l(\alpha) = 1$, one can conclude that, for any given function $f(x)$, it is a self-high-order generalized permutational fractional Fourier function under the "basic transform" $\mathcal{R}$ if and only if it is the self-transform function under the HGPFRFT $\mathcal{R}^{1/N}$, where $N \geq 2$ is a natural number. Naturally, if the function $f(x)$ is a self-high-order generalized permutational fractional Fourier function under $\mathcal{R}$, then for any real fractional order α, the $f(x)$ is also the self-transform function under $\mathcal{R}^{\alpha}$.

Secondly, the Euclidian norm of the complex vector $\rho^{(\alpha)} = (P_0(\alpha)\ P_1(\alpha)\ \cdots\ P_{M-1}(\alpha))^{\mathrm{T}}$ is always one, i.e. for any real fraction α and any order M of the generalized permutation matrix $\boldsymbol{P}(\alpha)$, $\sum_{l=0}^{(M-1)} |P_l(\alpha)|^2 = 1$. This means that, in the M-dimensional complex linear space, the vector $\rho^{(\alpha)}$ stands on the M-dimensional complex unit hyper-spherical surface. Therefore the vector $\rho^{(\alpha)}$ stands on the common part of the M-dimensional complex hyper-plane and the unit hyper-sphere.

Thirdly, to understand the geometrical property of the coefficients $P_l(\alpha)$, $0 \leq l \leq 4$, one can illustrate the geometrical locus of these coefficients in the complex plane and three-dimensional real space. For the coefficients associated with the HGPFRFT proposed in this paper, an interesting property discovered is that all the coefficients, as the complex functions for the fraction α, are the same closed curve in the complex plane, for any given integer $M \geq 3$, which crosses the real unit (1,0) one time and the origin (0,0) $(M-1)$ times and the average position $(1/M,0)$ $(M-2)$ times, but when the fraction α increases from 0 to M, their jumping-off points are different from each other. Figure 3 shows the curve of the five coefficients of HGPFRFT with $M = 5$ and $0 \leq \alpha \leq 1$ in the complex plane. In fact, the closed curve consisting of the five curves is the complete geometrical locus of the coefficient $P_0(\alpha)$ with $0 \leq \alpha \leq 5$, it crosses the real unit (1,0) one time and the origin (0,0) four times and the average position (0.2,0) three times. For the coefficients $P_1(\alpha), \cdots, P_4(\alpha)$, this result is true too.

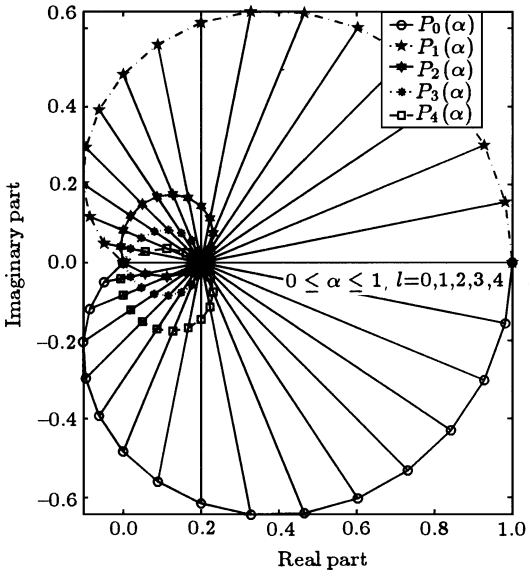


Fig.3. The curve of the coefficients of HGPFRFT with $M = 5$ and $0 \leq \alpha \leq 1$.

Figure 4 demonstrates the three-dimensional locus of these coefficients in the three-dimensional real space whose vertical axis is the fractional order α. In the three-dimensional space, these five curves do not synchronously intersect, but four curves intersect at the points $(0, 0, l)$, $l = 0, 1, 2, 3, 4, 5$.

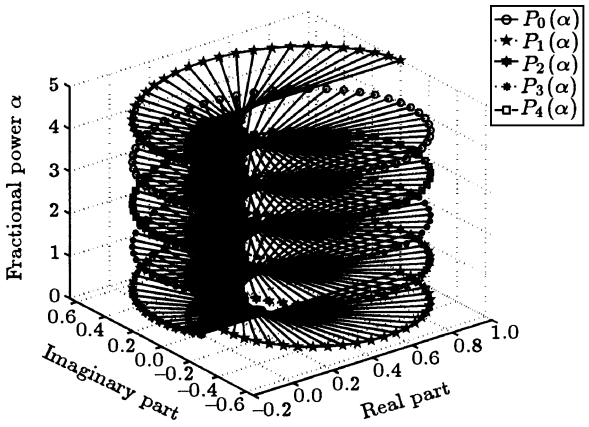


Fig.4. Three-dimensional locus of the coefficients of HGPFRFT with $M = 5$ and $0 \leq \alpha \leq 5$.

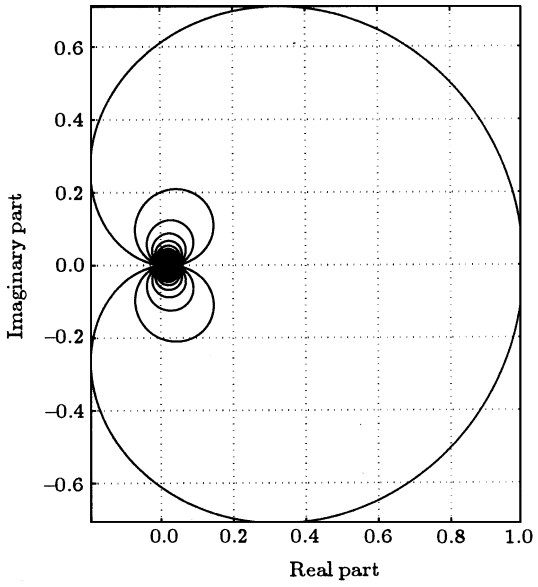


Fig.5. The geometrical symmetry of the coefficient curve of HGPFRFT with $M = 25$ and $0 \leq \alpha \leq 25$.

Figure 5 illustrates the geometrical symmetry of the coefficient curve in the definition of HGPFRFT. The plot in Fig.5 is the curve of the coefficient $P_0(\alpha)$ of HGPFRFT with $M = 25$ and $0 \leq \alpha \leq 25$; it has geometrical symmetry with respect to the real axis.

Fourthly, it is ingenious that for any integer $M \geq 3$ and any fractional order α, these coefficients, in the definition of HGPFRFT, lie on a line. In fact, because

$$\begin{aligned} P_l(\alpha) =& \exp\left(-\frac{\pi(M-1)(\alpha-l)\mathrm{i}}{M} \right) \\ & \times \frac{\sin(\pi(\alpha-l))}{M\sin(\pi(\alpha-l)/M)} \\ =& \mathrm{Re}_l(\alpha) + \mathrm{i}\,\mathrm{Im}_l(\alpha) \end{aligned}$$

and the average position of these coefficients is $Q_M = (1/M, 0)$, by computing the slope of the line which passes through the two points $P_l(\alpha)$ and Q_M, one can obtain

$$\hbar(\alpha) = \frac{\mathrm{Im}_l(\alpha)}{\mathrm{Re}_l(\alpha) - 1/M} = \frac{-\sin(\pi(M-1)(\alpha-l)/M)\sin(\pi(\alpha-l))}{\cos(\pi(M-1)(\alpha-l)/M)\sin(\pi(\alpha-l)) - \sin(\pi(\alpha-l)/M)} = \tan(-\alpha\pi),$$

thus the $(M+1)$ points, i.e. $P_l(\alpha)$, l=0,1, $\cdots$, $(M-1)$, and Q_M, lie on line which passes through the point Q_M and forms an angle $\alpha\pi$ with the real axis in a clockwise direction. Figures 1, 3 and 4 demonstrate this property. Therefore, the coefficients associated with the definition of HGPFRFT have the rotational characteristics. These results may find many applications in, for example, improving the image quality or signal processing with additional real space information.

5. Conclusions

We have generalized the definition of FRFT to HGPFRFT by taking an M-periodic "basic transform". The several representative properties of HGPFRFT are that, (1) its eigenvalue is M-periodic not only with respect to the fractional order of the transform but also to the order of the Hermite–Gaussian function basis which is the eigenfunction of the transform, (2) its structure is the simple generalized permutation matrix, the coefficients in the definition of HGPFRFT have some important properties, for example, one may associate with the concept of the self-transform function, and especially, the rotational one which may improve the multichannel optimal filter. HGPFRFT is extensive and generalized enough to include Shih's FRFT, GFRFT and the original FRFT as the three especial cases which correspond to $M = 4$, $M = 4k$ (k is a natural number) and $M = \infty$, respectively.

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