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New explicit exact solutions to a nonlinear dispersive–dissipative equation *

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Using the first-integral method, we obtain a series of new explicit exact solutions such as exponential function solutions, triangular function solutions, singular solitary wave solution and kink solitary wave solution of a nonlinear dispersive–dissipative equation, which describes weak nonlinear ion-acoustic waves in plasma consisting of cold ions and warm electrons.

Keywords: exponential function solution, triangular function solution, solitary wave solution, first integral method, nonlinear dispersive–dissipative equation

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1. Introduction

Searching for new exact solutions of nonlinear partial differential equations is always an interesting and important topic, especially for those which appear in many branches of nonlinear science. Many new methods have been developed by mathematicians and physicists to find analytical solutions of nonlinear partial differential equations. Some of the most important new methods are the homogeneous balance method,[1,2] tanh function expansion method,[3,4] variable separation method,[5,6] exploratory method,[7] and Jacobi elliptic function expansion method,[8] etc.

In this letter, we consider a nonlinear dispersive–dissipative equation

$$u_t + uu_x + bu_{xxx} - a(u_t + muu_x)_x = 0, \qquad (1)$$

where a, b, m are real constants with $abm \neq 0$. Equation (1) describes weak nonlinear ion-acoustic waves in plasma. It has been derived by Kakutani and Kawahara [9] by analysing a two-fluid plasma model, consisting of cold ions and warm electrons. Solving this equation is some how difficult, because the equation is nonintegrable. Up to now, Malfliet [10] has derived a travelling wave solution of Eq.(1) for $m = 0$ with the tanh method. Isidore [11] has given an exact solution of Eq.(1) by using invariant formalism of Painlevé analysis. The purpose of this letter is to seek some new travelling wave solutions of Eq.(1). To this end one assumes that Eq.(1) has travelling wave solution in the form $u(x,t) = u(\xi)$, $\xi = x - vt$, where v is the velocity of the travelling wave. Substituting into Eq.(1) and performing one integration yields

$$u_{\xi\xi} = b_1 uu_\xi + b_2 u_\xi + b_3 u^2 + b_4 u + b_5, \qquad (2)$$

where $b_1 = \dfrac{am}{b}$, $b_2 = -\dfrac{av}{b}$, $b_3 = -\dfrac{1}{2b}$, $b_4 = \dfrac{v}{b}$, and $b_5 = \dfrac{c}{b}$ is an arbitrary integration constant. Let $x = u, y = u_\xi$, then Eq.(2) is equivalent to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= b_1 xy + b_2 y + b_3 x^2 + b_4 x + b_5. \end{aligned} \qquad (3)$$

It is easily seen that Eq.(3) cannot be integrated directly. This indicates that the usual direct integral method cannot be used to solve this equation. If we can find two first integrals to Eq.(3) under the same conditions, then the general solutions to Eq.(3) can be solved directly. However, in general, it is difficult to realize this even for one first integral. Recently, applying the theory of commutative algebra, Feng [12–14] proposed a very concise and effective new method—the first-integral method. The

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key idea of this method is applying the division theorem in commutative algebra to obtain the first integral to Eq.(3), then using the direct integral method to derive the exact solutions of Eq.(1). In this paper, using the new method, we comprehensively study the exact solutions to the Eq.(1), and obtain a series of new explicit exact solutions such as exponential function solutions, triangular function solutions, singular solitary wave solution and kink solitary wave solution of a nonlinear dispersive–dissipative equation. In Section 2, using the first-integral method, we study the exact solutions of the nonlinear dispersive–dissipative equation. Section 3 gives a brief conclusion.

2. Exact solutions to the nonlinear dispersive–dissipative equation

According to the first-integral method, we suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to Eq.(3), and $p(x,y) = \sum_{i=0}^{n} a_i(x)y^i$ is an irreducible polynomial in $C[x,y]$ such that

$$p[x(\xi), y(\xi)] = \sum_{i=0}^{n} a_i(x)y^i = 0, \tag{4}$$

where $a_i(x)(i = 0, 1, 2, \cdots, n)$ are polynomials of x and $a_n(x) \neq 0$. Expression (4) is also called the first integral to Eq.(3). We start our study by assuming $n = 2$ in Eq.(4). Note that $\mathrm{d}p/\mathrm{d}\xi$ is a polynomial in x and y, and $p[x(\xi), y(\xi)] = 0$ implies $\mathrm{d}p/\mathrm{d}\xi = 0$. According to the division theorem, there exists a polynomial $h(x,y) = \alpha(x) + \beta(x)y$ in $C[x,y]$ such that

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}\xi} =& \frac{\partial p}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \xi} \\ =& \sum_{i=0}^{2}[a_i'(x)y^{i+1}] + \sum_{i=0}^{2}[ia_i(x)y^{i-1}] \\ & \times (b_1xy + b_2y + b_3x^2 + b_4x + b_5) \\ =& [\alpha(x) + \beta(x)y]\left[\sum_{i=0}^{2} a_i(x)y^i\right].\end{aligned} \tag{5}$$

On equating the coefficients of $y^i(i = 3, 2, 1, 0)$ on both sides of Eq.(5), we have

$$a_2'(x) = \beta(x)a_2(x), \tag{6}$$

$$\begin{aligned}& a_1'(x) + 2b_1a_2(x)x + 2b_2a_2(x) \\ =& \alpha(x)a_2(x) + \beta(x)a_1(x),\end{aligned} \tag{7}$$

$$\begin{aligned}& a_0'(x) + b_1a_1(x)x + b_2a_1(x) \\ & + 2b_3a_2(x)x^2 + 2b_4a_2(x)x + 2b_5a_2(x) \\ =& \alpha(x)a_1(x) + \beta(x)a_0(x),\end{aligned} \tag{8}$$

$$b_3a_1(x)x^2 + b_4a_1(x)x + b_5a_1(x) = \alpha(x)a_0(x). \tag{9}$$

Since $a_i(x)(i = 0, 1, 2)$ and $\beta(x)$ are polynomials, from Eq.(6), we deduce that $a_2(x)$ is a constant and $\beta(x) = 0$. For simplification, taking $a_2(x) = 1$, then Eqs.(7)–(9) becomes

$$a_1'(x) + 2b_1x + 2b_2 = \alpha(x), \tag{10}$$

$$\begin{aligned}& a_0'(x) + b_1a_1(x)x + b_2a_1(x) \\ & + 2b_3x^2 + 2b_4x + 2b_5 = \alpha(x)a_1(x),\end{aligned} \tag{11}$$

$$b_3a_1(x)x^2 + b_4a_1(x)x + b_5a_1(x) = \alpha(x)a_0(x). \tag{12}$$

Balancing the degrees of $a_1(x)$ and $\alpha(x)$, we conclude that when $\deg a_1(x) = 1$, then $\deg \alpha(x) = 1$; when $\deg a_1(x) = 2$, then $\deg \alpha(x)$=0 or 1. Next we discuss the three following cases.

2.1. In case I

In the case $\deg a_1(x) = 1$ and $\deg \alpha(x) = 1$, assume $a_1(x) = r_1x + r_0$ $(r_1 \neq 0)$ and $\alpha(x) = \alpha_0 + \alpha_1x$ $(\alpha_1 \neq 0)$. From Eqs.(10) and (11), we deduce that $r_1 = \alpha_0 - 2b_2$, $\alpha_1 = 2b_1$ and $a_0(x) = d + \frac{1}{3}(\alpha_0b_1 - 2b_1b_2 - 2b_3)x^3 + \frac{1}{2}(\alpha_0^2 - 3\alpha_0b_2 + r_0b_1 + 2b_2^2 - 2b_4)x^2 + (\alpha_0r_0 - r_0b_2 - 2b_5)x$, where d is an arbitrary integration constant. Substituting $a_1(x)$, $a_0(x)$ and $\alpha(x)$ into Eq.(12) and setting all coefficients of $x^i(i = 4, 3, 2, 1, 0)$ to zero then yields

$$2b_1^2b_2 + 2b_1b_3 - \alpha_0b_1^2 = 0, \tag{13}$$

$$\begin{aligned}& 6b_1b_2^2 - 6b_1b_4 + 3r_0b_1^2 - 11\alpha_0b_1b_2 \\ & - 2\alpha_0b_3 + 4\alpha_0^2b_1 + 6b_2b_3 - 3\alpha_0b_3 = 0,\end{aligned} \tag{14}$$

$$\begin{aligned}& 8b_1b_5 + 4r_0b_1b_2 - 2\alpha_0b_2^2 + 2\alpha_0b_4 - 5\alpha_0r_0b_1 \\ & + 3\alpha_0^2b_2 - \alpha_0^3 - 4b_2b_4 + 2r_0b_3 + 2b_4\alpha_0 = 0,\end{aligned} \tag{15}$$

$$\begin{aligned}& 2db_1 - 2\alpha_0b_5 - \alpha_0r_0b_2 + r_0\alpha_0^2 \\ & + 2b_2b_5 - r_0b_4 - \alpha_0b_5 = 0,\end{aligned} \tag{16}$$

$$r_0b_5 - d\alpha_0 = 0. \tag{17}$$

Solving these algebraic equations, we obtain

$$\begin{aligned} b_4 &= \frac{b_1b_2b_3+b_3^2}{b_1^2}+\frac{b_1^2b_5}{b_1b_2+b_3},\\ \alpha_0 &= 2b_2+\frac{2b_3}{b_1},\\ r_0 &= \frac{2b_1b_5}{b_1b_2+b_3},\\ d &= \frac{r_0^2}{4} \quad (b_1b_2+b_3\neq 0) \end{aligned} \tag{18}$$

and

$$\begin{aligned} b_2 &= -\frac{b_3}{b_1},\\ \alpha_0 &= 2b_2+\frac{2b_3}{b_1},\\ r_0 &= \frac{2b_4}{b_1},\\ d &= \frac{r_0^2}{4},\\ b_5 &= 0. \end{aligned} \tag{19}$$

In the case of Eq.(18), from Eq.(4) we obtain

$$y = -\frac{b_1b_5}{b_1b_2+b_3}-\frac{b_3}{b_1}x. \tag{20}$$

Expression (20) is the first integral of Eq.(3). Combining Eq.(3) with Eq.(20), we easily obtain the exact solution to Eq.(2) as follows:

$$u(\xi) = \frac{\exp\left(-\frac{b_3}{b_1}(\xi+\xi_0)\right)-b_1^2b_5}{b_1b_2b_3+b_3^2}, \tag{21}$$

where ξ_0 is an arbitrary integration constant. Noting that $b_1=\frac{am}{b}$, $b_2=-\frac{av}{b}$, $b_3=-\frac{1}{2b}$, $b_5=\frac{c}{b}$ and $\xi=x-vt$, the explicit and exact solution to Eq.(1) can be written as

$$u(x,t) = \frac{4b^3\exp\left(\frac{1}{2am}(x-vt+\xi_0)\right)-4a^2m^2c}{b+2a^2mv}, \tag{22}$$

where velocity

$$v = \frac{b(1-m)\pm m\sqrt{b^2+8a^4m^2c-16a^4m^3c}}{2a^2m(2m-1)}$$

and c, ξ_0 are arbitrary integration constants. Exponential function solution (22) is a new solution to Eq.(1) obtained by using the present method. This solution develops singularity at a finite point, i.e. for any fixed $t=t_0$, there exists x_0 at which this solution blows up. There is much current interest in the formation of so-called "blow ups" or "hot spots" of the solution.[15]

As for the case of Eq.(19), from Eq.(4) we obtain

$$y = -\frac{b_4+b_3x}{b_1}. \tag{23}$$

So, the exact solution to Eq.(1) is

$$u(x,t) = 2v-2b\exp\left(\frac{1}{2am}(x-vt+\xi_0)\right), \tag{24}$$

where velocity

$$v = -\frac{b}{2a^2m}$$

and ξ_0 is an arbitrary integration constant. This is also a new solution to Eq.(1) for integration constant $c=0$.

2.2. In case II

In the case $\deg a_1(x)=2$ and $\deg\alpha(x)=0$, assume $a_1(x)=r_2x^2+r_1x+r_0$ $(r_2\neq 0)$ and $\alpha(x)=\alpha_0$. From Eqs.(10) and (11), we deduce that $r_2=-b_1$,$r_1=\alpha_0-2b_2$, and

$$\begin{aligned} a_0(x) =& d+\frac{1}{4}b_1^2x^4+\frac{1}{3}(3b_1b_2-2\alpha_0b_1-2b_3)x^3\\ &+\frac{1}{2}(\alpha_0^2-3\alpha_0b_2-r_0b_1+2b_2^2-2b_4)x^2\\ &+(\alpha_0r_0-r_0b_2-2b_5)x, \end{aligned}$$

where d is an arbitrary integration constant. Substituting $a_1(x)$, $a_0(x)$ and $\alpha(x)$ into Eq.(12) and setting all coefficients of $x^i(i=4,3,2,1,0)$ to zero yields

$$b_1\alpha_0+4b_3=0, \tag{25}$$

$$3b_1b_2\alpha_0-2b_3\alpha_0-2b_1\alpha_0^2+6b_2b_3+3b_1b_4-3b_3\alpha_0=0, \tag{26}$$

$$\begin{aligned} &2b_2^2\alpha_0-2b_4\alpha_0-b_1r_0\alpha_0-3b_2\alpha_0^2\\ &+\alpha_0^3+4b_2b_4+2b_1b_5-2b_3r_0-2b_4\alpha_0=0, \end{aligned} \tag{27}$$

$$2b_5\alpha_0+b_2r_0\alpha_0-r_0\alpha_0^2-2b_2b_5+b_4r_0+b_5\alpha_0=0, \tag{28}$$

$$r_0b_5-d\alpha_0=0. \tag{29}$$

Solving algebraic equations (25)–(29), we obtain

$$\begin{aligned} b_4 &= \frac{2b_3(b_1b_2+2b_3)}{b_1^2},\\ \alpha_0 &= -\frac{4b_3}{b_1},\\ r_0 &= -\frac{b_1b_5}{b_3},\\ d &= \frac{r_0^2}{4}. \end{aligned} \tag{30}$$

In the case of Eq.(30), from Eq.(4) we obtain

$$y = \frac{b_1b_5}{2b_3}+\left(b_2+\frac{2b_3}{b_1}\right)x+\frac{b_1}{2}x^2. \tag{31}$$

Expression (31) is the first-integral to Eq.(3). Combining Eq.(3) with Eq.(31), we can integrate Eq.(31) directly and obtain the exact solutions to Eq.(1) as follows:

$$u(x,t)=v-\sqrt{2c+v^2}\tanh\left[\frac{am\sqrt{2c+v^2}}{2b}(x-vt+\xi_0)\right] \quad (\text{if } v^2>-2c), \tag{32}$$

$$u(x,t)=v+\sqrt{-(2c+v^2)}\tan\left[\frac{am\sqrt{-(2c+v^2)}}{2b}(x-vt+\xi_0)\right] \quad (\text{if } v^2<-2c), \tag{33}$$

or

$$u(x,t)=v-\sqrt{2c+v^2}\coth\left[\frac{am\sqrt{2c+v^2}}{2b}(x-vt+\xi_0)\right] \quad (\text{if } v^2>-2c), \tag{34}$$

$$u(x,t)=v-\sqrt{-(2c+v^2)}\cot\left[\frac{am\sqrt{-(2c+v^2)}}{2b}(x-vt+\xi_0)\right] \quad (\text{if } v^2<-2c), \tag{35}$$

where velocity

$$v=\frac{b}{a^2m(m-1)}$$

and c,ξ_0 are arbitrary integration constants. Expression (32) is a kink solitary wave solution, which is substantially the same as that given in Ref.[3]. But we should notice the condition $v^2>-2c$ given here, which is the existent condition of kink solitary wave. The solutions (33) and (35) are triangular function solutions, and Eq.(34) is a singular solitary wave solution. Such solutions are, in fact, diverging shock-typed solutions, which represent a collapsible state.[16]

2.3. In case III

In the case $\deg a_1(x)=2$ and $\deg\alpha(x)=1$, assume $a_1(x)=r_2x^2+r_1x+r_0$ $(r_2\neq 0)$ and $\alpha(x)=\alpha_1x+\alpha_0$ $(\alpha_1\neq 0)$. From Eqs.(10) and (11), we deduce that $r_2=\frac{\alpha_1}{2}-b_1$, $r_1=\alpha_0-2b_2$, $\alpha_1=b_1$ and

$$\begin{aligned}a_0(x)=&d+\frac{1}{6}(b_1b_2-\alpha_0b_1-4b_3)x^3\\&+\frac{1}{2}(\alpha_0^2-3\alpha_0b_2+2b_2^2-2b_4)x^2\\&+(\alpha_0r_0-r_0b_2-2b_5)x,\end{aligned}$$

where d is an arbitrary integration constant. Substituting $a_1(x)$, $a_0(x)$ and $\alpha(x)$ into Eq.(12) and setting all coefficients of $x^i(i=4,3,2,1,0)$ to zero yields

$$b_1^2b_2-4b_1b_3-b_1^2\alpha_0+3b_1b_3=0, \tag{36}$$

$$\begin{aligned}&3b_1b_2^2-3b_1b_4-4b_1b_2\alpha_0-2b_3\alpha_0+b_1\alpha_0^2\\&+6b_2b_3+\frac{3}{2}b_1b_4-3b_3\alpha_0=0,\end{aligned} \tag{37}$$

$$\begin{aligned}&2b_1b_5+b_1b_2r_0-b_2^2\alpha_0+b_4\alpha_0-b_1r_0\alpha_0+\frac{3}{2}b_2\alpha_0^2\\&-\frac{1}{2}\alpha_0^3-2b_2b_4-\frac{1}{2}b_1b_5+b_3r_0+b_4\alpha_0=0,\end{aligned} \tag{38}$$

$$b_1d-2b_5\alpha_0-b_2r_0\alpha_0+r_0\alpha_0^2+2b_2b_5-b_4r_0-b_5\alpha_0=0, \tag{39}$$

$$r_0b_5-d\alpha_0=0. \tag{40}$$

Solving the above algebraic equations, we obtain

$$\begin{aligned}b_4&=\frac{2b_3(b_1b_2+2b_3)}{b_1^2},\\b_5&=\frac{b_1^2b_2^2b_3+4b_1b_2b_3^2-5b_3^3}{b_1^4},\\\alpha_0&=b_2-\frac{b_3}{b_1},\\r_0&=\frac{-b_1^2b_2^2-2b_1b_2b_3+15b_3^2}{2b_1^3},\\d&=\frac{-b_1^3b_2^3b_3-7b_1^2b_2^2b_3^2+5b_1b_2b_3^3+75b_3^4}{2b_1^6},\end{aligned} \tag{41}$$

and

$$\begin{aligned}b_4&=\frac{2b_3(b_1b_2+2b_3)}{b_1^2},\\b_5&=\frac{b_1^2b_2^2b_3+4b_1b_2b_3^2+3b_3^3}{b_1^4},\\\alpha_0&=b_2-\frac{b_3}{b_1},\\r_0&=\frac{-b_1^2b_2^2-2b_1b_2b_3+3b_3^2}{2b_1^3},\\d&=-\frac{b_1^3b_2^3b_3+7b_1^2b_2^2b_3^2+15b_1b_2b_3^3+9b_3^4}{2b_1^6}.\end{aligned} \tag{42}$$

In the case of Eq.(41), from Eq.(4) we obtain

$$\begin{aligned}y_1=&\frac{1}{4b_1^3}(b_1b_2+5b_3+b_1^2x)\Big[(b_1b_2-3b_3+b_1^2x)\\&+\sqrt{(b_1b_2+5b_3+b_1^2x)(b_1b_2-3b_3+b_1^2x)}\Big],\end{aligned} \tag{43}$$

$$\begin{aligned}y_2=&\frac{1}{4b_1^3}(b_1b_2+5b_3+b_1^2x)\Big[(b_1b_2-3b_3+b_1^2x)\\&-\sqrt{(b_1b_2+5b_3+b_1^2x)(b_1b_2-3b_3+b_1^2x)}\Big].\end{aligned} \tag{44}$$

Integrating Eqs.(43) and (44), we obtain some exponential function solutions to Eq.(1) as follows:

For Eq.(43), we have

$$u_1(x,t) = \frac{v}{m} + \frac{5b}{2a^2m^2} + \frac{4b}{a^2m^2}\frac{1}{\exp\left(-\frac{1}{am}(x-vt+\xi_0)\right) - 2\exp\left(-\frac{1}{2am}(x-vt+\xi_0)\right)}, \tag{45}$$

$$u_2(x,t) = \frac{v}{m} + \frac{5b}{2a^2m^2} + \frac{4b}{a^2m^2}\frac{1}{\exp\left(-\frac{1}{am}(x-vt+\xi_0)\right) + 2\exp\left(-\frac{1}{2am}(x-vt+\xi_0)\right)}. \tag{46}$$

For Eq.(44), we have

$$u_1(x,t) = \frac{v}{m} + \frac{5b}{2a^2m^2} + \frac{b^3}{2a^2m^2}\frac{1}{2\exp\left(-\frac{1}{am}(x-vt+\xi_0)\right) - b\exp\left(-\frac{1}{2am}(x-vt+\xi_0)\right)}, \tag{47}$$

$$u_2(x,t) = \frac{v}{m} + \frac{5b}{2a^2m^2} + \frac{b^3}{2a^2m^2}\frac{1}{2\exp\left(-\frac{1}{am}(x-vt+\xi_0)\right) + b\exp\left(-\frac{1}{2am}(x-vt+\xi_0)\right)}, \tag{48}$$

where velocity

$$v = \frac{b}{a^2m(m-1)}$$

and ξ_0 is an arbitrary integration constant. These solutions are all new solutions obtained by us.

As for the case of Eq.(42), from Eq.(4) we obtain

$$y_1 = -\frac{b_3}{b_1^3}(b_1b_2 + 3b_3 + b_1^2x), \tag{49}$$

$$y_2 = \frac{b_2^2}{2b_1} + \frac{2b_2b_3}{b_1^2} + \frac{3b_3^2}{2b_1^3} + \left(b_2 + \frac{2b_3}{b_1}\right)x + \frac{b_1}{2}x^2. \tag{50}$$

It is easily found that the first-integral Eqs.(49) and (50) are a special case of Eqs.(20) and (31) for the integration constant

$$b_5 = \frac{b_1^2b_2^2b_3 + 4b_1b_2b_3^2 + 3b_3^3}{b_1^4}.$$

So we will omit detailed calculation of this case.

3. Conclusion

In summary, we have found various kinds of new exact solutions such as exponential function solutions, triangular function solutions, singular solitary wave solution and kink solitary wave solution of the nonlinear dispersive–dissipative equation, which describes weak nonlinear ion-acoustic waves in plasma consisting of cold ions and warm electrons, by using the first-integral method. We verified each solution by putting them back into the original equation using Mathematica. Through the discussion of this paper, we also see that the first-integral method is one of the most effective methods to get some exact solutions especially for nonintegrable models. It has the merit of being widely applicable.

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