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A subalgebra of loop algebra $\tilde{A}_2$ and its applications

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A subalgebra of loop algebra $\tilde{A}_2$ and its expanding loop algebra $\overline{G}$ are constructed. It follows that both resulting integrable Hamiltonian hierarchies are obtained. As a reduction case of the first hierarchy, a generalized nonlinear coupled Schrödinger equation, the standard heat-conduction and a formalism of the well known Ablowitz, Kaup, Newell and Segur hierarchy are given, respectively. As a reduction case of the second hierarchy, the nonlinear Schrödinger and modified Korteweg de Vries hierarchy and a new integrable system are presented. Especially, a coupled generalized Burgers equation is generated.

Keywords: loop algebra, integrable hierarchy, Hamiltonian structure
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1. Introduction

As we know, the loop algebra $\tilde{A}_1$ with the basis described as follows is used constantly:

$$h(n)=\begin{pmatrix}\lambda^n & 0\\ 0 & -\lambda^n\end{pmatrix},\quad e(n)=\begin{pmatrix}0 & \lambda^n\\ 0 & 0\end{pmatrix},\quad f(n)=\begin{pmatrix}0 & 0\\ \lambda^n & 0\end{pmatrix}, \tag{1}$$

$$[h(m),e(n)]=2e(m+n),\quad [h(m),f(n)]=-2f(m+n), \tag{2}$$

$$[e(m),f(n)]=h(m+n),\quad \deg h(n)=\deg e(n)=\deg f(n)=n, \tag{3}$$

from which the well known Ablowitz, Kaup, Newell and Segur (AKNS) hierarchy, Kaup and Newell (KN) hierarchy etc. were obtained.[1]

Guo [2,3] once proposed an approach to engender Hamiltonian hierarchies with multiple potential functions and obtained some integrable systems. Although some integrable systems with multiple potential functions were given by the method of integrable couplings,[4−7] a great deal of the calculation was tedious. Therefore, Guo set forth a simple method, i.e. forcing the power margins of spectral parameter λ in a spectral matrix to be more than two. For this sake, he constructed a few subalgebras of loop algebra $\tilde{A}_1$ with the margins of λ powers to be more than two. In what follows, by use of Guo's idea, we take a subalgebra of loop algebra $\tilde{A}_2$ and its expanding loop algebra $\overline{G}$ to establish two new Lax pairs, from which two new integrable Hamiltonian hierarchies are obtained. As reduction cases of the first hierarchy, a generalized nonlinear coupled Schrödinger equation, the standard heat conduction equation and a formalism of AKNS hierarchy are obtained. As a reduction case of the second hierarchy, the nonlinear Schrödinger and modified Korteweg de Vries (NLS-MKdV) hierarchy [8] and a new integrable system are given. Especially, a coupled generalized Burgers equation is presented. In another sense, this paper also provides a method for generating the expansion of integrable systems of the known hierarchies.

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2. Two new integrable hierarchies

2.1. A subalgebra of the loop algebra $\tilde{A}_2$

Consider the following subalgebra of loop algebra $\tilde{A}_2$:

$$h(n)=\begin{pmatrix}0&0&0\\0&0&-\mathrm{i}\lambda^{2n}\\0&\mathrm{i}\lambda^{2n}&0\end{pmatrix},\tag{4a}$$

$$e(n)=\begin{pmatrix}0&0&\mathrm{i}\lambda^{2n+1}\\0&0&0\\-\lambda^{2n+1}&0&0\end{pmatrix},\tag{4b}$$

$$f(n)=\begin{pmatrix}0&-\mathrm{i}\lambda^{2n+1}&0\\\mathrm{i}\lambda^{2n+1}&0&0\\0&0&0\end{pmatrix},\tag{4c}$$

$$[h(m),e(n)]=f(m+n),\tag{4d}$$

$$[h(m),f(n)]=e(m+n),\tag{4e}$$

$$[f(m),e(n)]=-h(m+n+1),\tag{4f}$$

$$\deg h(n)=2n,\quad \deg e(n)=\deg f(n)=2n+1.\tag{4g}$$

In terms of Eqs.(4a)–(4c), we design an isospectral problem

$$\varphi_x=U\varphi,\quad \lambda_t=0,$$
$$U=\begin{pmatrix}0&-\mathrm{i}r\lambda&q\lambda\\\mathrm{i}r\lambda&0&-\mathrm{i}\lambda^2-\mathrm{i}s\\-q\lambda&\mathrm{i}\lambda^2+\mathrm{i}s&0\end{pmatrix}.\tag{5}$$

Let

$$V=\begin{pmatrix}0&-\mathrm{i}c\lambda&b\lambda\\\mathrm{i}c\lambda&0&-\mathrm{i}a\\-b\lambda&\mathrm{i}a&0\end{pmatrix},$$

where

$$a=\sum_{m\geq 0}a_m\lambda^{-2m},$$
$$b=\sum_{m\geq 0}b_m\lambda^{-2m},$$
$$c=\sum_{m\geq 0}c_m\lambda^{-2m}.$$

Solving the auxiliary equation

$$V_x=[U,V],\tag{6}$$

gives rise to the recurrence relations as follows:

$$\begin{aligned}a_{mx}=&qc_{m+1}-rb_{m+1}\\=&qb_{mx}-rc_{mx}-qsc_m+rsb_m,&\text{(7a)}\\b_{mx}=&c_{m+1}-ra_m+sc_m,&\text{(7b)}\\c_{mx}=&b_{m+1}-qa_m+sb_m,&\text{(7c)}\\a_0=&c=\text{const},\quad b_0=c_0=0,&\text{(7d)}\\a_1=&-\frac{c}{2}(r^2-q^2),\quad b_1=cq,\\c_1=&cr,\quad b_1=cq,&\text{(7e)}\\a_2=&-crq_x+cqr_x-cs(q^2-r^2)\\&+\frac{3c}{8}(r^2-q^2)^2,&\text{(7f)}\\b_2=&cr_x-\frac{c}{2}q(r^2-q^2)-cqs,&\text{(7g)}\\c_2=&cq_x-\frac{c}{2}r(r^2-q^2)-csr.&\text{(7h)}\end{aligned}$$

Substituting $\lambda^{2n}V=\left(\lambda^{2n}\sum_{m=0}^{n}V_m\right)+\left(\lambda^{2n}\sum_{m=n+1}^{\infty}V_m\right)=(\lambda^{2n}V)_+ +(\lambda^{2n}V)_-$ into Eq.(6) yields

$$-(\lambda^{2n}V)_{+x}+[U,(\lambda^{2n}V)_+]=(\lambda^{2n}V)_{-x}-[U,(\lambda^{2n}V)_-].\tag{8}$$

It is easy to find that the terms on the left-hand side in Eq.(8) are of degree ≥ 0, while the terms on the right-hand side are of degree ≤ 1. Therefore, the terms of both sides of Eq.(8) are of degree 0,1. Thus, we have

$$\begin{aligned}&-(\lambda^{2n}V)_{+x}+[U,(\lambda^{2n}V)_+]\\=&-b_{n+1}f(0)-c_{n+1}e(0)-qc_{n+1}h(0)+rb_{n+1}h(0).\end{aligned}$$

Set $V^{(n)}=(\lambda^{2n}V)_+$, then a new Lax pair $\varphi_x=U\varphi,\varphi_t=V^{(n)}\varphi$ leads to the zero-curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0,\tag{9}$$

from which a new integrable system is presented as follows:

$$u_{t_n}=\begin{pmatrix}q\\r\\s\end{pmatrix}_{t_n}=\begin{pmatrix}c_{n+1}\\b_{n+1}\\rb_{n+1}-qc_{n+1}\end{pmatrix}$$
$$=\frac{1}{2}\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-\partial\end{pmatrix}G_n=JG_n.\tag{10}$$

Here $G_n=(-2b_{n+1},2c_{n+1},2a_n)^{\mathrm{T}}$, J is a Hamiltonian operator. From Eqs.(7a), we find $a_{nx}=qb_{nx}-rc_{nx}-$

$s(qc_n - rb_n) = qb_{nx} - rc_{nx} - sa_{n-1x}$. Thus, a recurrence operator is shown as

$$L = \begin{pmatrix} -s + q\partial^{-1}q\partial & -\partial + q\partial^{-1}r\partial & q\partial^{-1}s\partial \\ -\partial - r\partial^{-1}q\partial & -s - r\partial^{-1}r\partial & -r\partial^{-1}s\partial \\ -\partial^{-1}q\partial & -\partial^{-1}r\partial & -\partial^{-1}s\partial \end{pmatrix}.$$

Therefore, Eq.(10) may be written as

$$u_{t_n} = JL^nG_0 = JL^n \begin{pmatrix} -2cq \\ 2cr \\ 2c \end{pmatrix}. \tag{11}$$

As a reduction case of Eq.(11), taking $n = 2$, we have generalized nonlinear coupled Schrödinger equations

$$\begin{aligned} q_{t_2} =& cr_{xx} - \frac{c}{2}(q(r^2 - q^2))_x - c(qs)_x \\ & - cr^2q_x + cqrr_x + \frac{3c}{2}rs(r^2 - q^2) \\ & + \frac{3c}{8}r(r^2 - q^2)^2 - csq_x + crs^2, \\ r_{t_2} =& cq_{xx} - \frac{c}{2}(r(r^2 - q^2))_x - c(rs)_x \\ & + cq^2r_x - crqq_x + \frac{3c}{2}qs(r^2 - q^2) \\ & + \frac{3c}{8}q(r^2 - q^2)^2 - csr_x + cqs^2, \\ s_{t_2} =& crq_{xx} - cqr_{xx} - c(s(q^2 - r^2))_x \\ & + \frac{3c}{2}(r^2 - q^2)(rr_x - qq_x). \end{aligned} \tag{12}$$

Therefore, we call Eq.(11) a generalized Schrödinger hierarchy.

Taking $s = 0$ in Eq.(11) engenders a formalism of AKNS hierarchy as follows:

$$\tilde{u}_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \tilde{J}\tilde{L}^n\tilde{G}_0, \tag{13}$$

where

$$\tilde{J} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\tilde{L} = \begin{pmatrix} q\partial^{-1}q\partial & \partial + q\partial^{-1}r\partial \\ -\partial - r\partial^{-1}q\partial & -r\partial^{-1}r\partial \end{pmatrix},$$

$$\tilde{G}_0 = \begin{pmatrix} -2cq \\ 2cr \end{pmatrix}.$$

Remark 1 Here $\tilde{L}$ is different from that of the well-known AKNS hierarchy.[1] Hence, the system (13) is not an AKNS hierarchy.

Similar to Eq.(12), in terms of Eq.(13), we have a simple coupled Schrödinger equation

$$\begin{aligned} q_{t_2} =& cr_{xx} - \frac{c}{2}(q(r^2 - q^2))_x - cr^2q_x \\ & + cqrr_x + \frac{3c}{8}r(r^2 - q^2)^2, \\ r_{t_2} =& cq_{xx} - \frac{c}{2}(r(r^2 - q^2))_x + cq^2r_x \\ & - cqrq_x + \frac{3c}{8}q(r^2 - q^2)^2. \end{aligned} \tag{14}$$

Let $q = r = u, t_2 = t$ in Eq.(14); the standard linear heat conduction equation $u_t = cu_{xx}$ is given. A direct calculation shows

$$\begin{aligned} &< V, \frac{\partial U}{\partial q} >= -2b\lambda^2, \\ &< V, \frac{\partial U}{\partial r} >= 2c\lambda^2, \\ &< V, \frac{\partial U}{\partial s} >= 2a, \\ &< V, \frac{\partial U}{\partial \lambda} >= (4a - 2qb + 2rc)\lambda. \end{aligned} \tag{15}$$

Inserting Eq.(15) into the trace identity gives

$$\frac{\delta}{\delta u}(4a - 2qb + 2rc)\lambda = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix} -2b\lambda^2 \\ 2c\lambda^2 \\ 2a \end{pmatrix}. \tag{16}$$

Comparing the coefficients of λ^{-2n-1}yields

$$\frac{\delta}{\delta u}(4a_{n+1} - 2qb_{n+1} + 2rc_{n+1}) = (-2n + \gamma)G_n. \tag{17}$$

Taking $n = 0$ in Eq.(17) gives $\gamma = 0$. Thus,

$$\frac{\delta H_n}{\delta u} = G_n, \tag{18}$$

which are just the adjoint symmetries,[8] where

$$H_n = \frac{4a_{n+1} - 2qb_{n+1} + 2rc_{n+1}}{-4n}$$

are Hamiltonian functions.

In terms of Eq.(18), we may establish binary symmetry constraints to obtain nonlinearization of the system (11). We shall discuss this problem in another paper.

From Eq.(18), we have the bi-Hamiltonian structure of the system (11) as follows:

$$u_{t_n} = J\frac{\delta H_n}{\delta u} = K\frac{\delta H_{n-1}}{\delta u}, \tag{19}$$

where $K = JL$.

It is easy to verify that $JL = L^*J$. Therefore, the system (19) is integrable in the Liouville sense.

2.2. An expanding loop algebra

Let us consider an expanding loop algebra $\overline{G}$ of the loop algebra (4) as follows:

$$h(j,n)=\begin{pmatrix}0&0&0\\0&0&-\mathrm{i}\lambda^{2n+j}\\0&\mathrm{i}\lambda^{2n+j}&0\end{pmatrix},\tag{20a}$$

$$e(j,n)=\begin{pmatrix}0&0&\lambda^{2n+j}\\0&0&0\\-\lambda^{2n+j}&0&0\end{pmatrix},\tag{20b}$$

$$f(j,n)=\begin{pmatrix}0&-\mathrm{i}\lambda^{2n+j}&0\\\mathrm{i}\lambda^{2n+j}&0&0\\0&0&0\end{pmatrix},\tag{20c}$$

$$[h(j,m),e(k,n)]=\begin{cases}f(j+k,m+n), & j+k<2,\\ f(0,m+n+1), & j+k\geq 2,\end{cases}\tag{20d}$$

$$[h(j,m),f(k,n)]=\begin{cases}e(j+k,m+n), & j+k<2,\\ e(0,m+n+1), & j+k\geq 2,\end{cases}\tag{20e}$$

$$[e(j,m),f(k,n)]=\begin{cases}h(j+k,m+n), & j+k<2,\\ h(0,m+n+1), & j+k\geq 2,\end{cases}\tag{20f}$$

$$\deg h(j,n)=\deg e(j,n)=\deg f(j,n)=2n+j,\quad j=0,1.\tag{20g}$$

From Eqs.(20a)–(20c), an isospectral problem is established:

$$\phi_x=U\phi,\quad \lambda_t=0,\quad \phi=(\phi_1,\phi_2,\phi_3)^{\mathrm{T}},$$
$$U=\begin{pmatrix}0&-\mathrm{i}u_2-\mathrm{i}u_4\lambda&u_1+u_3\lambda\\ \mathrm{i}u_2+\mathrm{i}u_4\lambda&0&-\mathrm{i}\lambda^2\\ -u_1-u_3\lambda&\mathrm{i}\lambda^2&0\end{pmatrix}=h(0,1)+u_1e(0,0)+u_2f(0,0)+u_3e(1,0)+u_4f(1,0).\tag{21}$$

Let

$$V=\begin{pmatrix}0&-\mathrm{i}c(0)-\mathrm{i}c(1)\lambda&b(0)+b(1)\lambda\\ \mathrm{i}c(0)+\mathrm{i}c(1)\lambda&0&-\mathrm{i}a(0)-\mathrm{i}a(1)\lambda\\ -b(0)-b(1)\lambda&\mathrm{i}a(0)+\mathrm{i}a(1)\lambda&0\end{pmatrix}$$
$$=\sum_{m=0}^{\infty}\sum_{j=0}^{1}(a(j,m)h(j,-m)+b(j,m)e(j,-m)+c(j,m)f(j,-m)),$$

where

$$a(0)=\sum_{m\geq 0}a(0,m)\lambda^{-2m},\quad a(1)=\sum_{m\geq 0}a(1,m)\lambda^{-2m},\cdots$$

Solving the stationary zero-curvature equation

$$V_x=[U,V],\tag{22}$$

yields

$$\begin{aligned}a_x(0,m)=&u_1c(0,m)-u_2b(0,m)+u_3c(1,m+1)-u_4b(1,m+1)\\ =&u_1c(0,m)-u_2b(0,m)+u_3b_x(1,m)\\ &-u_4c_x(1,m)+(u_2u_3-u_1u_4)a(1,m),\end{aligned}\tag{23a}$$
$$b_x(0,m)=c(0,m+1)-u_2a(0,m)-u_4a(1,m+1),\tag{23b}$$
$$c_x(0,m)=b(0,m+1)-u_1a(0,m)-u_3a(0,m),\tag{23c}$$
$$a_x(1,m)=u_1c(1,m)-u_2b(1,m)+u_3c(0,m)-u_4b(0,m),\tag{23d}$$
$$b_x(1,m)=c(1,m+1)-u_2a(1,m)-u_4a(0,m),\tag{23e}$$
$$c_x(1,m)=b(1,m+1)-u_1a(1,m)-u_3a(0,m),\tag{23f}$$
$$a(0,0)=\alpha,\tag{23g}$$
$$a(1,0)=b(1,0)=c(1,0)=b(0,0)=c(0,0)=0,\tag{23h}$$

$$a(1,1) = 0, \tag{23i}$$
$$b(1,1) = \alpha u_3, \tag{23j}$$
$$c(1,1) = \alpha u_4, \tag{23k}$$
$$a(0,1) = \frac{\alpha}{2}(u_3^2 - u_4^2), \tag{23l}$$
$$b(0,1) = \alpha u_1, \tag{23m}$$
$$c(0,1) = \alpha u_2, \tag{23n}$$
$$a(1,2) = \alpha\left[u_{4x}u_3 - u_4u_{3x} + \frac{3}{8}(u_3^2 - u_4^2)\right], \tag{23o}$$
$$c(1,2) = \alpha u_{3x} + \frac{\alpha}{2}u_4(u_3^2 - u_4^2), \tag{23p}$$
$$b(1,2) = \alpha u_{4x} + \frac{\alpha}{2}u_3(u_3^2 - u_4^2). \tag{23q}$$

Note that

$$V_+^{(n)} = \sum_{m=0}^{n}\sum_{j=0}^{1}(a(j,m)h(j,-m) + b(j,m)e(j,-m) + c(j,m)f(j,-m)),$$
$$V_-^{(n)} = \lambda^{2n}V - V_+^{(n)}.$$

Then Eq.(22) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}].$$

A direct calculation gives

$$\begin{aligned} &-V_{+x}^{(n)} + [U, V_+^{(n)}] \\ =&-[b(0,n+1) - u_3a(1,n+1)]f(0,0) \\ &-[c(0,n+1) - u_4a(1,n+1)]e(0,0) \\ &-b(1,n+1)f(1,0) - c(1,n+1)e(1,0) \\ &-[u_3c(1,n+1) - u_4b(1,n+1)]h(0,0). \end{aligned}$$

Taking a modified term $\Delta_n = \partial^{-1}(u_4b(1,n+1) - u_3c(1,n+1))h(0,0)$, with $V^{(n)} = V_+^{(n)} + \Delta_n$, then the zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{24}$$

admits the Lax integrable system

$$\begin{aligned} u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t &= \frac{1}{2}\begin{pmatrix} u_2\partial^{-1}u_2 - \partial & u_2\partial^{-1}u_1 & 0 & 0 \\ u_1\partial^{-1}u_2 & \partial + u_1\partial^{-1}u_1 & 0 & 0 \\ 0 & 0 & -u_4\partial^{-1}u_4 & 1 - u_4\partial^{-1}u_3 \\ 0 & 0 & -1 - u_3\partial^{-1}u_4 & -u_3\partial^{-1}u_3 \end{pmatrix}\begin{pmatrix} -2b(0,n) \\ 2c(0,n) \\ -2b(1,n+1) \\ 2c(1,n+1) \end{pmatrix} \\ &= J_1\begin{pmatrix} -2b(0,n) \\ 2c(0,n) \\ -2b(1,n+1) \\ 2c(1,n+1) \end{pmatrix} = J_1G_{1n} \\ &= \frac{1}{2}\begin{pmatrix} 0 & 0 & u_2\partial^{-1}u_2 - \partial & u_2\partial^{-1}u_1 \\ 0 & 0 & u_1\partial^{-1}u_2 & \partial + u_1\partial^{-1}u_1 \\ u_2\partial^{-1}u_2 - \partial & u_2\partial^{-1}u_1 & u_2\partial^{-1}u_4 - u_4\partial^{-1}u_2 & u_2\partial^{-1}u_3 + u_4\partial^{-1}u_1 \\ u_1\partial^{-1}u_2 & \partial + u_1\partial^{-1}u_1 & u_1\partial^{-1}u_4 + u_3\partial^{-1}u_2 & u_1\partial^{-1}u_3 + u_3\partial^{-1}u_1 \end{pmatrix}\begin{pmatrix} -2b(1,n) \\ 2c(1,n) \\ -2b(0,n) \\ 2c(0,n) \end{pmatrix} \\ &= J_2\begin{pmatrix} -2b(1,n) \\ 2c(1,n) \\ -2b(0,n) \\ 2c(0,n) \end{pmatrix} = J_2G_{2n}. \end{aligned} \tag{25}$$

From Eqs.(23a)–(23f), a recurrence operator is presented as

$$L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix},$$

where

$$
\begin{aligned}
A_1 &= u_3\partial^{-1}u_3\partial - (u_1 + u_3\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_2,\\
B_1 &= -\partial + u_4\partial^{-1}u_4\partial - (u_1 + u_3\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_1,\\
C_1 &= -u_3\partial^{-1}u_2 - (u_1 + u_3\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_4,\\
D_1 &= -u_3\partial^{-1}u_1 - (u_1 + u_3\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_3,\\
A_2 &= \partial + u_4\partial^{-1}u_3 - (u_2 + u_4\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_2,\\
B_2 &= u_4\partial^{-1}u_4\partial - (u_2 + u_4\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_1,\\
C_2 &= -u_4\partial^{-1}u_2 - (u_2 + u_4\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_4,\\
D_2 &= -(u_2 + u_4\partial^{-1}(u_2u_3 - u_1u_4))\partial^{-1}u_3.
\end{aligned}
$$

L satisfies

$$G_{1n} = LG_{2n}, \quad J_1L = L^*J_1 = J_2. \tag{26}$$

A direct calculation yields

$$
\begin{aligned}
< V, \frac{\partial U}{\partial u_1} > &= -2b(0) - 2b(1)\lambda,\\
< V, \frac{\partial U}{\partial u_2} > &= 2c(0) + 2c(1)\lambda,\\
< V, \frac{\partial U}{\partial u_3} > &= -2b(0)\lambda - 2b(1)\lambda^2,\\
< V, \frac{\partial U}{\partial u_4} > &= 2c(0)\lambda + 2c(1)\lambda^2,\\
< V, \frac{\partial U}{\partial \lambda} > &= 2u_4(0) - 2u_3b(0)\\
&\quad + [2u_4c(1) - 2u_3b(1)\\
&\quad + 4a(0)]\lambda + 4a(1)\lambda^2.
\end{aligned}
$$

Substituting the above formulae into the trace identity, we have

$$\frac{\delta}{\delta u}\left(< V, \frac{\partial U}{\partial \lambda} > \right) = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\begin{pmatrix} < V, \frac{\partial U}{\partial u_1} > \\ < V, \frac{\partial U}{\partial u_2} > \\ < V, \frac{\partial U}{\partial u_3} > \\ < V, \frac{\partial U}{\partial u_4} > \end{pmatrix}. \tag{27}$$

Comparing the coefficients of λ^{-2n-1} in Eq.(27), we obtain

$$
\begin{aligned}
&\frac{\delta}{\delta u}(2u_4c(1, n+1) - 2u_3b(1, n+1)\\
&+ 4a(0, n+1)) = (\gamma - 2n)G_{1n}.
\end{aligned}
\tag{28}
$$

Comparing the coefficients of λ^{-2n} in Eq.(27) leads to

$$
\begin{aligned}
&\frac{\delta}{\delta u}(2u_4c(0, n) - 2u_3b(0, n)\\
&+ 4a(1, n+1)) = (\gamma - 2n + 1)G_{2n}.
\end{aligned}
\tag{29}
$$

Inserting the initial values in Eqs.(23g)–(23q) into Eqs.(27) and (28), we find $\gamma = 0$. Thus, we have

$$
\begin{aligned}
&\frac{\delta}{\delta u}H(1, 2n)\\
=&G_{1n},\\
&H(1, 2n)\\
=&\frac{u_3b(1, n+1) - u_4c(1, n+1) - 2a(0, n+1)}{n};
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&\frac{\delta}{\delta u}H(2, 2n-1)\\
=&G_{2n},\\
&H(2, 2n-1)\\
=&\frac{2u_3b(0, n) - 2u_4c(0, n) - 4a(1, n+1)}{2n-1}.
\end{aligned}
\tag{31}
$$

Hence, Eq.(25) can be written as

$$
\begin{aligned}
u_t =& J_1\frac{\delta H(1, 2n)}{\delta u}\\
=& J_1L\frac{\delta H(2, 2n-1)}{\delta u}\\
=& J_2\frac{\delta H(2, 2n-1)}{\delta u}\\
=& J_2L\frac{\delta H(1, 2n-2)}{\delta u}\\
=& J_1L^n\begin{pmatrix} 0 \\ 0 \\ -2\alpha u_3 \\ 2\alpha u_4 \end{pmatrix}.
\end{aligned}
\tag{32}
$$

Equation (26) implies that Eq.(32) is Liouville integrable.

Some reduction cases of Eq.(32) are given as follows.

i) Taking $u_3 = u_4 = 0$, Eq.(32) reduces to the NLS-MKdV hierarchy [8]

$$
\begin{aligned}
u_t &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t \\
&= \frac{1}{2}\begin{pmatrix} -\partial + u_2\partial^{-1}u_2 & u_2\partial^{-1}u_1 \\ u_1\partial^{-1}u_2 & \partial + u_1\partial^{-1}u_1 \end{pmatrix}\begin{pmatrix} -u_1\partial^{-1}u_2 & -\partial - u_1\partial^{-1}u_1 \\ -\partial + u_2\partial^{-1}u_1 & u_2\partial^{-1}u_1 \end{pmatrix}^{n-1}\begin{pmatrix} -2\alpha u_1 \\ 2\alpha u_2 \end{pmatrix} \\
&= \tilde{J}_1\tilde{L}_1^{n-1}\begin{pmatrix} -2\alpha u_1 \\ 2\alpha u_2 \end{pmatrix}, \qquad (33)
\end{aligned}
$$

where $\tilde{J}_1$ is a symplectic operator.

ii) Taking $u_1 = u_2 = 0$, we set $c(0,n) = b(0,n) = a(1,n) = 0$, then Eq.(32) reduces to

$$
\begin{aligned}
\tilde{u}_t &= \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}_t \\
&= \frac{1}{2}\begin{pmatrix} u_4\partial^{-1}u_4 & 1 - u_4\partial^{-1}u_3 \\ -1 - u_3\partial^{-1}u_4 & -u_3\partial^{-1}u_3 \end{pmatrix}\begin{pmatrix} u_3\partial^{-1}u_3\partial & -\partial + u_3\partial^{-1}u_4\partial \\ -\partial - u_4\partial^{-1}u_3\partial & -u_4\partial^{-1}u_4\partial \end{pmatrix}\begin{pmatrix} -2b(1,n) \\ 2c(1,n) \end{pmatrix} \\
&= \tilde{J}_2\tilde{L}_2\begin{pmatrix} -2b(1,n) \\ 2c(1,n) \end{pmatrix} \qquad (34)
\end{aligned}
$$

where $\tilde{J}_2$ is a Hamiltonian operator.

In particular, taking $n = 2, t_2 = t$ in Eq.(34), we present coupled generalized Burgers equations

$$
\begin{aligned}
u_{3t} =& \alpha\Big(u_{4xx} + u_4(u_{3x}u_4 - u_3u_{4x}) \\
&+ \frac{3}{2}(u_3^2 - u_4^2)u_{3x}\Big), \\
u_{4t} =& \alpha\Big(u_{3xx} + u_3(u_{3x}u_4 - u_3u_{4x}) \\
&+ \frac{3}{2}(u_3^2 - u_4^2)u_{4x}\Big).
\end{aligned}
$$

The system (34) is a new integrable hierarchy of evolution equations, to the best of our knowledge.

Remark 2 We can further expand the loop algebra (20) into other high-dimensional loop algebras by replacing $2n+j$ with $Nn+j, N = 3, 4, \cdots$, so that new integrable Hamiltonian systems could be worked out. In addition, we can use the method in Refs.[6,7] to obtain some expanding integrable models of the system (32). By making use of the approach of K symmetry and τ symmetry theory in Ref.[9], we can deduce a nonisospectral integrable Hamiltonian hierarchy and an infinite-dimensional Lie algebra. These two problems will be discussed in another paper.

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