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A hierarchy of lattice soliton equations and its Darboux transformation*

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A new discrete spectral problem is introduced and the corresponding hierarchy of the lattice soliton equations are derived by means of the trace identity. We find a new Darboux transformation of the lattice soliton equation, through which the explicit solutions are shown.

Keywords: discrete lattice equation, trace identity, Darboux transformation
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1. Introduction

Much progress has been made on the theory of continuous integrable systems since the trace identity was proposed.[1,2] Many new integrable systems with good properties have been given,[3−6] such as the Darboux transformation (DT) and the nonlinearization.[7−9] In recent years discrete integrable systems have become the focus of common concern because of their application in physics, mathematics and many other branches of science as well.[10−12]

Usually we consider the spectral problem

$$E\phi = U(u,\lambda)\phi, \quad \phi_t = V\phi, \tag{1}$$

where $U(u,\lambda)$ is a matrix and

$$u(n,t) = (u_1(n,t), u_2(n,t))^{\mathrm{T}},$$

$$u_i = u_i(n,t) \equiv u_{ni}$$

are the potentials depending on integers $n \in Z$ and $t \in R$. λ is the spectral parameter and $\lambda_t = 0$. For the function $f(n,t) = f_n$ the shift operator E and the difference operator D are defined by

$$Ef(n,t) = f(n+1,t) \equiv f_{n+1},$$
$$E^{-1}f(n,t) = f(n-1,t) \equiv f_{n-1},$$
$$Df(n,t) = (E-1)f(n,t) = f(n+1,t) - f(n,t).$$

The eigenfunction $\phi = (\phi^1, \phi^2)^{\mathrm{T}}$ is a function depending on n and t. So Eq.(1) also can be rewritten as

$$\phi_{n+1} = U_n\phi_n,$$
$$\phi_{nt} = V_n\phi_n. \tag{2}$$

The compatibility condition of Eq.(1) is a discrete zero-curvature equation

$$U(u,\lambda)_{mt} = (EV(u,\lambda))U(u,\lambda) - U(u,\lambda)V(u,\lambda), \tag{3}$$

which gives a hierarchy lattice equations

$$U_{mt} = K_m(u(n,t)), \tag{4}$$

where $K_m(u(n,t)) = K_m(u, Eu, E^{-1}u, \Lambda)$.

In the continuous case many good approaches have been used, such as the homogeneous balance method,[13] the hyperbolic function method,[14] the Jacobi elliptic function method,[15−20] etc. For lattice soliton equations, there are fewer successful solution approaches. However, if the lattice soliton equation is obtained from a spectral problem which is formed properly, the DT can be found easily.

By virtue of the Lax pair of Eq.(4), we define the transformation

$$\tilde{\phi}_n = T_n\phi_n, \tag{5}$$

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which can turn Eq.(2) into

$$\tilde{\phi}_{n+1} = \tilde{U}_n \tilde{\phi}_n,$$
$$\tilde{\phi}_{nt} = \tilde{V}_n \tilde{\phi}_n,$$

with $(u_1(n,t), u_2(n,t)) \to (\tilde{u}_1(n,t), \tilde{u}_2(n,t))$.

After the original solutions of Eq.(4) are properly selected we can find the new solutions by the DT.

2. New lattice soliton equations

In this paper we introduce a discrete spectral problem

$$E\phi = U\phi, \tag{6}$$
$$U = \begin{pmatrix} \lambda & \lambda p \\ q & 1+pq \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

where $p = p(n,t)$ and $q = q(n,t)$ are two potentials, and λ is a spectral parameter with $\lambda_t = 0$.

We assume that

$$\Gamma = \begin{pmatrix} A & \lambda B \\ C & -A \end{pmatrix}.$$

The stationary zero-curvature equation

$$(E\Gamma)U - U\Gamma = 0 \tag{7}$$

gives

$$\begin{aligned} &\lambda A^{(1)} + \lambda q B^{(1)} - \lambda A - \lambda pC = 0, \\ &\lambda p A^{(1)} + \lambda pq B^{(1)} + \lambda B^{(1)} - \lambda^2 B + \lambda pA = 0, \\ &\lambda C^{(1)} - qA^{(1)} - qA - pqC - C = 0, \\ &\lambda pC^{(1)} - pqA^{(1)} - A^{(1)} - \lambda qB + pqA + A = 0. \end{aligned} \tag{8}$$

Substituting the expansions

$$A = \sum_{m=0}^{\infty} A_m \lambda^{-m},$$
$$B = \sum_{m=0}^{\infty} B_m \lambda^{-m},$$
$$C = \sum_{m=0}^{\infty} C_m \lambda^{-m}$$

so

$$\begin{aligned} U_{t_m} &= (E(\Gamma\lambda^m)_+)U - U(\Gamma\lambda^m)_+ \\ &= \begin{pmatrix} 0 & \lambda(pA_m^{(1)} + pqB_m^{(1)} + B_m^{(1)} + pA_m) \\ -qA_m^{(1)} - qA_m - pqC_m - C_m & pqA_m + A_m - pqA_m^{(1)} - A_m^{(1)} \end{pmatrix}. \end{aligned} \tag{11}$$

Then the Lax integrable lattice soliton equations are

$$\begin{aligned} q_t &= -qA_m^{(1)} - qA_m - pqC_m - C_m, \\ p_t &= pA_m^{(1)} + pqB_m^{(1)} + B_m^{(1)} + pA_m. \end{aligned} \tag{12}$$

When $m = 1$, the first system is

$$\begin{aligned} q_{t_1} &= q^2 p^{(1)} - q^{(-1)}, \\ p_{t_1} &= p^{(1)} - p^2 q^{(-1)}. \end{aligned} \tag{13}$$

into Eq.(8), we get the recurrence relations

$$\begin{aligned} &A_m^{(1)} + qB_m^{(1)} - A_m - pC_m = 0, \\ &pA_m^{(1)} + pqB_m^{(1)} + B_m^{(1)} - B_{m+1} + pA_m = 0, \\ &C_{m+1}^{(1)} - qA_m^{(1)} - qA_m - pqC_m - C_m = 0, \\ &pC_{m+1}^{(1)} - pqA_m^{(1)} - A_m^{(1)} - qB_{m+1} + pqA_m + A_m = 0. \end{aligned} \tag{9}$$

The initial values are taken as

$$A_0 = \frac{1}{2}, \quad B_0 = C_0 = 0,$$

and we assume that Eq.(9) has additional initial condition $A_j\,|_{[u]=0} = B_j\,|_{[u]=0} = C_j\,|_{[u]=0} = 0$ $(j \geq 1)$. Then A_j, B_j and C_j are uniquely determined by Eq.(9). The first few A_j, B_j and C_j are

$$\begin{aligned} &B_1 = p, \quad C_1 = q^{(-1)}, \quad A_1 = -pq^{(-1)}, \\ &B_2 = p^{(1)} - p^2 q^{(-1)}, \\ &C_2^{(1)} = q^{(-1)} - q^2 p^{(1)}, \\ &A_2 = -p^{(1)} q^{(-1)} - pq^{(-1)} + p^2 (q^{(-1)})^2. \end{aligned}$$

That is to say the recurrence relations in the following are right:

$$\begin{aligned} &B_{m+1} = pA_m^{(1)} + pqB_m^{(1)} + B_m^{(1)} + pA_m, \\ &C_{m+1}^{(1)} = qA_m^{(1)} + qA_m + pqC_m + C_m, \\ &A_{m+1}^{(1)} - A_{m+1} = pC_{m+1} - qB_{m+1}^{(1)}. \end{aligned} \tag{10}$$

Let

$$(\Gamma\lambda^m)_+ = \sum_{i=0}^{m} \begin{pmatrix} A_i & \lambda B_i \\ C_i & -A_i \end{pmatrix} \lambda^{m-i},$$

When $m=2$, the second system is

$$\begin{aligned} q_{t_2} =& -(p^{(1)})^2q^3+(p^{(2)}+p^{(1)})q^2 \\ &+p^{(1)}q^{(-1)}q-q^{(-2)}+p(q^{(-1)})^2, \\ p_{t_2} =& p^3(q^{(-1)})^2-p^2q^{(-1)} \\ &-pp^{(1)}(q+q^{(-1)})+p^{(2)}-(p^{(1)})^2q. \end{aligned} \quad (14)$$

We introduce the notation

$$\langle A,B\rangle = \mathrm{tr}(AB).$$

We need the following equations, which are easy to calculate:

$$\left\langle V,\frac{\partial U}{\partial p}\right\rangle = C,$$
$$\left\langle V,\frac{\partial U}{\partial q}\right\rangle = \lambda B-2pA-p^2C,$$
$$\left\langle V,\frac{\partial U}{\partial \lambda}\right\rangle = \frac{1}{\lambda}(A+pC+2pqA-\lambda qB+p^2qC),$$

where $V=\Gamma U^{-1}$.

Substituting them into the trace identity,[10]

$$\begin{aligned} &\frac{\delta}{\delta u}\frac{1}{\lambda}(A+pC+2pqA-\lambda qB+p^2qC) \\ &=\lambda^{-\tau}\left(\frac{\partial}{\partial\lambda}\lambda^{\tau}\begin{pmatrix} C \\ \lambda B-2pA-p^2C\end{pmatrix}\right), \end{aligned} \quad (15)$$

and comparing the coefficient of λ^{-m-1} on both sides, we have

$$\begin{aligned} &\frac{\delta}{\delta u}(A_m+pC_m+2pqA_m-qB_{m+1}+p^2qC_m) \\ &=(\tau-m)\begin{pmatrix} C_m \\ B_{m+1}-2pA_m-p^2C_m\end{pmatrix}. \end{aligned}$$

From Eq.(10) we know that

$$B_{m+1}-2pA_m-p^2C_m=B_m^{(1)}.$$

To fix the constant τ, we set $m=1$, and obtain $\tau=0$. So

$$\begin{pmatrix} C_m \\ B_m^{(1)}\end{pmatrix}=\frac{\delta H_m}{\delta u},$$

with

$$H_m=\sum_{n\in Z}\left(-\frac{A_m+pC_m+2pqA_m-qB_{m+1}-p^2qC_m}{m}\right)(n).$$

Then Eq.(7) can be shown as the following:

$$\begin{pmatrix} p \\ q\end{pmatrix}_{t_m}=J\begin{pmatrix} C_m \\ B_m^{(1)}\end{pmatrix}=J\frac{\delta H_m}{\delta u}=JL\frac{\delta H_{m-1}}{\delta u}=JL^{m-1}\frac{\delta H_1}{\delta u}, \quad (16)$$

in which

$$J=\begin{pmatrix} p(E+1)D^{-1}p & 1+pq-p(E+1)D^{-1}q \\ -1-pq-q(E+1)D^{-1}p & q(E+1)D^{-1}q\end{pmatrix}$$

is the skew-symmetry operator and

$$L=\begin{pmatrix} E^{-1}(1+pq+q(E+1)D^{-1}p) & -E^{-1}(q(E+1)D^{-1}q) \\ E(p(E+1)D^{-1}p) & E(1+pq-p(E+1)D^{-1}q)\end{pmatrix}$$

is the recurrence operator.

It is easy to obtain

$$L=\begin{pmatrix} 0 & -E^{-1} \\ E & 0\end{pmatrix}J.$$

3. Darboux transformation

We take Eq.(13) as an example to derive the corresponding DT. Equation (13) can be rewritten as

$$\begin{aligned} p_{nt} &= p_{n+1}-p_n^2q_{n-1}, \\ q_{nt} &= q_n^2p_{n+1}-q_{n-1}. \end{aligned} \quad (17)$$

Its Lax pairs are

$$\begin{aligned} Y_{n+1} &= U_nY_n, \\ Y_{nt} &= V_nY_n, \end{aligned} \quad (18)$$

with

$$U_n = \begin{pmatrix} \lambda & \lambda p_n \\ q_n & p_n q_n + 1 \end{pmatrix},$$
$$V_n = \begin{pmatrix} \frac{\lambda}{2} - p_n q_{n-1} & \lambda p_n \\ q_{n-1} & -\frac{\lambda}{2} + p_n q_{n-1} \end{pmatrix}.$$

We assume there is a gauge transformation

$$\tilde{Y}_n = T_n Y_n,$$

which can transform two spectral problems into

$$\begin{aligned} \tilde{Y}_{n+1} &= \tilde{U}_n \tilde{Y}_n, \\ \tilde{Y}_{nt} &= \tilde{V}_n \tilde{Y}_n. \end{aligned} \tag{19}$$

with

$$\begin{aligned} \tilde{U}_n &= T_{n+1} U_n T_n^{-1}, \\ \tilde{V}_n &= (T_{nt} + T_n V_n) T_n^{-1}; \end{aligned}$$

at the same time $\tilde{U}_n$, $\tilde{V}_n$ and U_n, V_n have the same schedule.

Let

$$\phi_n = \begin{pmatrix} \phi_n^1 \\ \phi_n^2 \end{pmatrix}, \quad \psi_n = \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}$$

be two basic solutions of Eq.(13) and use ϕ_n, ψ_n to define a matrix T_n by

$$T_n = \begin{pmatrix} \lambda + a_n & \lambda b_n \\ c_n & \lambda + d_n \end{pmatrix}.$$

Let λ_1 and λ_2 be two solutions of $\det T_n = 0$. When $\lambda = \lambda_i$,

$$\tilde{Y}_n = T_n Y_n = \begin{pmatrix} \lambda_i \phi_n^1 + a_n \phi_n^1 + \lambda_i b_n \phi_n^2 & \lambda_i \psi_n^1 + a_n \psi_n^1 + \lambda_i b_n \psi_n^2 \\ c_n \phi_n^1 + \lambda_i \phi_n^2 + d_n \phi_n^2 & c_n \psi_n^1 + \lambda_i \psi_n^2 + d_n \psi_n^2 \end{pmatrix}.$$

Two rows of $\tilde{Y}_n$ are linear-correlated, and let the coefficient be γ_i. So the following are right:

$$\begin{aligned} &\lambda_i \phi_n^1 + a_n \phi_n^1 + \lambda_i b_n \phi_n^2 \\ =&\gamma_i (\lambda_i \psi_n^1 + a_n \psi_n^1 + \lambda_i b_n \psi_n^2), \\ &c_n \phi_n^1 + \lambda_i \phi_n^2 + d_n \phi_n^2 \\ =&\gamma_i (c_n \psi_n^1 + \lambda_i \psi_n^2 + d_n \psi_n^2). \end{aligned}$$

These can be written as

$$\begin{aligned} &\lambda_i + a_n + \lambda_i b_n \alpha_i(n) = 0, \\ &c_n + \lambda_i \alpha_i(n) + d_n \alpha_i(n) = 0, \end{aligned} \tag{20}$$

with

$$\alpha_i(n) = \frac{\phi_n^2 - \gamma_i \psi_n^2}{\phi_n^1 - \gamma_i \psi_n^1} \quad (i = 1, 2).$$

Solving Eq.(20) we get

$$\begin{aligned} a_n &= \frac{\lambda_1 \lambda_2 (\alpha_1(n) - \alpha_2(n))}{\lambda_2 \alpha_2 - \lambda_1 \alpha_1}, \\ b_n &= \frac{\lambda_1 - \lambda_2}{\lambda_2 \alpha_2(n) - \lambda_1 \alpha_1(n)}, \\ c_n &= \frac{(\lambda_1 - \lambda_2) \alpha_1(n) \alpha_2(n)}{\alpha_1(n) - \alpha_2(n)}, \\ d_n &= \frac{\lambda_2 \alpha_2(n) - \lambda_1 \alpha_1(n)}{\alpha_1(n) - \alpha_2(n)}. \end{aligned} \tag{21}$$

Using Eq.(21), we know that

$$\alpha_i(n+1) = \frac{\mu_i(n)}{\nu_i(n)},$$

with

$$\begin{aligned} \mu_i(n) &= q_n + p_n q_n \alpha_i(n) + \alpha_i(n), \\ \nu_i(n) &= \lambda_i + \lambda_i p_n \alpha_i(n). \end{aligned}$$

We also can get

$$\begin{aligned} a_{n+1} &= \frac{\lambda_1 \lambda_2 [\mu_1(n) \nu_2(n) - \mu_2(n) \nu_1(n)]}{\lambda_2 \mu_2 \nu_1 - \lambda_1 \mu_1 \nu_2}, \\ b_{n+1} &= \frac{(\lambda_1 - \lambda_2) \nu_1(n) \nu_2(n)}{\lambda_2 \mu_2 \nu_1 - \lambda_1 \mu_1 \nu_2}, \\ c_{n+1} &= \frac{(\lambda_1 - \lambda_2) \mu_1(n) \mu_2(n)}{\mu_1(n) \nu_2(n) - \mu_2(n) \nu_1(n)}, \\ d_{n+1} &= \frac{\lambda_2 \mu_2(n) \nu(n)_1 - \lambda_1 \mu(n)_1 \nu(n)_2}{\mu_1(n) \nu_2(n) - \mu_2(n) \nu_1(n)}. \end{aligned} \tag{22}$$

Through direct calculation we can prove

$$(b_n - p_n) c_n + q_n b_{n+1} + a_{n+1} - a_n = 0, \tag{23a}$$

$$p_n a_{n+1} + b_{n+1} + p_n q_n b_{n+1} + b_n d_n - p_n d_n = 0. \tag{23b}$$

From Eq.(21) we know that

$$\det T_n = (\lambda - \lambda_1)(\lambda - \lambda_2).$$

From all of these, we prove the following assertion.

Proposition 1 The matrix $\tilde{U}_n$ defined by $\tilde{U}_n = T_{n+1} U_n T_n^{-1}$ has the same form as U_n, in which the old potentials p_n, q_n are mapped into $\tilde{p}_n$, $\tilde{q}_n$ according to

$$\begin{aligned} \tilde{p}_n &= p_n - b_n, \\ \tilde{q}_n &= q_n + c_{n+1}. \end{aligned} \tag{24}$$

The transformation (24) is usually called the DT from $(Y_n;\ p_n,\ q_n)$ to $(\tilde{Y}_n;\ \tilde{p}_n,\ \tilde{q}_n)$.

Proof

Let

$$T_n^{-1} = T_n^*/\det T_n,$$
$$T_{n+1}U_nT_n^* = \begin{pmatrix} f_{11}(\lambda,n) & f_{12}(\lambda,n) \\ f_{21}(\lambda,n) & f_{22}(\lambda,n) \end{pmatrix}.$$

Because

$$T_n^* = \begin{pmatrix} \lambda+d_n & -\lambda b_n \\ -c_n & \lambda+a_n \end{pmatrix},$$
$$T_{n+1} = \begin{pmatrix} \lambda+a_{n+1} & \lambda b_{n+1} \\ c_{n+1} & \lambda+d_{n+1} \end{pmatrix},$$

we know that $f_{11}(\lambda,n)$, $f_{12}(\lambda,n)$, $\lambda f_{21}(\lambda,n)$ and $\lambda f_{22}(\lambda,n)$ are cubic polynomials in λ through a direct calculation. We prove in the following that when $\lambda=\lambda_i$ (i=1,2),

$$f_{kl}(\lambda_i,n)=0 \quad (k,l=1,2).$$

Take $f_{11}(\lambda,n)$ as an example. From Eq.(20) we know that

$$\lambda_i+a_n=-\lambda_i b_n\alpha_i(n), \quad \lambda_i+d_n=-\alpha_i^{-1}(n)c_n,$$

so

$$\begin{aligned} &f_{11}(\lambda_i,n)\\ =&(\lambda_i+d_n)[\lambda_i(\lambda_i+a_{n+1})+\lambda_i q_n b_{n+1}]\\ &-\lambda_i p_n c_n(\lambda_i+a_{n+1})-\lambda_i b_{n+1}c_n(1+p_nq_n)\\ =&(-\alpha_i^{-1}(n)c_n)[\lambda_i(-\lambda_i b_{n+1}\alpha_i(n+1)+\lambda_i q_n b_{n+1}]\\ &-\lambda_i p_n c_n(-\lambda_i b_{n+1}\alpha_i(n+1))\\ &-\lambda_i b_{n+1}c_n(1+p_nq_n)=0. \end{aligned}$$

Then

$$T_{n+1}U_nT_n^*=(\det T_n)Q_n,$$

in which

$$Q_n=\begin{pmatrix} \lambda Q_n^{11}+Q_n^1 & \lambda Q_n^{12}+Q_n^2 \\ Q_n^{21} & Q_n^{22} \end{pmatrix}.$$

This also can be written as

$$T_{n+1}U_n=Q_nT_n.$$

After expanding, comparing the coefficients of λ^j (j=0,1,2), we obtain

$$\begin{aligned} Q_n^{11}&=1,\\ Q_n^1&=Q_n^2=0,\\ Q_n^{12}&=p_n-b_n=\tilde{p}_n,\\ Q_n^{21}&=c_{n+1}+q_n=\tilde{q}_n\\ Q_n^{22}&=p_nc_{n+1}+1+p_nq_n-c_{n+1}b_n-q_nb_n\\ &=1+(c_{n+1}+q_n)(p_n-b_n)=1+\tilde{p}_n\tilde{q}_n. \end{aligned}$$

So

$$\tilde{U}_n=Q_n.$$

The proof is completed.

Proposition 2 Under the transformation (24), the matrix $\tilde{V}_n$ has the following form:

$$\tilde{V}_n=\begin{pmatrix} \frac{\lambda}{2}-\tilde{p}_n\tilde{q}_{n-1} & \lambda\tilde{p}_n \\ \tilde{q}_{n-1} & \tilde{p}_n\tilde{q}_{n-1}-\frac{\lambda}{2} \end{pmatrix}.$$

Proof

Let

$$T_n^{-1}=T_n^*/\det T_n,$$
$$(T_{nt}+T_nV_n)T_n^*=\begin{pmatrix} g_{11}(\lambda,n) & g_{12}(\lambda,n) \\ g_{21}(\lambda,n) & g_{22}(\lambda,n) \end{pmatrix}.$$

Through direct calculation we know $g_{11}(\lambda,n)$, $g_{12}(\lambda,n)$, $g_{21}(\lambda,n)$, $g_{22}(\lambda,n)$ are cubic polynomials of λ, and when $\lambda=\lambda_i$ (i=1,2), g_{kl} (λ_i,n)=0 (k,l=1,2).

Taking $g_{11}(\lambda,n)$ as an example, from Eq.(20) we notice that

$$\begin{aligned} &\lambda_i+a_n=-\lambda_i b_n\alpha_i(n),\\ &\lambda_i+d_n=-c_n\alpha_i^{-1}(n),\\ &a_{nt}=-\lambda_i b_{nt}\alpha_i(n)-\lambda_i b_n\alpha_{it}(n),\\ &\alpha_{it}(n)=q_{n-1}+2p_nq_{n-1}\alpha_i(n)-\lambda_i\alpha_i(n)-\lambda_i p_n\alpha_i^2(n). \end{aligned}$$

So

$$\begin{aligned} g_{11}(\lambda_i,n)=&\Big[a_{nt}+(a_n+\lambda_i)\Big(\frac{\lambda_i}{2}-p_nq_{n-1}\Big)\\ &+\lambda_i b_nq_{n-1}\Big](\lambda_i+d_n)\\ &-\lambda_i^2b_{nt}p_nc_n+(a_n+\lambda_i)\lambda_ip_nc_n\\ &+\lambda_ib_n\Big(-\frac{\lambda_i}{2}+p_nq_{n-1}\Big)c_n=0. \end{aligned}$$

Therefore we assume that

$$(T_{nt}+T_nV_n)T_n^*=(\det T_n)R_n,$$

with

$$R_n=\begin{pmatrix} \lambda R_n^{11}+R_n^1 & \lambda R_n^{12}+R_n^2 \\ R_n^{21} & \lambda R_n^{22}+R_n^3 \end{pmatrix}.$$

It can also be written as

$$T_{nt} + T_n V_n = R_n T_n.$$

Comparing the coefficient of λ^j (j=0,1,2), we get

$$\begin{aligned}
R_n^{11} =& \frac{1}{2},\\
R_n^{22} =& -\frac{1}{2},\\
R_n^{2} =& 0,\\
R_n^{12} =& p_n - b_n = \tilde{p}_n,\\
R_n^{21} =& c_n + q_{n-1} = \tilde{q}_{n-1},\\
R_n^{1} =& -p_n q_{n-1} + q_{n-1} b_n - p_n c_n + b_n c_n\\
=& -(p_n - b_n)(c_n + q_{n-1}) = -\tilde{p}_n \tilde{q}_{n-1},\\
R_n^{3} =& (p_n - b_n)(c_n + q_{n-1}) = \tilde{p}_n \tilde{q}_{n-1}.
\end{aligned}$$

The proof is completed.

So we conclude that the transformation (20) can transform the Lax pair into another Lax pair with the same form.

Proposition 3 The transformation $(Y_n;\ p_n,\ q_n) \to (\tilde{Y}_n;\ \tilde{p}_n,\ \tilde{q}_n)$ is a DT of Eq.(17). Under the transformation

$$\begin{aligned}
\tilde{p}_n &= p_n - b_n,\\
\tilde{q}_n &= q_n + c_{n+1},
\end{aligned}$$

if p_n, q_n are two solutions of Eq.(17), then $\tilde{p}_n, \tilde{q}_n$ are also solutions of Eq.(17).

In the following the transformation is used to get the new solutions of Eq.(17).

Let $p_n = q_n = 1$, Then the Lax pair of Eq.(18) can be written as

$$\begin{aligned}
Y_{n+1} &= \begin{pmatrix} \lambda & \lambda \\ 1 & 2 \end{pmatrix} Y_n,\\
Y_{nt} &= \begin{pmatrix} \frac{1}{2}\lambda - 1 & \lambda \\ 1 & 1 - \frac{1}{2}\lambda \end{pmatrix} Y_n. \qquad (25)
\end{aligned}$$

Two real basic solutions of Eq.(25) are

$$\begin{aligned}
\phi_n &= \beta^n \exp\left(\frac{t}{2}\sqrt{\lambda^2+4}\right) \begin{pmatrix} \lambda \\ 1 - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2+4}}{2} \end{pmatrix},\\
\psi_n &= \beta^{-n} \exp\left(-\frac{t}{2}\sqrt{\lambda^2+4}\right) \begin{pmatrix} \lambda \\ 1 - \frac{\lambda}{2} - \frac{\sqrt{\lambda^2+4}}{2} \end{pmatrix},
\end{aligned}$$

with

$$\beta = 1 + \frac{\lambda}{2} + \frac{\sqrt{\lambda^2+4}}{2}.$$

From

$$\alpha_i(n) = \frac{\phi_n^2(\lambda_i) - \gamma_i \psi_n^2(\lambda_i)}{\phi_n^1(\lambda_i) - \gamma_i \psi_n^1(\lambda_i)},$$

$$\alpha_1(n) = \frac{\beta_1^{2n} \mathrm{e}^{\xi_1 t}\left(1 - \frac{\lambda_1}{2} + \frac{\xi_1}{2}\right) - \gamma_1\left(1 - \frac{\lambda_1}{2} - \frac{\xi_1}{2}\right)}{\beta_1^{2n} \mathrm{e}^{\xi_1 t}\lambda_1 - \gamma_1 \lambda_1},$$

$$\alpha_2(n) = \frac{\beta_2^{2n} \mathrm{e}^{\xi_2 t}\left(1 - \frac{\lambda_2}{2} + \frac{\xi_2}{2}\right) - \gamma_2\left(1 - \frac{\lambda_2}{2} - \frac{\xi_2}{2}\right)}{\beta_2^{2n} \mathrm{e}^{\xi_2 t}\lambda_2 - \gamma_2 \lambda_2}.$$

so

$$\begin{aligned}
\tilde{p}_n =& 1 + b_n = 1 + \frac{\lambda_1 - \lambda_2}{\lambda_2\alpha_2 - \lambda_1\alpha_1},\\
\tilde{q}_n =& 1 + c_{n+1} = 1 + \frac{(\lambda_1 - \lambda_2)(1 + 2\alpha_1 + 2\alpha_2 + 4\alpha_1\alpha_2)}{(\lambda_2 - \lambda_1) + (\lambda_2\alpha_2 - \lambda_1\alpha_1) + 2(\lambda_2\alpha_1 - \lambda_1\alpha_2) + 2(\lambda_2 - \lambda_1)\alpha_1\alpha_2},
\end{aligned}$$

with

$$\begin{aligned}
\beta_i &= 1 + \frac{\lambda_i}{2} + \frac{\sqrt{\lambda_i^2+4}}{2},\\
\xi_i &= \sqrt{\lambda_i^2+4},
\end{aligned}$$

are new solutions of Eq.(17).

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