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A new type of Lie symmetrical non-Noether conserved quantity for nonholonomic systems*

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For a nonholonomic system, a new type of Lie symmetrical non-Noether conserved quantity is given under general infinitesimal transformations of groups in which time is variable. On the basis of the invariance theory of differential equations of motion under infinitesimal transformations for t and q_s, we construct the Lie symmetrical determining equations, the constrained restriction equations and the additional restriction equations of the system. And a new type of Lie symmetrical non-Noether conserved quantity is directly obtained from the Lie symmetry of the system, which only depends on the variables t, q_s and $\dot{q}_s$. A series of deductions are inferred for a holonomic nonconservative system, Lagrangian system and other dynamical systems in the case of vanishing of time variation. An example is given to illustrate the application of the results.

Keywords: nonholonomic system, Lie symmetry, non-Noether conserved quantity
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1. Introduction

Symmetries and conserved quantities for a dynamical system play an important role in the fields of mathematics, mechanics and physics. There are three main symmetry methods used to seek the conserved quantities of a dynamical system: Noether symmetry, Lie symmetry and Mei's form invariance. The Noether symmetry is an invariance of the Hamiltonian's action under the infinitesimal transformations of groups.[1−6] The Lie symmetry is an invariance of the differential equations under the infinitesimal transformations of groups.[6,7] The Mei form invariance means that the form of the equations of motion remains invariant under the infinitesimal transformations of the dynamical function of a system. It is different from the Noether symmetry and the Lie symmetry and can also lead to a conserved quantity under certain conditions.[8−11]

In 1979, Lutzky applied the Lie theory to the differential equations of motion for a mechanical system, and studied Lie symmetries and conserved quantities of the dynamical system.[7] In the last 20 years, researchers have developed the method of Lie symmetry and obtained many important results (see Refs.[4–7, 12–43]). In general, a Lie symmetry does not imply a conserved quantity. If and only if the infinitesimal transformations of Lie symmetry satisfy the structure equation (i.e. Noether identity), does the Lie symmetry lead to a conserved quantity,[6,7,12−30] which is then called the Noetherian. In 1992, Hojman[31] gave the direct method of finding conserved quantities by using Lie symmetry. This direct method is generalized by González-Gascón[32] and Lutzky.[33,34] Pillay and Leach[35] demonstrated that the invariant derived from Hojman's conservation law is trivial for all Noether symmetries. Therefore, by using Hojman's method, one can find the non-trivial conserved quantity which is led to by the non-Noether symmetry. The conserved quantity is called the non-Noetherian. In recent years, the Lie symmetrical non-Noether conserved quantities of some dynamical systems have been studied by using this direct method in Refs.[36–43], and a series of important results are obtained. But, the direct method of finding conserved quantity mentioned above is utilized only to the spe-

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cial infinitesimal transformations of groups in which the time is invariant.

This paper studies a new type of conserved quantity of Lie symmetry for a nonholonomic system under general infinitesimal transformations of groups in which the time is variable. On the basis of the invariance theory of differential equations of motion under the infinitesimal transformations, we construct the Lie symmetrical determining equations, the constrained restriction equations and the additional restriction equations of the system. And a new type of Lie symmetrical non-Noether conserved quantity is directly obtained from the Lie symmetry of the system, which only depends on the variables t, q_s and $\dot{q}_s$, and a series of deductions are given for the holonomic nonconservative system, the Lagrangian system and other systems in the case of vanishing of time variation. Also, an example is given to illustrate the results.

2. Lie symmetry of nonholonomic systems

Here we consider a mechanical system whose configuration is determined by the n generalized coordinates $q_s (s = 1, \ldots, n)$. Let its motion be subjected to the g ideal nonlinear nonholonomic constraints of Chetaev type

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \qquad (\beta = 1, \ldots, g). \tag{1}$$

Then its motion can be described by the differential equations as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s = 1, \ldots, n), \tag{2}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the nonpotential generalized forces, λ_β are the multipliers of constraint, and the repeated subscripts represent the summation. In general, it is supposed that the system (2) is nonsingular, i.e.

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0. \tag{3}$$

Before integrating Eq.(2), we can find λ_β as functions of t, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, i.e.

$$\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (\beta = 1, \ldots, g). \tag{4}$$

Substituting this into Eq.(2), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s \quad (s = 1, \ldots, n), \tag{5}$$

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}. \tag{6}$$

Equations (5) are called the equations of motion of the holonomic system corresponding to the nonholonomic system expressed by Eqs.(1) and (2). From Eqs.(5), we can find all the generalized accelerations as

$$\ddot{q}_s = h_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \ldots, n). \tag{7}$$

We introduce the general infinitesimal transformations for t and q_s as

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \end{aligned} \tag{8}$$

where ε is an infinitesimal parameter, and ξ_0, ξ_s are infinitesimal generators. Equation (8) is a one-parameter Lie group of transformations.

Taking the infinitesimal generator vector

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}, \tag{9}$$

and its first prolonged vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}, \tag{10}$$

the invariance of Eq.(7) under the infinitesimal transformations (8) can satisfy the determining equations

$$\ddot{\xi}_s - \dot{q}_s\ddot{\xi}_0 - 2\dot{\xi}_0 h_s = X^{(1)}(h_s) \quad (s = 1, \ldots, n), \tag{11}$$

and the invariance of the constraint Eq.(1) under the infinitesimal transformations (8) can be made to satisfy the constrained restriction equations

$$X^{(1)}(f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})) = 0 \quad (\beta = 1, \ldots, g). \tag{12}$$

Considering the Appell–Chetaev condition

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0 \quad (\beta = 1, \ldots, g; s = 1, \ldots, n) \tag{13}$$

and

$$\delta q_s = \Delta q_s - \dot{q}_s \Delta t = \varepsilon(\xi_s - \dot{q}_s\xi_0), \tag{14}$$

we have the following additional restriction equations:

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0 \quad (\beta = 1, \ldots, g; s = 1, \ldots, n). \tag{15}$$

If the generators ξ_0, ξ_s satisfy the determining equations (11), then the transformations are called the Lie symmetrical transformations of the corresponding holonomic system (5).

If the generators ξ_0 and ξ_s satisfy the determining equations (11) and the constrained restriction equations (12), then the transformations are called the

weakly Lie symmetrical transformations of the nonholonomic system (1) and (2).

If the generators ξ_0, ξ_s satisfy the determining equations (11), the constrained restriction equations (12) and the additional restriction equations (15), then the transformations are called the strongly Lie symmetrical transformations of the nonholonomic system (1) and (2).

3. A new type of Lie symmetrical non-Noether conserved quantity of nonholonomic systems

We now obtain directly the non-Noether conserved quantities from the Lie symmetries of the nonholonomic system, which only depend on the variables t, q_s and $\dot{q}_s$.

Theorem 1 For the holonomic system (5) corresponding to the nonholonomic system (1) and(2), if the generators ξ_0 and ξ_s of infinitesimal transformations (8) satisfy the determining equations (11) and the following condition:

$$\ddot{\xi}_0 = 0, \tag{16}$$

and there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\frac{\partial h_s}{\partial \dot{q}_s} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\mu = 0 \quad (s = 1, \ldots, n), \tag{17}$$

then the corresponding holonomic system (5) possesses the following conserved quantity:

$$\begin{aligned} I_L =& \frac{1}{\mu}\frac{\partial}{\partial t}(\mu\xi_0) + \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) \\ &+ \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}[\mu(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)] = \text{const}. \end{aligned} \tag{18}$$

Proof We have

$$\begin{aligned} \frac{\mathrm{d}I_L}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\right] \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0), \end{aligned} \tag{19}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} &= \frac{\partial\dot{\xi}_0}{\partial t} - \frac{\partial h_k}{\partial t}\frac{\partial\xi_0}{\partial\dot{q}_k}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s} &= \frac{\partial\dot{\xi}_s}{\partial q_s} - \frac{\partial h_k}{\partial q_s}\frac{\partial\xi_s}{\partial\dot{q}_k}, \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) &= \frac{\partial}{\partial\dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) - \frac{\partial}{\partial q_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) \\ &\quad - \frac{\partial h_k}{\partial\dot{q}_s}\frac{\partial}{\partial\dot{q}_k}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0). \end{aligned} \tag{20}$$

Substituting Eqs.(20) into Eq.(19) and using Eqs.(11), we obtain

$$\begin{aligned} \frac{\mathrm{d}I_L}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right) \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\right] + \ddot{\xi}_0 + \frac{\partial h_s}{\partial\dot{q}_s}\dot{\xi}_0 \\ &+ \frac{\partial^2 h_s}{\partial\dot{q}_s\partial t}\xi_0 + \frac{\partial^2 h_s}{\partial\dot{q}_s\partial q_k}\xi_k \\ &+ \frac{\partial^2 h_s}{\partial\dot{q}_s\partial\dot{q}_k}(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0). \end{aligned} \tag{21}$$

Taking the partial derivatives of condition (17) for t, q_k and $\dot{q}_k$, substituting them into Eq.(21), and using condition (16) and Eq.(11), we obtain

$$\frac{\mathrm{d}I_L}{\mathrm{d}t} = 0. \tag{22}$$

Theorem 2 For the nonholonomic system (1) and (2), if the generators ξ_0 and ξ_s of infinitesimal transformations (8) satisfy the determining equations (11) and the constrained restriction equations (12), and the conditions (16) and (17) hold, then the system possesses the weakly Lie symmetrical conserved quantity (18).

Theorem 3 For the nonholonomic system (1) and (2), if the generators ξ_0 and ξ_s of infinitesimal transformations (18) satisfy the determining equations (11), the constrained restriction equations (12) and the additional restriction equations (15), and the conditions (16) and (17) hold, then the system possesses the strongly Lie symmetrical conserved quantity (18).

4. Deductions

From Theorem 1, we have the following deductions:

Deduction 1 For the ideal holonomic nonconservative system (5) satisfying $\Lambda_s \equiv 0$, if the generators ξ_0 and ξ_s of infinitesimal transformations (8) satisfy the determining equations (11), and the conditions (16) and (17) hold, then the system possesses the Lie symmetrical conserved quantity (18).

Deduction 2 For the Lagrangian system (5) satisfying $\Lambda_s \equiv 0$ and $Q_s \equiv 0$, if the generators ξ_0 and ξ_s of infinitesimal transformations (8) satisfy the determining equations (11), and the conditions (16) and

(17) hold, then the system possesses the Lie symmetrical conserved quantity (18).

If we only introduce the special infinitesimal transformations for q_s, then we have $\Delta t = 0$ and $\xi_0 = 0$, and Eqs.(8)–(11) and (18) become, respectively,

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{23}$$

$$X^{(0)} = \xi_s \frac{\partial}{\partial q_s}, \quad X^{(1)} = X^{(0)} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s}, \tag{24}$$

$$\ddot{\xi}_s = X^{(1)}(h_s) = \xi_k \frac{\partial h_s}{\partial q_k} + \dot{\xi}_k \frac{\partial h_s}{\partial \dot{q}_k}, \tag{25}$$

$$I_L = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}(\mu\dot{\xi}_s) = \text{const.} \tag{26}$$

We then have the following deduction.

Deduction 3 For the holonomic system (5) corresponding to the nonholonomic system (1) and (2), if the generators ξ_s of infinitesimal transformations (23) satisfy the determining equations (25), and there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the condition (17), then the system possesses the Lie symmetrical conserved quantity (26).

Deduction 4 For the nonholonomic system (1) and (2), if the generators ξ_s of infinitesimal transformations (23) satisfy the determining equations (25), and there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the condition (17), then the system possesses the weakly Lie symmetrical conserved quantity (26).

Deduction 5 For the nonholonomic system (1) and (2), if the generators ξ_s of infinitesimal transformations (23) satisfy the determining equations (25), the constrained restriction equations (12) and the additional restriction equations (15), and there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the condition (17), then the system possesses the strongly Lie symmetrical conserved quantity (26).

For the ideal holonomic nonconservative system (5) satisfying $\Lambda_s \equiv 0$ and the Lagrangian system (5) satisfying $\Lambda_s \equiv 0$ and $Q_s \equiv 0$, Deduction 3 gives the conserved quantity (26), respectively.

For the Lagrangian system, if there is a function $\mu = \mu(\boldsymbol{q})$ satisfying condition (17), then we can obtain the Hojman conserved quantity (26).[31]

5. Example

As an example, we study the conserved quantities of Lie symmetry of the Appell–Hamel problem, of which the Lagrangian, the constraint equation and the nonpotential generalized forces are, respectively,[2−6]

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3, \tag{27}$$

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0, \tag{28}$$

$$Q_1 = Q_2 = Q_3 = 0. \tag{29}$$

Equation (2) gives

$$\begin{aligned} &m\ddot{q}_1 = 2\lambda\dot{q}_1, \quad m\ddot{q}_2 = 2\lambda\dot{q}_2, \\ &m\ddot{q}_3 = -mg - 2\lambda\dot{q}_3. \end{aligned} \tag{30}$$

From Eqs.(28) and (30), we have

$$\lambda = -\frac{mg}{4\dot{q}_3}, \tag{31}$$

and

$$\begin{aligned} \ddot{q}_1 &= -\frac{g\dot{q}_1}{2\dot{q}_3} = h_1, \\ \ddot{q}_2 &= -\frac{g\dot{q}_2}{2\dot{q}_3} = h_2, \\ \ddot{q}_3 &= -\frac{g}{2} = h_3. \end{aligned} \tag{32}$$

The determining Eqs.(11) give

$$\begin{aligned} &\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 + g\frac{\dot{q}_1}{\dot{q}_3}\dot{\xi}_0 \\ &= -(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0)\frac{g}{2\dot{q}_3} + (\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0)\frac{g\dot{q}_1}{2\dot{q}_3^2}, \\ &\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 + g\frac{\dot{q}_2}{\dot{q}_3}\dot{\xi}_0 \\ &= -(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)\frac{g}{2\dot{q}_3} + (\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0)\frac{g\dot{q}_2}{2\dot{q}_3^2}, \\ &\ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 + g\dot{\xi}_0 = 0, \end{aligned} \tag{33}$$

and these have solutions

$$\xi_0 = 1, \ \xi_1 = \xi_2 = \xi_3 = 0, \tag{34}$$

$$\xi_0 = 0, \ \xi_1 = \xi_2 = 0, \ \xi_3 = 1, \tag{35}$$

$$\xi_0 = 1, \ \xi_1 = \xi_2 = 0, \ \xi_3 = 1. \tag{36}$$

The generators (34), (35) and (36) satisfy respectively the condition (16). The condition (17) gives

$$-\frac{g}{\dot{q}_3} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\mu = 0, \tag{37}$$

and the above equation has following solutions

$$\mu = t^{-2}, \tag{38}$$

$$\mu = \left[-\frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + mgq_3\right]t^{-2}. \tag{39}$$

According to Theorem 1, from Eqs.(34), (38) and (39), we obtain, respectively,

$$I_L = -2t^{-1} = \text{const.} \tag{40}$$

From Eqs.(35),(38) and (39), we obtain, respectively,

$$I_L = 0, \tag{41}$$

$$I_L = -g\left[\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-gq_3\right]^{-1} = \text{const}. \tag{42}$$

From Eqs.(36),(38) and (39), we obtain, respectively,

$$I_L = -2t^{-1} = \text{const}, \tag{43}$$

$$I_L = -2t^{-1} - g\left[\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-gq_3\right]^{-1} = \text{const}. \tag{44}$$

The constrained restriction equations (12) give

$$(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0)\dot{q}_1+(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0)\dot{q}_2-(\dot{\xi}_3-\dot{q}_3\dot{\xi}_0)\dot{q}_3=0, \tag{45}$$

and the additional restriction equations (15) give

$$(\xi_1-\dot{q}_1\xi_0)\dot{q}_1+(\xi_2-\dot{q}_2\xi_0)\dot{q}_2-(\xi_3-\dot{q}_3\xi_0)\dot{q}_3=0. \tag{46}$$

The generators (34) satisfy Eqs.(45) and (46). From theorem 3, we know that the conserved quantity (40) is strongly Lie symmetrical. The generators (35) and (36) satisfy Eq.(45) and they do not satisfy Eq.(46). From Theorem 2, we know that the conserved quantities (42) and (44) are all weakly Lie symmetrical.

Using the inverse Noether theorem,[2−4] we know that the conserved quantities (40), (42) and (44) are non-Noether ones.

References

[1] Noether A E 1918 *Nachr. Akad. Wiss. Math. Phys.* **2** 235
[2] Li Z P 1993 *Classical and Quantal Dynamics of Constrained Systems and Their Symmetry Properties* (Beijing: Beijing Polytechnic University Press) (in Chinese)
[3] Li Z P 1999 *Constrained Hamiltonian Systems and Their Symmetry Properties* (Beijing: Beijing Polytechnic University Press) (in Chinese)
[4] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[5] Mei F X 2000 *ASME Appl. Mech. Rev.* **53** 283
[6] Mei F X 2000 *Int. J. Non-linear Mech.* **35** 229
[7] Lutzky M 1979 *J. Math. Phys. A: Math. Gen.* **12** 973
[8] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120
[9] Mei F X 2001 *Chin. Phys.* **10** 177
[10] Mei F X 2001 *J. Beijing Inst. Technol.* **10** 138
[11] Wang S Y and Mei F X 2002 *Chin. Phys.* **11** 5
[12] Prince G E and Elizer C J 1981 *J. Phys. A: Math. Gen.* **14** 587
[13] Thompson G 1986 *J. Phys. A: Math. Gen.* **19** 105
[14] Zhao Y Y 1994 *Acta Mech. Sin.* **26** 380 (in Chinese)
[15] Mei F X, Wu R H and Zhang Y F 1998 *Acta Mech. Sin.* **30** 468
[16] Mei F X, Zhang Y F and Shang M 1999*Mech. Res. Commun.* **26** 7
[17] Zhang Y and Mei F X 2000 *Chin. Sci. Bull.* **45** 1354
[18] Luo S K 2002 *Chin. Phys. Lett.* **19** 449
[19] Luo S K 2002 *Chin. Phys.* **11** 429
[20] Fu J L, Chen X W and Luo S K 2000 *Appl. Math. Mech.* **21** 549
[21] Fu J L and Wang X M 2000 *Acta Phys. Sin.* **49** 1023 (in Chinese)
[22] Zhang Y and Xue Y 2001 *Acta Phys. Sin.* **50** 816 (in Chinese)
[23] Zhang H B 2002 *Chin. Phys.* **11** 765
[24] Li Y C 2002 *Acta. Phys. Sin.* **51** 2186 (in Chinese)
[25] Qiao Y F 2001 *Acta. Phys. Sin.* **50** 1 (in Chinese)
[26] Fang J H 2002 *Chin. Phys.* **11** 313
[27] Fang J H 2002 *Appl. Math. Mech.* **23** 1215
[28] Fang J H 2003 *Commun. Theor. Phys.* **40** 269
[29] Luo S K 2003 *Acta. Phys. Sin.* **52** 2941 (in Chinese)
[30] Luo S K 2004 *Acta. Phys. Sin.* **53** 5 (in Chinese)
[31] Hojman S A 1992 *J. Phys. A: Math. Gen.* **25** L 291
[32] González-Gascón F 1994 *J. Phys. A: Math. Gen.* **27** L59
[33] Lutzky M 1998 *Int. J. Non-linear Mech.* **33** 393
[34] Lutzky M 1999 *Int. J. Non-linear Mech.* **34** 387
[35] Pillay T and Leach P G L 1996 *J. Phys. A: Math. Gen.* **29** 6999
[36] Mei F X 2002 *Chin. Sci. Bull.* **47** 2049
[37] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)
[38] Zhang Y 2002 *Chin. Quart. Mech.* **23** 392 (in Chinese)
[39] Zhang Y 2002 *Acta. Phys. Sin.* **51** 461 (in Chinese)
[40] Luo S K 2003 *Chin. Phys. Lett.* **20** 597
[41] Luo S K and Jia L Q 2003 *Commun. Theor. Phys.* **40** 265
[42] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255
[43] Luo S K and Mei F X 2004 *Acta Phys. Sin.* **53** 666 (in Chinese)