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Lattice Boltzmann simulation of transverse wave travelling in Maxwell viscoelastic fluid*

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A nine-velocity lattice Boltzmann method for Maxwell viscoelastic fluid is proposed. Travelling of transverse wave in Maxwell viscoelastic fluid is simulated. The instantaneous oscillating velocity, transverse shear speed and decay rate agree with theoretical results very well.

Keywords: lattice Boltzmann method, viscoelastic fluid
PACC: 4750

1. Introduction

The study of viscoelastic fluid has attracted much attention since its applications in studying the blood flow, polymeric fluids, etc. Studies on the numerical methods to simulate viscoelastic fluid are important in these fields. Howard *et al*[1] have simulated particulate sedimentation in Oldroyd-B viscoelastic fluid[2] using the arbitrary Lagrangian–Eulerian technique. They have observed drafting–kissing–chaining of two particles, in contrast to the drafting–kissing–tumbling in a Newtonian fluid. In the last decade, a new computational approach, called the lattice Boltzmann method (LBM),[3−5] has been developed as an alternative numerical method for simulating fluid flows in various physical and engineering systems. The LBM is based on the discrete Boltzmann equation[6] and inherited the advantage of local nature of the computation. The LBM has been proved to achieve second-order numerical accuracy in both space and time. The applications of the LBM include fluid turbulence,[7] multiphase and multi-component flows,[8−10] heat transfer,[11] reaction-diffusion,[12] particle suspensions in fluid,[13,14] blood flow[15−18] and flow in porous media.[19,20] Qian and Deng[21] were the first to extend lattice Boltzmann method to simulate Kelvin–Voigt-like viscoelastic fluid. Their model does not include memory of accumulated shear strain. Lallemand *et al* proposed a complex three-dimensional 32-velocity LBM for viscoelastic fluid, and simulated relaxation of longitude and transverse waves.[22] Recently Ispolatov and Grant[23] proposed an approach by incorporating viscoelastic effects into lattice Boltzmann simulations for a six-velocity two-dimensional BGK model on hexagonal lattice and successfully simulated the wave spectrum and resonance of Maxwell viscoelastic fluid. In this paper, we will extend Ispolatov's model to a nine-velocity LBM on a square lattice. The accuracy and the robustness of the model are demonstrated by numerical simulating the travelling of transverse wave in Maxwell viscoelastic fluid. The instantaneous oscillating velocity, transverse shear speed and decay rate are well consistent with theoretical results.

2. Nine-velocity LBM for the Maxwell viscoelastic fluid

Let $f_i(\boldsymbol{x},t)$ be a non-negative real number describing the distribution function of the fluid density at site $\boldsymbol{x}$ at time t moving in direction $\boldsymbol{e}_i$. Here $\boldsymbol{e}_0=(0,0)$, $\boldsymbol{e}_i=(\cos\pi(i-1)/2,\ \sin\pi(i-1)/2), i=1,2,3,4,$ and $\boldsymbol{e}_i=(\cos\pi(2i-1)/4, \sin\pi(2i-1)/4)$, for $i=5,6,7,8$ are the nine possible velocity vectors. The distribution functions evolve according to a Boltzmann equation that is discrete in both space and

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time,[5]

$$f_i(\boldsymbol{x}+\boldsymbol{e}_i,t+1)-f_i(\boldsymbol{x},t)=-\frac{1}{\tau}(f_i-f_i^{\rm eq}). \quad (1)$$

The density ρ and macroscopic velocity $\boldsymbol{v}$ are defined by

$$\rho=\sum_i f_i, \quad \rho\boldsymbol{v}=\sum_i f_i\boldsymbol{e}_i, \quad (2)$$

and the equilibrium distribution functions $f_i^{\rm eq}$ are usually supposed to be dependent only on the local density ρ and flow velocity $\boldsymbol{v}$. A suitable choice is[5]

$$f_i^{\rm eq}=\rho\alpha_i\left[1+3\boldsymbol{e}_i\cdot\boldsymbol{v}+\frac{9}{2}(\boldsymbol{e}_i\cdot\boldsymbol{v})^2-\frac{3}{2}v^2\right], \quad (3)$$

where $\alpha_0=4/9$, $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1/9$, and $\alpha_5=\alpha_6=\alpha_7=\alpha_8=1/36$. The macroscopic equations can be obtained by a Chapman–Enskog procedure.[5] The pressure p and the viscosity ν are defined by the equations $p=c_s^2\rho$ with $c_s^2=1/3$ and $\nu=(2\tau-1)/6$, respectively.

The constitutive equation of Maxwell viscoelastic model is

$$\tau\frac{\partial \Pi_{ij}^{\rm el}}{\partial t}=-\Pi_{ij}^{\rm el}+\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right), \quad (4)$$

where $\Pi_{ij}^{\rm el}$ is the viscoelastic stress tensor, μ is an elastic coefficient and τ is a memory time. From the definition of body force, we obtain

$$\int_V F_{{\rm el},i}{\rm d}v=\int_S -\Pi_{ij}^{\rm el}{\rm d}s_j, \quad (5)$$

where V is the bulk of the block of fluid, and S is the surface area of the block of fluid. That is,

$$F_{{\rm el},i}=-\frac{\partial}{\partial x_j}\Pi_{ij}^{\rm el}. \quad (6)$$

Taking Eq.(4) into account, we obtain

$$\boldsymbol{F}_{\rm el}(\mathbf{r},t)=\frac{\mu}{\tau}\int_{-\infty}^{t}{\rm d}t'{\rm e}^{-(t-t')/\tau}\nabla^2\boldsymbol{v}(\mathbf{r},t'), \quad (7)$$

where $\nabla^2=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)$. In order to reproduce the elastic term of Eq.(7), the lattice Boltzmann equation is modified as

$$\begin{aligned}&f_i(\boldsymbol{x}+\boldsymbol{e}_i,t+1)-f_i(\boldsymbol{x},t)\\ =&\frac{1}{6}\mathbf{F}_{\rm el}\cdot\boldsymbol{e}_i-\frac{1}{\lambda}(f_i-f_i^{\rm eq}).\end{aligned} \quad (8)$$

Equation (7) is implemented with partial difference quotient of t, then discretized,

$$\boldsymbol{F}_{\rm el}(\boldsymbol{r},t+1)=\left(1-\frac{1}{\tau}\right)\boldsymbol{F}_{\rm el}(\boldsymbol{r},t)+\frac{\mu}{\tau}\nabla^2\boldsymbol{v}(\boldsymbol{r},t). \quad (9)$$

The central difference formula of $\dfrac{\partial v_i}{\partial x_j}$ is

$$\frac{\partial v_i}{\partial x_j}=v_i\left(x_j+\frac{1}{2}\right)-v_i\left(x_j-\frac{1}{2}\right). \quad (10)$$

The central difference of $\dfrac{\partial^2 v_i}{\partial x_j^2}$ is

$$\frac{\partial^2 v_i}{\partial x_j^2}=v_i(x_j+1)+v_i(x_j-1)-2v_i(x_j). \quad (11)$$

From Eq.(11), for nine-velocity LBM, $\nabla^2\boldsymbol{v}$ is discretized in space,

$$\nabla^2\boldsymbol{v}=\frac{1}{3}\sum_{i=1}^{9}[\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{e}_i)-\boldsymbol{v}(\boldsymbol{r})]. \quad (12)$$

Eq.(9) and Eq.(12) define the evolution of vector $\boldsymbol{F}_{\rm el}$. Equation (8) defines the evolution of distribution function f_i and Eq.(2) defines the density and velocity of fluid. We will use this model to investigate transverse wave travelling in viscoelastic fluid.

Both sides of Eq.(8) are multiplied by vector $\boldsymbol{e}_i$ and summed from 0 to 9. Taking Eq.(2) into account, the Navier–Stokes equation with Maxwell elastic term reads as

$$\begin{aligned}\frac{\partial\boldsymbol{v}}{\partial t}=&-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{v}\\ &+\frac{\mu}{\rho\tau}\int_{-\infty}^{t}{\rm d}t'{\rm e}^{-(t-t')/\tau}\nabla^2\boldsymbol{v}(\boldsymbol{r},t'),\end{aligned} \quad (13)$$

where $p=\dfrac{1}{3}\rho$. Under incompressible conditions, the gradient of ρ is much smaller than the other items and is negligible, and Eq.(3) becomes

$$\frac{\partial\boldsymbol{v}}{\partial t}=\nu\nabla^2\mathbf{v}+\frac{\mu}{\rho\tau}\int_{-\infty}^{t}{\rm d}t'{\rm e}^{-(t-t')/\tau}\nabla^2\boldsymbol{v}(\boldsymbol{r},t'). \quad (14)$$

The nontrivial solution of Eq.(14) has the form

$$v_x(x,t)=0, \quad v_y(x,t)=v_0\exp({\rm i}(\omega t-kx)), \quad (15)$$

with

$$k^2=\frac{\omega(\omega\tau-i)}{\nu+\dfrac{\mu}{\rho}+{\rm i}\omega\nu\tau}. \quad (16)$$

The fitted solution is

$$\begin{aligned}k=&\sqrt{\frac{\omega}{2\left[\left(\nu+\dfrac{\mu}{\rho}\right)^2+(\nu\omega\tau)^2\right]}}\\ &\times\left\{\sqrt{\omega\frac{\mu}{\rho}\tau+\sqrt{\left(\omega\frac{\mu}{\rho}\tau\right)^2+\left[\nu(1+\omega^2\tau^2)+\frac{\mu}{\rho}\right]^2}}\right.\\ &\left.-{\rm i}\sqrt{-\omega\frac{\mu}{\rho}\tau+\sqrt{\left(\omega\frac{\mu}{\rho}\tau\right)^2+\left[\nu(1+\omega^2\tau^2)+\frac{\mu}{\rho}\right]^2}}\right\}.\end{aligned} \quad (17)$$

The plane transverse wave solution reads as

$$v_y = v_0 \exp[\mathrm{Im}(k)x] \cos[\mathrm{i}(\omega t - \mathrm{Re}(k)x)]. \quad (18)$$

Because $\mathrm{Im}(k) < 0$, v_y will decrease with increasing x.

3. Simulation and results

To demonstrate the robustness and accuracy of the present model in the simulation of the Maxwell viscoelastic fluid, we performed the following numerical simulation. The system size is $W \times H = 4000 \times 40$. $W \gg 20|\mathrm{Im}(k)|$ so that the right boundary has no influence on the decay of v_y. Periodic boundary condition is applied in the y direction. In the x direction and y direction at $x = W$, the non-slip boundary condition is implemented. The moving boundary condition is implemented in the x direction at $x = 0$ with y component velocity $v_y(0,t) = v_0 \cos(\omega t)$, and x component velocity $v_x \equiv 0$. Both boundary conditions in x and y direction use the technique proposed by Fillipova and Hanel.[25] The other parameters are $\lambda = 0.545$ ($\nu = 0.015$), $\mu = 0.3$, $\tau = 1000$, $\rho = 1.0$, and the period of oscillating boundary $T = 2000$ ($\omega = 2\pi/T$). Initial distributions were set to settled homogeneous equilibrium of density $\rho = 1.0$. In order to eliminate the influence of initial distribution, the calculation of physical quantities always begin with $20T$ initial preliminary steps. The simulation results are shown in Fig.1. It should be noted that Iaroslav and Martin only showed the average velocity $\langle v_y(x,t_n)\rangle_{y,n}$ measured at times $t_n = nT$, $n = 1, \ldots$ with $n = 1, 2, \ldots$.[23] We have displayed $v_y(x,t)$ at different times $t = 0, T/4, 2T/4, 3T/4$ in Fig.1 without averaging. The excellent agreement between the simulation result and the theoretical prediction verify the accuracy of the present lattice Boltzmann model.

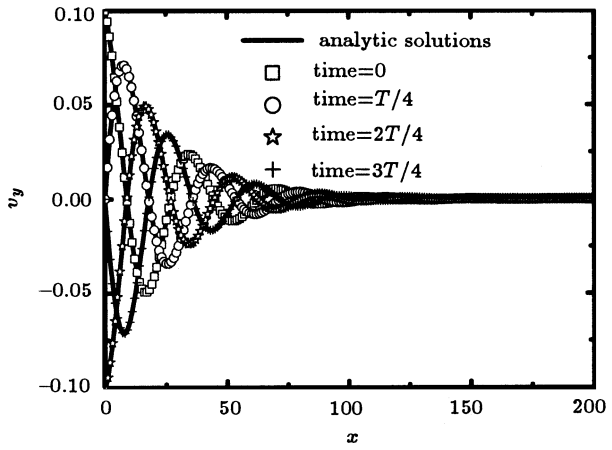


Fig.1. Plots of the y-component of the velocity measured at t=0, $T/4$, $2T/4$ and $3T/4$, vs the distance from the driving wall x=0 for ν=0.015, μ=0.3, T=2000, and τ=1000. The solid lines correspond to theoretical results $v_y = v_0 \exp(\mathrm{Im}(k)x) \cos(\mathrm{i}(\omega t - \mathrm{Re}(k)x)$.

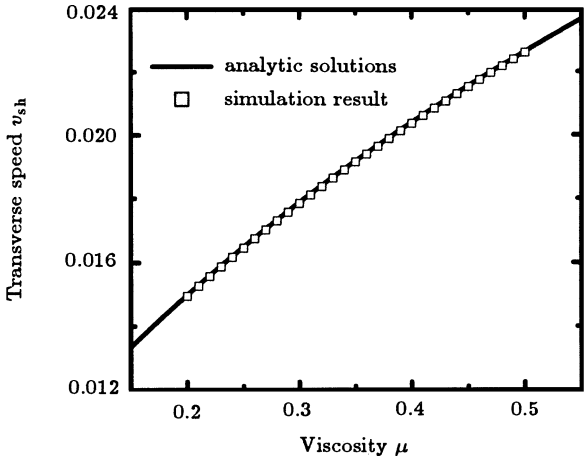


Fig.2. Plot of transverse shear speed vs μ for $\nu = 0.015$, $T = 2000$, and $\tau = 1000$, and the solid line corresponds to theoretical result.

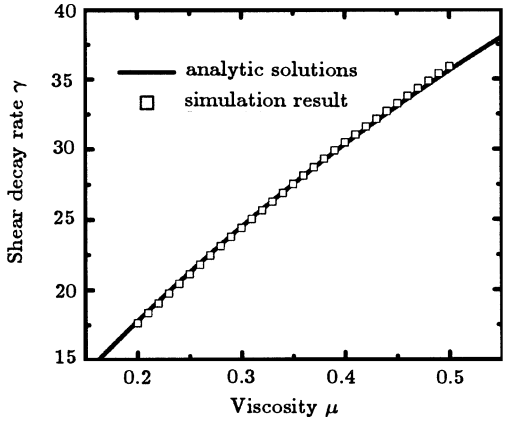


Fig.3. Plot of decay rates γ vs μ for $\nu = 0.015$, $T = 2000$, memory time $\tau = 1000$, and the solid line corresponds to the theoretical result.

The transverse shear speed $v_{\mathrm{sh}} = \dfrac{2\pi}{\mathrm{Re}(k)T}$ and decay rates $\gamma = -\dfrac{1}{\mathrm{Im}(k)}$ as functions of parameter μ can also be obtained from simulation, which are shown in Fig.2 and Fig.3, respectively. It is clear that our simulations also agree with theoretical results excellently.

4. Conclusion

We have extended a hexagonal six-velocity LMB of Maxwell viscoelastic fluid to square nine-velocity one. The accuracy and the robustness of the model have been demonstrated by numerical simulating the

travelling of transverse wave in Maxwell viscoelastic fluid. We note that Iaroslav and Martin only showed the average velocity $\langle v_y(x,t_n)\rangle_{y,n}$ measured at times $t_n = nT$, $n = 1,\ldots$ with $n = 1,2,\ldots$[23] while the agreement between our simulation and the theoretical prediction achieve a high accuracy at different times.

References

[1] Howard H H, Patankar N A and Zhu M Y 2001 *J. Comput. Phys.* **169** 427

[2] Bird R B, Armstrong R C and Hassager O 1987 *Dynamics of Polymeric Liquids* (Wiley-Interscience, New York) p1

[3] McNamara G R and Zanetti G 1988 *Phys. Rev. Lett.* **61** 2332

[4] Chen S Y, Chen H D, Martinez D O and Matthaeus W H 1991 *Phys. Rev. Lett.* **67** 3776

[5] Qian Y H, d'Humiéres D and Lallemand P 1992 *Europhys. Lett.* **17** 479

[6] Shan X W and He X Y 1998 *Phys. Rev. Lett.* **80** 65

[7] Chen H D *et al* 2003 *Science* **301** 633

[8] Shan X W and Chen H D 1993 *Phys. Rev.* E **47** 1815

[9] Fang H P *et al* 2001 *Int. J. Mod. Phys.* B **15** 1287

[10] Fang H P, Wan R Z and Fan L W 2000 *Chin. Phys.* **9** 515

[11] D'Orazio A, Succi S and Arrighetti C 2003 *Phys. Fluids* **15** 2778

[12] Dawson S P, Chen S Y and Doolen G 1993 *J. Chem. Phys.* **98** 1514

[13] Ladd A J C 1994 *J. Fluid Mech.* **271** 285

[14] Aidun C K, Lu Y N and Ding E J 1998 *J. Fluid Mech.* **373** 287

[15] Migliorini C *et al* 2002 *Biophys. J.* **83** 1834

[16] Hoekstra A G *et al* 2003 *Lect. Notes Comput. Sc.* **2657** 997

[17] Fang H P, Wang Z W, Lin Z F and Liu M R 2002 *Phys. Rev.* E **65** 051925

[18] Li H B, Lu X Y, Fang H P and Lin Z F *Prog. Comput. Fluid Dynam.* to appear

[19] Guo Z L and Zhao T S 2002 *Phys. Rev.* E **66** 036304

[20] Xu Y S and Xu X Z 2002 *Chin. Phys.* **11** 583

[21] Qian Y H and Deng Y F 1997 *Phys. Rev. Lett.* **79** 2742

[22] Lallemand P, d'Humiéres D, Lou L S and Rubinstein R 2003 *Phys. Rev.* E **67** 021203

[23] Ispolatov I and Grant M 2002 *Phys. Rev.* E **65** 056704

[24] He X Y and Luo L S 1997 *Phys. Rev.* E **56** 6811

[25] Filippova O and Hanel D 1997 *Comput. Fluids* **26** 697