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A general mapping approach and new travelling wave solutions to the general variable coefficient KdV equation*

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A general mapping deformation method is applied to a generalized variable coefficient KdV equation. Many new types of exact solutions, including solitary wave solutions, periodic wave solutions, Jacobian and Weierstrass doubly periodic wave solutions and other exact excitations are obtained by the use of a simple algebraic transformation relation between the generalized variable coefficient KdV equation and a generalized cubic nonlinear Klein–Gordon equation.

Keywords: generalized variable coefficient KdV equation, general mapping approach, travelling wave solution

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1. Introduction

Recently, using a deformation mapping method, which was first presented by Lou and Ni[1], Li[2] has obtained many soliton solutions and periodic solutions of nonlinear models through finding some relations between the travelling wave solutions of the given models and those of the cubic nonlinear Klein–Gordon (NKG) system which has been studied widely. Fan *et al*[3−11] have improved the mapping method and applied it in some physical models, and obtained a lot of new travelling wave solutions. This deformation mapping method can be extended to solving variable coefficients nonlinear equation systems. The basic idea of the algorithm is as follows. For a given (1+1)-dimensional nonlinear model, the equation reads

$$P(u, u_x, u_t, u_{xt}, u_{xx}, u_{tt}, \ldots) = 0, \tag{1}$$

where P is in general a polynomial function of its arguments, and their subscripts denote the partial derivatives. By using the travelling wave transformation, Eq.(1) possesses the following ansatz:

$$u = u(\xi),\ \xi = p(t)x + q(t), \tag{2}$$

where $p(t)$ and $q(t)$ are arbitrary functions of variable t to be determined. Substituting Eq.(2) into Eq.(1) yields an ordinary differential equation (ODE) $O(u(\xi), u(\xi)_\xi, u(\xi)_{\xi\xi}, \ldots)$=0. Then $u(\xi)$ is expanded into a polynomial in $g(\xi)$

$$u(\xi) = F(g(\xi)) = \sum_{j=0}^{m} a_j(t) g^j(\xi), \tag{3}$$

where $a_j(t)$ are functions of variable t to be determined and m is fixed by balancing the linear term of the highest order with the nonlinear term in Eq.(1). If we suppose $g(\xi) = \tanh(\xi)$, $g(\xi) = \mathrm{sech}(\xi)$, and $g(\xi) = \mathrm{sn}(\xi)$ or $g(\xi) = \mathrm{cn}(\xi)$ respectively, then the corresponding approach is usually called the tanh-function method, the sech-function method and the Jacobian-function method.[12−16] The main idea of the mapping approach is that $g(\xi)$ is not assumed to be a specific function, such as tanh, sech, sn and cn, etc, but a solution to the cubic nonlinear KG equation

$$\begin{aligned} g_\xi^2 &= c + \lambda g^2 + \frac{\mu}{2} g^4, \\ g_{\xi\xi} &= \lambda g + \mu g^3, \end{aligned} \tag{4}$$

where c, λ and μ are all arbitrary constants. Using the mapping relation (3) and the solutions to Eq.(4),

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one can obtain many exact travelling wave solutions to Eq.(1). In our present paper, along this line, we further assume that the mapping function $g(\xi)$ is an excitation of the generalized cubic nonlinear KG equation

$$g_\xi^2 = (c_0 + c_1 g + c_2 g^2 + c_3 g^3 + c_4 g^4),$$
$$g_{\xi\xi} = (c_1 + 2c_2 g + 3c_3 g^2 + 4c_4 g^3)/2, \qquad (5)$$

where $c_i, i \in (1,2,3,4)$ are all arbitrary constants. Since the generalized cubic nonlinear KG equation possesses more solutions, we may derive more travelling wave solutions to Eq.(1).

The KdV equation is encountered in many apparently unrelated phenomena in various physical systems such as plasmas, fluids, and lattice vibrations of a crystal at low temperatures. All these applications start from a more or less general physical model and end up in the KdV equation by considering a specific limit of the physical problem concerned. In this sense, the KdV equation is universal. In this paper, we apply the general mapping approach to the generalized variable coefficient KdV equation, which has been studied by Zhang and Cheng[17] and Li and Zhang.[18]

2. The application of the general mapping approach to the generalized variable coefficient KdV equation

Now we use the general deformation mapping method to find some novel travelling wave solutions to the generalized variable coefficient KdV equation[18]

$$u_t + 2\beta(t)u + [\alpha(t) + \beta(t)x]u_x - 3c\gamma(t)uu_x + \gamma(t)u_{xxx} = 0, \qquad (6)$$

where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are time-dependent coefficients. The importance of Eq.(6) is well known and there is an increasingly high level of interest in the construction of exact solutions to its special forms. Equation (6) can be reduced to a cylindrical KdV equation,

$$u_t + 6u + u_{xxx} + \frac{u}{2t} = 0, \qquad (7)$$

which describes solitary waves in a channel with slowly varying depth.

With the aid of the general mapping relation and the solutions to the generalized cubic NKG equation, many new exact travelling wave solutions to the generalized variable coefficient KdV equation can be obtained.

Now, substituting Eq.(2) into Eq.(6) and balancing the highest order derivative term u_{xxx} with the highest power nonlinear term uu_x gives leading order m=2. Therefore, we can choose the solution to Eq.(6) in the form

$$u(\xi) = a_0(t) + a_1(t)g(\xi) + a_2(t)g^2(\xi),$$
$$\xi = p(t)x + q(t). \qquad (8)$$

Substituting Eq.(8) into Eq.(6), and setting the coefficients of $g^i g'^j x^k (i = 1,2,3; j = 0,1; k = 0,1)$ to be zero, a system of nonlinear ordinary differential equations with variable coefficients are obtained as follows:

$$a_0' + 2\beta a_0 = 0,$$
$$a_1' + 2\beta a_1 = 0,$$
$$a_2' + 2\beta a_2 = 0,$$
$$a_1(p' + \beta p) = 0,$$
$$2a_2(p' + \beta p) = 0, \qquad (9)$$
$$6\gamma a_2 p(4c_4 p^2 - ca_2) = 0,$$
$$3\gamma p(-3ca_1a_2 + 2c_4 a_1 p^2 + 5c_3 a_2 p^2) = 0,$$
$$2\alpha p a_2 - 6c\gamma a_0 a_2 p + 3c_3\gamma a_1 p^3 + 8c_2\gamma a_2 p^3 - 3c\gamma a_1^2 p + 2a_2 q' = 0,$$
$$\alpha a_1 p - 3c\gamma a_0 a_1 p + c_2\gamma a_1 p^3 + 3c_1\gamma a_2 p^3 + a_1 q' = 0.$$

From Eq.(9), we finally find five kinds of solutions, namely

(A) $c_3 = c, c_4 = 0, a_0(t) = a_2(t) = 0, a_1(t) = \exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], p(t) = \exp\left[-\int_a^t \beta(s)\mathrm{d}s\right],$

$$q(t) = -\int_a^t (\alpha(s)\exp\left[-\int_a^s \beta(v)\mathrm{d}v\right] + c_2\gamma(s)\exp\left[-3\int_a^s \beta(v)\mathrm{d}v\right])\mathrm{d}s, \qquad (10)$$

where c_0, c_1, c_2 are arbitrary constants.

(B) $c_3 = c, c_4 = 0, a_0(t) = a_1(t) = \exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], a_2(t) = 0, p(t) = \exp\left[-\int_a^t \beta(s)\mathrm{d}s\right],$

$$q(t) = -\int_a^t (\alpha(s)\exp\left[-\int_a^s \beta(v)\mathrm{d}v\right] + (c_2 - 3c)\gamma(s)\exp\left[-3\int_a^s \beta(v)\mathrm{d}v\right])\mathrm{d}s, \qquad (11)$$

where c_0, c_1, c_2 are arbitrary constants.

(C) $c_1 = c_3 = 0, c_4 = c/4, a_0(t) = a_1(t) = 0, a_2(t) = \exp\Big[-2\int_a^t \beta(s)\mathrm{d}s\Big], p(t) = \exp\Big[-\int_a^t \beta(s)\mathrm{d}s\Big],$

$$q(t) = -\int_a^t (\alpha(s)\exp\Big[-\int_a^s \beta(v)\mathrm{d}v\Big] + 4c_2\gamma(s)\exp\Big[-3\int_a^s \beta(v)\mathrm{d}v\Big])\mathrm{d}s, \tag{12}$$

where c_0, c_2 are arbitrary constants.

(D) $c_1 = c_3 = 0, c_4 = c/4, a_1(t) = 0, a_0(t) = a_2(t) = \exp\Big[-2\int_a^t \beta(s)\mathrm{d}s\Big], p(t) = \exp\Big[-\int_a^t \beta(s)\mathrm{d}s\Big],$

$$q(t) = -\int_a^t (\alpha(s)\exp\Big[-\int_a^s \beta(v)\mathrm{d}v\Big] + (4c_2 - 3c)\gamma(s)\exp\Big[-3\int_a^s \beta(v)\mathrm{d}v\Big])\mathrm{d}s, \tag{13}$$

where c_0, c_2 are arbitrary constants.

(E) $3c_1 + c_2 = c/4,\ c_3 = c/2, c_4 = c/4, a_0(t) = 0, a_1(t) = a_2(t) = \exp\Big[-2\int_a^t \beta(s)\mathrm{d}s\Big],$

$$p(t) = \exp\Big[-\int_a^t \beta(s)\mathrm{d}s\Big],$$

$$q(t) = -\int_a^t (\alpha(s)\exp\Big[-\int_a^s \beta(v)\mathrm{d}v\Big] + c/4\gamma(s)\exp\Big[-3\int_a^s \beta(v)\mathrm{d}v\Big])\mathrm{d}s, \tag{14}$$

where c_0 are arbitrary constants.

From the above derivations, we can see that if $g(\xi)$ is determined, then the corresponding travelling solution u can be easily fixed.

3. Some new exact travelling wave excitations to the generalized variable coefficient KdV equation

In Ref.[5], many exact solutions to the generalized cubic NKG equation have been given. By using the mapping relation, we can obtain many new exact travelling wave solutions to the generalized variable coefficient KdV equation, which contain solitary solutions, periodic wave solutions, Jacobian and Weierstrass functions and other travelling wave solutions. In what follows we only discuss some new or interesting expressions of g and its corresponding excitations, rather than all of it.

3.1. Weierstrass elliptic doubly periodic-type wave excitation

According to Eq.(10), as c_2=0, $c_3 = c > 0$, $a_1(t) = \exp\Big[-2\int_a^t \beta(s)\mathrm{d}s\Big]$, we can obtain a Weierstrass elliptic doubly periodic-type wave solution

$$g_1 = \wp(\frac{\sqrt{c}}{2}\xi, G_2, G_3), \tag{15}$$

where $G_2 = -4c_1/c$ and $G_3 = -4c_0/c$ are called invariants of the Weierstrass elliptic function. Then the corresponding Weierstrass elliptic doubly periodic-type wave excitation of Eq.(6) reads as

$$u_1 = \wp(\frac{\sqrt{c}}{2}\xi, G_2, G_3)\exp\Big[-2\int_a^t \beta(s)\mathrm{d}s\Big], \tag{16}$$

where ξ denotes $p(t)x+q(t)$, with $p(t)$ and $q(t)$ showed by Eq.(10).

3.2. Jacobian elliptic doubly periodic wave excitation

Based on Eq.(12), as $c_3 = 0$ (corresponding $c_1 = 0$), three Jacobian elliptic doubly periodic wave solutions can be derived:

$$g_2 = \varepsilon\sqrt{-\frac{c_2m^2}{c_4(m^2+1)}}\,\mathrm{sn}\Big(\sqrt{-\frac{c_2}{m^2+1}}\xi, m\Big), \quad \Big(c_0 = \frac{2c_2^2m^2}{c_4(m^2+1)^2}, c_2 < 0, c_4 > 0, \varepsilon^2 = 1\Big), \tag{17}$$

$$g_3 = \sqrt{-\frac{c_2m^2}{c_4(2m^2-1)}}\,\mathrm{cn}\Big(\sqrt{\frac{c_2}{2m^2-1}}\xi, m\Big), \quad \Big(c_0 = \frac{2c_2^2m^2(m^2-1)}{c_4(2m^2-1)^2}, c_2 > 0, c_4 < 0\Big), \tag{18}$$

$$g_4 = \sqrt{-\frac{c_2}{c_4(2-m^2)}}\,\mathrm{dn}\Big(\sqrt{\frac{c_2}{2-m^2}}\xi, m\Big), \quad \Big(c_0 = \frac{2c_2^2(1-m^2)}{c_4(2-m^2)^2}, c_2 > 0, c_4 < 0\Big), \tag{19}$$

where m is modulus of Jacobian function. Then the corresponding Jacobian elliptic function solutions to

Eq.(6) read

$$u_2 = -\frac{4c_2m^2}{c(m^2+1)}\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi, m\right) \times \exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (20)$$

$$u_3 = -\frac{4c_2m^2}{c(2m^2-1)}\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi, m\right) \times \exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (21)$$

$$u_4 = -\frac{4c_2}{c(2-m^2)}\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi, m\right) \times \exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (22)$$

where ξ denotes $p(t)x+q(t)$, with $p(t)$ and $q(t)$ showed by Eq.(12).

3.3. Solitary-like wave excitation

Similarly, as $c_0 = c_1 = 0$, the solitary-like wave solutions to Eq.(5) are

$$g_5 = \frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\varepsilon\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}, \ (c_2 > 0); \quad (23)$$

as $c_0 = c_1 = 0, c_2 > 0$,

$$g_6 = -\frac{c_2}{c}\mathrm{sech}^2(\frac{\sqrt{c_2}}{2}\xi); \quad (24)$$

as $c_0 = c_1 = c_3 = 0, c_2 > 0$,

$$g_7 = \sqrt{-\frac{4c_2}{c}}\mathrm{sech}(\sqrt{c_2}\xi); \quad (25)$$

as $c_0 = c_2^2/c, c_1 = c_3 = 0, c_2 < 0$,

$$g_8 = \sqrt{-\frac{2c_2}{c}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right). \quad (26)$$

Then the corresponding hyperbolic elliptic function solutions to Eq.(6) read as

$$u_5 = \frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\varepsilon\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3} \times \left(1 + \frac{c_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{c_2}\xi\right)}{2\varepsilon\sqrt{c_2c_4}\tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}\right) \times \exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (27)$$

where $\varepsilon = \pm 1, \omega = p(t)x+q(t)$, $p(t)$ and $q(t)$ are shown by Eq.(13);

$$u_6 = -\frac{c_2}{c}\mathrm{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right)\exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (28)$$

where $\xi = p(t)x + q(t)$, $p(t)$ and $q(t)$ are shown by Eq.(10);

$$u_7 = -\frac{4c_2}{c}\mathrm{sech}^2(\sqrt{c_2}\xi)\exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (29)$$

where $\xi = p(t)x + q(t)$, $p(t)$ and $q(t)$ are shown by Eq.(12);

$$u_8 = -\frac{2c_2}{c}\tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right)\exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (30)$$

where $\xi = p(t)x + q(t)$, $p(t)$ and $q(t)$ are shown by Eq.(12).

3.4. Periodic wave excitation

Now we focus on the periodic wave solutions. As $c_0 = c_1 = 0, c_2 < 0$, the periodic wave solution to Eq.(5) is

$$g_9 = -\frac{c_2}{c}\mathrm{sec}^2\left(\frac{\sqrt{-c_2}}{2}\xi\right); \quad (31)$$

as $c_0 = c_2^2/c, c_1 = c_3 = 0, c_2 > 0$,

$$g_{10} = \frac{4c_2}{c}\tan^2\left(\sqrt{\frac{c_2}{2}}\xi\right). \quad (32)$$

The corresponding Jacobian elliptic function solutions to Eq.(6) read

$$u_9 = -\frac{c_2}{c}\mathrm{sec}^2\left(\frac{\sqrt{-c_2}}{2}\xi\right)\exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (33)$$

where $\xi = p(t)x + q(t)$, $p(t)$ and $q(t)$ are showed by Eq.(10);

$$u_{10} = \frac{4c_2}{c}\tan^2\left(\sqrt{\frac{c_2}{2}}\xi\right)\exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (34)$$

where $\xi = p(t)x + q(t)$, $p(t)$ and $q(t)$ are shown by Eq.(12).

3.5. Other new excitation

Besides the above excitations, there exist abundant other new solutions to g like rational solutions, exponential solutions and polynomial solutions, etc, for there are arbitrary constants in Eq.(5). As a detailed discussion has been given in Ref.[4], we will not investigate all the cases and only cite a special example. For example, as $c_0 = 1/36, c_3 = c_4 = 0, c_2 > 0$, the exponential solution to Eq.(5) is

$$g_{11} = 1/6 + \exp(\varepsilon\sqrt{c_2}\xi), \ (\varepsilon^2 = 1). \quad (35)$$

Then the corresponding solution to Eq.(1) reads

$$u_{11} = (1/6 + \exp(\varepsilon\sqrt{c_2}\xi))(1 + 1/6 + \exp(\varepsilon\sqrt{c_2}\xi)) \times \exp\left[-2\int_a^t \beta(s)\mathrm{d}s\right], \quad (36)$$

where $\xi = p(t)x + q(t)$, $p(t)$ and $q(t)$ are showed by Eq.(13).

4. Summary

Using the general mapping deformation method, we obtain many new types of travelling wave solutions to the generalized variable coefficient KdV equation. To our knowledge, many excitations were not reported in the previous studies like Refs.[17] and [18]. The approach we have used here is standard and simple and can be used in various types of nonlinear equations no matter they are integrable or not. Therefore the proposed method is worthy of further study and may help us find more new excitations (especially the solitary wave excitations).

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